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Integrazione di alcune funzioni particolari

1 Funzioni razionali con radici

Sia

$$R\left(x, \sqrt{\frac{ax+b}{cx+d}}\right)$$

una funzione razionale che dipende da x e $\sqrt{\frac{ax+b}{cx+d}}$.

Gli integrali del tipo

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) \mathrm{d}x$$

si risolvono con la sostituzione

$$\begin{aligned} t &= \sqrt[n]{\frac{ax+b}{cx+d}} \\ t^n &= \frac{ax+b}{cx+d} \\ nt^{n-1}\,\mathrm{d}t &= \left(\frac{ax+b}{cx+d}\right)' \mathrm{d}x \\ &= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2}\,\mathrm{d}x \end{aligned}$$

1.1 Esempio

$$\begin{aligned}
&\int \frac{1}{(1-x)\sqrt{1+x}}\,\mathrm{d}x \\
&\quad= \int \frac{2t}{(1-t^2+1)t}\,\mathrm{d}t \qquad\qquad t=\sqrt{1+x} \implies \begin{cases} t^2 = 1+x \implies x = t^2-1 \\ 2t\,\mathrm{d}t = 1\,\mathrm{d}x \end{cases} \\
&\quad= 2\int \frac{1}{2-t^2}\,\mathrm{d}t \\
&\quad= 2\int \frac{1}{\left(\sqrt{2}-t\right)\left(\sqrt{2}+t\right)}\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\left(\sqrt{2}-t\right)\left(\sqrt{2}+t\right)} &= \frac{A}{\sqrt{2}-t} + \frac{B}{\sqrt{2}+t} \\
&= \frac{A\sqrt{2}+At+B\sqrt{2}-Bt}{\left(\sqrt{2}-t\right)\left(\sqrt{2}+t\right)} \\
&= \frac{(A-B)t + A\sqrt{2}+B\sqrt{2}}{\left(\sqrt{2}-t\right)\left(\sqrt{2}+t\right)}
\end{aligned}$$

$$\begin{cases} A-B=0 \\ A\sqrt{2}+B\sqrt{2}=1 \end{cases} \implies \begin{cases} A=B \\ 2A\sqrt{2}=1 \end{cases} \implies A=B=\frac{1}{2\sqrt{2}}$$

$$\begin{aligned}
2\int \frac{1}{\left(\sqrt{2}-t\right)\left(\sqrt{2}+t\right)}\,\mathrm{d}t &= \frac{2}{2\sqrt{2}}\int \frac{1}{\sqrt{2}-t}\,\mathrm{d}t + \frac{2}{2\sqrt{2}}\int \frac{1}{\sqrt{2}+t}\,\mathrm{d}t \\
&= -\frac{1}{\sqrt{2}}\int \frac{1}{t-\sqrt{2}}\,\mathrm{d}t + \frac{1}{\sqrt{2}}\int \frac{1}{t+\sqrt{2}}\,\mathrm{d}t \\
&= -\frac{1}{\sqrt{2}}\log\left|t-\sqrt{2}\right| + \frac{1}{\sqrt{2}}\log\left|t+\sqrt{2}\right| + c \\
&= \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{1+x}+\sqrt{2}}{\sqrt{1+x}-\sqrt{2}}\right| + c
\end{aligned}$$

2 Funzioni razionali con seni e coseni

Gli integrali

$$\int R(\sin x, \cos x)\,\mathrm{d}x$$

si risolvono con la sostituzione:

$$t = \operatorname{tg} \frac{x}{2}$$

$$\begin{aligned}
\implies \sin x &= \sin\left(2 \cdot \frac{x}{2}\right) = 2 \sin \frac{x}{2} \cos \frac{x}{2} \\
&= 2 \sin \frac{x}{2} \cos \frac{x}{2} \cdot \frac{\cos \frac{x}{2}}{\cos \frac{x}{2}} = 2 \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \cdot \cos^2 \frac{x}{2} \\
&= 2 \frac{\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}}{\frac{1}{\cos^2 \frac{x}{2}}} = 2 \frac{\operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} \qquad \text{perché } \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
\implies \cos x &= \cos\left(2 \cdot \frac{x}{2}\right) = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \\
&= \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right) \frac{\cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \left(1 - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}\right) \cos^2 \frac{x}{2} \\
&= \frac{1 - \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}}{\frac{1}{\cos^2 \frac{x}{2}}} = \frac{1 - \operatorname{tg}^2 \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} \\
&= \frac{1-t^2}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
&\implies \mathrm{d}t = \frac{1}{2}\left(1 + \operatorname{tg}^2 \frac{x}{2}\right) \mathrm{d}x = \frac{1}{2}(1+t^2)\,\mathrm{d}x \\
&\implies \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t
\end{aligned}$$

2.1 Esempio

$$\begin{aligned}\int \frac{1}{\sin x}\,\mathrm{d}x &= \int \frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}\,\mathrm{d}t \qquad\qquad t=\operatorname{tg}\frac{x}{2} \implies \begin{cases}\sin x = \dfrac{2t}{1+t^2}\\ \mathrm{d}x = \dfrac{2}{1+t^2}\,\mathrm{d}t\end{cases}\\ &= \int \frac{1}{t}\,\mathrm{d}t\\ &= \log|t| + c\\ &= \log\left|\operatorname{tg}\frac{x}{2}\right| + c\end{aligned}$$

3 Prodotti di potenze di seni e coseni

Gli integrali del tipo

$$\int \sin^m x \cos^n x\,\mathrm{d}x$$

si risolvono:

- per sostituzione se m e/o n è dispari;
- per parti se m e n sono entrambi pari.

