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Chapter 5, Parametric Estimation, part 1

Topics: Estimation from the Correlogram

Estimation : correlogram based estimate

Also known as "Yule Walker" estimate:
this estimate is based on the autocorrelation function (the correlogram) and is obtained matching the theoretical and the sample moments (also "method of moments" estimate).

AR(1) ($|\phi| < 1$):

$$\gamma_0 = \frac{\sigma^2}{1-\phi^2}, \quad \gamma_1 = \frac{\sigma^2}{1-\phi^2}\phi, \quad \rho_1 = \phi$$

the estimates are obtained replacing the theoretical moments by the sample ones: compute the sample moments $\widehat{\gamma}_0$ and $\widehat{\gamma}_1$, and then $\widehat{\rho}_1$: the estimate of ϕ and of σ^2 are

$$\widehat{\phi} = \widehat{\rho}_1$$

$$\widehat{\sigma^2} = \left(1 - \widehat{\phi}^2\right)\widehat{\gamma}_0$$

AR(p) (stationary)

Recall the autocovariance function

$$\gamma_0 = \phi_1\gamma_1 + \ldots + \phi_p\gamma_p + \sigma^2$$
$$\gamma_{j\geq 1} = \phi_1\gamma_{j-1} + \ldots + \phi_p\gamma_{j-p}$$

For $j = 1, \ldots, p$, stack the equations for $\gamma_{j\geq 1}$ as

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_{p-1} \\ \gamma_p \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-2} & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{p-3} & \gamma_{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p-2} & \gamma_{p-3} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{p-1} & \gamma_{p-2} & \cdots & \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_{p-1} \\ \phi_p \end{pmatrix}$$

This can be seen as a linear system in $\phi_1, \ldots, \phi_p$.

Replacing the theoretical moments $\gamma_0, \ldots, \gamma_p$ with the sample ones, $\widehat{\gamma}_0, \ldots, \widehat{\gamma}_p$, the estimates $\widehat{\phi}_1,...,\widehat{\phi}_p$ are

$$\begin{pmatrix} \widehat{\phi}_1 \\ \widehat{\phi}_2 \\ \cdots \\ \widehat{\phi}_{p-1} \\ \widehat{\phi}_p \end{pmatrix} = \begin{pmatrix} \widehat{\gamma}_0 & \widehat{\gamma}_1 & \cdots & \widehat{\gamma}_{p-2} & \widehat{\gamma}_{p-1} \\ \widehat{\gamma}_1 & \widehat{\gamma}_0 & \cdots & \widehat{\gamma}_{p-3} & \widehat{\gamma}_{p-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \widehat{\gamma}_{p-2} & \widehat{\gamma}_{p-3} & \cdots & \widehat{\gamma}_0 & \widehat{\gamma}_1 \\ \widehat{\gamma}_{p-1} & \widehat{\gamma}_{p-2} & \cdots & \widehat{\gamma}_1 & \widehat{\gamma}_0 \end{pmatrix}^{-1} \begin{pmatrix} \widehat{\gamma}_1 \\ \widehat{\gamma}_2 \\ \cdots \\ \widehat{\gamma}_{p-1} \\ \widehat{\gamma}_p \end{pmatrix}$$

Given these estimates $\widehat{\phi}_1, \ldots, \widehat{\phi}_p$ and given the sample moments $\widehat{\gamma}_0, \ldots, \widehat{\gamma}_p$, we can then compute the estimate of σ^2

$$\widehat{\sigma^2} = \widehat{\gamma}_0 - \widehat{\phi}_1 \widehat{\gamma}_1 - \ldots - \widehat{\phi}_p \widehat{\gamma}_p$$

MA(1)

We can apply the same principle to the MA(1). Recall

$$\gamma_0 = (1+\theta^2)\sigma^2, \gamma_1 = \theta\sigma^2, \rho_1 = \frac{\theta}{1+\theta^2}$$

then we can replace ρ_1 by the corresponding sample moment and estimate θ solving

$$\widehat{\rho}_1 = \frac{\theta}{1+\theta^2},$$

$$\widehat{\rho}_1\theta^2 - \theta + \widehat{\rho}_1 = 0$$

$$\widehat{\theta}_1 = \frac{1-\sqrt{1-4\widehat{\rho}_1^2}}{2\widehat{\rho}_1}, \widehat{\theta}_2 = \frac{1+\sqrt{1-4\widehat{\rho}_1^2}}{2\widehat{\rho}_1}$$

If $\theta \neq 1$ we cannot say if $\theta < 1$ or $\theta > 1$; however, if $|\widehat{\rho}_1| < 1/2$ and we want an invertible model for Y_t, choose $\widehat{\theta} = \widehat{\theta}_1$. Next, having $\widehat{\theta}$ and $\widehat{\gamma}_0$, estimate σ^2 as

$$\widehat{\sigma^2} = \frac{\widehat{\gamma}_0}{1+\widehat{\theta}^2}$$

ARMA(1, 1)

We can apply the same procedure for the ARMA(1, 1): since

$$\gamma_0 = \sigma^2 \left(1 + \frac{(\phi + \theta)^2}{1 - \phi^2} \right)$$

$$\rho_1 = \frac{(\theta + \phi)(1 + \theta\phi)}{1 + \theta^2 + 2\phi\theta}$$

$$\rho_2 = \phi\rho_1$$

we can compute the sample moments $\widehat{\gamma}_0$, $\widehat{\rho}_1$ and $\widehat{\rho}_2$ and, assuming that there is no common factor ($\theta \neq -\phi$ in this case) estimate

$$\widehat{\phi} = \frac{\widehat{\rho}_2}{\widehat{\rho}_1},$$

then estimate θ solving

$$\widehat{\rho}_1 = \frac{\left(\theta + \widehat{\phi}\right)\left(1 + \theta\widehat{\phi}\right)}{1 + \theta^2 + 2\widehat{\phi}\theta}$$

and finally σ^2 as

$$\widehat{\sigma^2} = \widehat{\gamma}_0 \left(1 + \frac{\left(\widehat{\phi} + \widehat{\theta}\right)^2}{1 - \widehat{\phi}^2} \right)^{-1}$$

ARMA(p, q)

We can use the same technique to estimate any other ARMA(p, q) (again, assuming no common factor).

Examples of correlogram based estimates

suppose that $Y_1, \ldots, Y_T$ are observed and let

$$\bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t, \; \hat{\gamma}_j = \frac{1}{T}\sum_{t=1}^{T-j}(Y_t - \bar{Y})(Y_{t+j} - \bar{Y}), \; \hat{\rho}_j = \frac{\hat{\gamma}_j}{\hat{\gamma}_0}$$

Example 1. AR(1)

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t, \; \varepsilon_t \; iid(0, \sigma^2)$$

$$\bar{Y} = 1.5, \; \hat{\gamma}_0 = 3, \; \hat{\gamma}_1 = 2.1$$

For the AR(1), $\rho_1 = \phi$, so

$$\hat{\phi} = \hat{\rho}_1$$

and $\hat{\phi} = 2.1/3 = 0.7$; from $\gamma_0 = \frac{\sigma^2}{1-\phi^2}$ then

$$\widehat{\sigma^2} = \hat{\gamma}_0\left(1 - \hat{\phi}^2\right)$$

and compute $\widehat{\sigma^2} = 3(1 - 0.7^2) = 1.53$. Finally, from $\mu = \frac{c}{1-\phi}$ then

$$\hat{c} = \hat{\mu}\left(1 - \hat{\phi}\right)$$

and using $\hat{\mu} = \bar{Y}$, compute $\hat{c} = 1.5(1 - 0.7) = 0.45$.