Analogamente per integrali del tipo

$$\int \sin^m x\,\mathrm{d}x \qquad \int \cos^m x\,\mathrm{d}x$$

3.1 Esempio con esponente dispari

$$\begin{aligned}
\int \sin^2 x \cos^3 x \,\mathrm{d}x &= \int \sin^2 x \cos^2 x \cos x \,\mathrm{d}x \\
&= \int \sin^2 x (1 - \sin^2 x) \cos x \,\mathrm{d}x \\
&= \int (\sin^2 x - \sin^4 x) \cos x \,\mathrm{d}x \\
&= \int (t^2 - t^4) \,\mathrm{d}t && t = \sin x \implies \mathrm{d}t = \cos x \,\mathrm{d}x \\
&= \frac{t^3}{3} - \frac{t^5}{5} + c \\
&= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c
\end{aligned}$$

3.2 Esempio con esponenti pari

$$\begin{aligned}
\int \cos^4 x \sin^2 x \,\mathrm{d}x &= \int \cos^4 x (1 - \cos^2 x) \,\mathrm{d}x \\
&= \int (\cos^4 x - \cos^6 x) \,\mathrm{d}x \\
&= \int \cos^4 x \,\mathrm{d}x - \int \cos^6 x \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
\int \cos^4 x \,\mathrm{d}x &= \int \underbrace{\cos^3 x}_{g}\underbrace{\cos x}_{f'} \,\mathrm{d}x \qquad\qquad g'(x) = -3\cos^2 x \sin x \\
&= \sin x \cos^3 x + \int 3\cos^2 x \sin^2 x \,\mathrm{d}x \\
&= \sin x \cos^3 x + 3\int \cos^2 x (1-\cos^2 x) \,\mathrm{d}x \\
&= \sin x \cos^3 x + 3\int \cos^2 x \,\mathrm{d}x - 3\int \cos^4 x \,\mathrm{d}x \\
&= \sin x \cos^3 x + 3\int \underbrace{\frac{1+\cos(2x)}{2}}_{\substack{=\cos^2 x \\ \text{(formula trig.)}}} \,\mathrm{d}x - 3\int \cos^4 x \,\mathrm{d}x \\
&= \sin x \cos^3 x + 3\int \frac{1}{2} \,\mathrm{d}x + 3\cdot\frac{1}{4}\int 2\cos(2x) \,\mathrm{d}x - 3\int \cos^4 x \,\mathrm{d}x \\
&= \sin x \cos^3 x + \frac{3}{2}x + \frac{3}{4}\sin(2x) - 3\int \cos^4 x \,\mathrm{d}x \\
\implies 4\int \cos^4 x \,\mathrm{d}x &= \sin x \cos^3 x + \frac{3}{2}x + \frac{3}{4}\sin(2x) \\
\int \cos^4 x \,\mathrm{d}x &= \frac{\sin x \cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x) + c
\end{aligned}
$$

$$
\begin{aligned}
\int \cos^6 x \,\mathrm{d}x &= \int \underbrace{\cos^5 x}_{g}\underbrace{\cos x}_{f'} \,\mathrm{d}x \qquad\qquad g'(x) = -5\cos^4 x \sin x \\
&= \sin x \cos^5 x + \int 5\sin^2 x \cos^4 x \,\mathrm{d}x \\
&= \sin x \cos^5 x + 5\int \cos^4 x (1-\cos^2 x) \,\mathrm{d}x \\
&= \sin x \cos^5 x + 5\int \cos^4 x \,\mathrm{d}x - 5\int \cos^6 x \,\mathrm{d}x \\
&= \sin x \cos^5 x + 5\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x)\right] - 5\int \cos^6 x \,\mathrm{d}x \\
\implies 6\int \cos^6 x \,\mathrm{d}x &= \sin x \cos^5 x + 5\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x)\right] \\
\int \cos^6 x \,\mathrm{d}x &= \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x)\right]
\end{aligned}
$$

$$
\begin{aligned}
\implies \int \cos^4 x \sin^2 x \, \mathrm{d}x &= \int \cos^4 x \, \mathrm{d}x - \int \cos^6 x \, \mathrm{d}x \\
&= \frac{\sin x \cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x) \\
&\quad - \frac{\sin x \cos^5 x}{6} - \frac{5}{6}\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{8}x + \frac{3}{16}\sin(2x)\right] \\
&\quad + c \\
&= \frac{\sin x \cos^3 x}{24} + \frac{3}{48}x + \frac{3}{96}\sin(2x) - \frac{\sin x \cos^5 x}{6} + c \\
&= \frac{\sin x \cos^3 x}{24} + \frac{1}{16}x + \frac{1}{32}\sin(2x) - \frac{\sin x \cos^5 x}{6} + c
\end{aligned}
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