Example 2. **AR**(2)

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t,\ \varepsilon_t\ iid(0, \sigma^2)$$

$$\overline{Y} = 6, \widehat{\gamma}_0 = 10, \widehat{\gamma}_1 = 5, \widehat{\gamma}_2 = 1$$

For the AR(2),

$$\rho_1 = \frac{\phi_1}{1-\phi_2}, \rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2}$$

so $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are the solutions of

$$\widehat{\rho}_1 = \frac{\widehat{\phi}_1}{1-\widehat{\phi}_2}, \widehat{\rho}_2 = \frac{\widehat{\phi}_1^2 + \widehat{\phi}_2 - \widehat{\phi}_2^2}{1-\widehat{\phi}_2}$$

so

$$\widehat{\phi}_2 = \frac{\widehat{\rho}_2 - \widehat{\rho}_1^2}{1-\widehat{\rho}_1^2}, \widehat{\phi}_1 = \widehat{\rho}_1\left(1-\widehat{\phi}_2\right)$$

and in this case $\widehat{\rho}_1 = 0.5$, $\widehat{\rho}_2 = 0.1$ so $\widehat{\phi}_1 = 0.6$, $\widehat{\phi}_2 = -0.2$.

Next, recalling that $\gamma_0 = \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2$,

$$\widehat{\sigma^2} = \widehat{\gamma}_0 - \widehat{\phi}_1\widehat{\gamma}_1 - \widehat{\phi}_2\widehat{\gamma}_2$$

so in this case $\widehat{\sigma^2} = 7.2$. Also, from $\mu = \frac{c}{1-\phi_1-\phi_2}$ then

$$\widehat{c} = \widehat{\mu}\left(1-\widehat{\phi}_1-\widehat{\phi}_2\right)$$

and using $\widehat{\mu} = \overline{Y}$, compute $\widehat{c} = 3.6$.

Example 3. MA(1)

$$Y_t = c + \varepsilon_t + \theta\varepsilon_{t-1},\ \varepsilon_t\ iid(0,\sigma^2)$$

$$\overline{Y} = 1, \widehat{\gamma}_0 = 5, \widehat{\gamma}_1 = 2$$

For the MA(1),

$$\rho_1 = \frac{\theta}{1+\theta^2},$$

note that $\widehat{\rho}_1\left(1+\widehat{\theta}^2\right) - \widehat{\theta} = 0$ is a second degree equation in $\widehat{\theta}$, so it has two solutions,

$$\widehat{\theta}_{i,ii} = \frac{1 \pm \sqrt{1-4\widehat{\rho}_1}}{2\widehat{\rho}_1}.$$

Taking

$$\widehat{\theta} = \widehat{\theta}_i = \frac{1 - \sqrt{1-4\widehat{\rho}_1}}{2\widehat{\rho}_1}$$

since in this case $\widehat{\rho}_1 = 0.4$, then $\widehat{\theta} = 0.5$.

From

$$\gamma_0 = (1+\theta^2)\sigma^2$$

we compute $\widehat{\sigma^2} = 5/(1+0.5^2) = 4$.

Finally, using $\widehat{\mu} = \overline{Y}$, we compute $\widehat{c} = 1$.

The fact that there are two values of $\widehat{\theta}$ that are associated with a given $\widehat{\rho}_1$ is not a surprise, as we know that the same autocorrelation structure is generated by two values of θ, say θ_i and θ_{ii}, which are such that $\theta_i = 1/\theta_{ii}$.

Although both $\widehat{\theta}_i$ and $\widehat{\theta}_{ii}$ are valid solutions of $\widehat{\rho}_1\left(1 + \widehat{\theta}^2\right) - \widehat{\theta} = 0$, one should present one estimate (only), so it is not acceptable to say that the estimate is

$$\widehat{\theta}_{i,ii} = \frac{1 \pm \sqrt{1 - 4\widehat{\rho}_1}}{2\widehat{\rho}_1}.$$

(as this would two estimates).

Here, I choose $\widehat{\theta}_i$ as this is the one that yields invertibility.