Notes Advanced Microeconomics

In economics we make a lot of assumption —> Main reason is that economics model are just a modellization of the reality. We need need something simpler and easier than the reality. Assumption should help us to simplify the problem.

Economics model are useful because when we analise data, we need to have some background theory to interpret the result.

Example
If i have some data on the sales or on the price of the good. Then we estimate a regression: you see how sales depends on the price. What kind of relation we are going to expect? Positive or negative? Negative.
How do we explain this negative relation? This relation is based on theory in which we assume we receive some satisfation(utility) from a given good. You can't compare satisfation with price that you're paying for that good. If price increase you buy less! If you have an income you can buy less unit of the good is lesser. Unit I can afford= Income/price. But there are example in which price increase and sells increase. The based theory is like this but is based on assumptions.

Ch.1 - Consumer Theory

First lectures will be manly on consumer choices.

Main topics: Consumer theory explains how consumer decides what to buy and how much of a good to buy.

In particolar, we will see the concept of preference and choice. We will main base the lecture on preference based-approach compared with the choice approach. We will see utilty function also and then introduce way to rationalise behaviours. In economics they care about their utilty but don't care about others so the concept of altruism will be implemented editing the concept of utilty function.

What is **preference**?
First, how consumer decides between two different goods.
Do you prefer an orange or an apple?
2 guys, one orange and one apple.
This choice are based on some preferences, so we will define is the preferences of individuals. This is an element with which can make choice. We will see the so called **consumption set** that is indicate with X.

Consumption set X: all set of alternatives that are available to the decision maker.
In this case the DM is the consumer.

So set of all possible choice means consumption set is very big. In the consumption set we will not only goods to buy but all possible combinations of quantities of these goods. (1 apple, 2 orange or 2 apples, 1 orange). In our exercise we will be mainly two goods with quantities to buy.

Preference based-approach: assume that I know the preference of consumers so according to the preference that i know i can predict what people will buy.
In the example i took i know she prefers oranges than apple and if i offer an orange or an apple she will decises to buy and orange.

Choice-based approach: the second approach is the choice-based approach. According to this approach i don't know her preference but I make her an offer and i offer her 1 orange and 1 apple. Based on the choice that she makes, she decides to buy the orange. So I infer that she prefer orange to apple. So I build the preference based on preferences on my observation of her consumption here. This is something closer to reality but it's harder to treat it analiticlaly. Sometime to change preference maybe. In the traditional theory we will use preference does not change over time. Main advantage of the choice-based approach is based on behaviours which is something I can observe while preference based approach rely on preferences we have to assume we know (even in the reality we don't) as assumption.

Preference-based approach

How preference defined? We should ask to the decision maker which are the possible alternatives which are all in the consumption set X.
Given two alternatives x and y that belong to the consumption set (X) what kind of ranking can you do? You can say that given this two, i can prefere an orange to an apple and the simple of preference is like a greater sign. I can say i prefer an apple to an orange o i could say I'm indifference (tilde symble).

A preference relation is an operator that allows you to do this ranking and the kind of presence relation we will be use must be complete.

Complete: Individuals must be able to compare every alternatives in the consumption set.
So for any two alternative you are able to choice between one of this.
Ex. I can compare even with good that i never bought.
In reality, If i ask you to compare one good to another you would say "i don't know".

A binary preference relation is a relation. In Mathematics a binary relation of a pair.
When we compare alternative we will compare what we called ordered pairs.

Binary relation: collection of ordered pairs(x,y) from a set x,y appartenete al consumption set X.

Esempio: this are two ordered pair.

THIS REFERS TO ORANGE GROUP

(1,2) (2,1)

N. ORANGE N. APPLES

BETTER BECAUSE I HAVE MORE QUANTITY

(2,1) (2,2)

THIS HOLDS IF ALL GOODS ARE BOTH DESIDERABLE
(POLLUTION SHOULD BE LESS)
BADS!

The symble of strict preferences.
Before we said that x is prefer to y but we don't say that x is strictly preferred to y. We are introducing the operator of **strict preferences**. Here we have three options: x strictly pref to y or y strictly pref to x or x ind to y. I can only choose one between the three.

The same relation could be define using another operator which is called **weak preference** operator where x is as good as y or x is weakly prefered to y.
Weak preference operator we will not have three options but only two. We could say that x is al least as good as y or y is at least as good as x. In this case, the difference in respect to the strict preference is that we can choose both. This mean the two goods could have the same value. In case you choice both option this mean that the two good are indifferent.

The same preferences can be describe using the two symbles. So to say that x is ind to y, in stric preference we said only check x ind y, while with the weak preference i check the two boxes.

This part is about: a way to define how to make choice and we introduce this operator to define preference between alternatives. So what is prefered to what and what is indifferent to what.

Reflexivity: any alternative x can be in a set of thing that I could buy and every alternative must be indifference to itself. (X is as good as X —> it's like a tautology.

Reflexivity implies that X ind to itself or using weak preference is as good as y.

In order to analise consumer behaviour we have to verify that relation is relational.
A preference relation if weak preference:
- Completeness: we are able to compare (to rank) every alternatives.
- Transitivity: implies 3 possible alternatives that are also referred as bundles of goods.

Bundle could be 2 orange and an apple. Or two oranges and two apples.

$$\underbrace{\underset{O\ \ A}{(2,1)}}_{x} \qquad \underbrace{\underset{O\ \ A}{(2,2)}}_{y}$$

If bundle like this we would have 3 goods:

$$\underset{O\ \ A\ \ S}{(2,1,3)}$$

But for the majority we will thread pair of goods.

Transitivity implies that you weakly prefered x to y and you weakly prefer y to z then implies you weakly prefer x to z. Even this is a stronger assumption so this mean if I ask you: you prefer orange to apple? You say orange. Then, you prefer apple to strawberry ? Then we can conclude that orange is weakly prefered to strawberries.
In theory holds but in the reality you may prefer strawberry to orange (but in the course we will use transitivity).

$$O \succsim A \quad A \succsim S \overset{\text{TRANSITIVITY}}{\Longrightarrow} O \succsim S$$

Rational preference mean that preference relation satisfy completeness (i can compare all possible alternatives) and transitivity.

One bundle is prefered to another but we don't say how to make choice so why x is prefered to y. One possible way of define preferences is one of the following and this is how preferences are described. X weakly prefered to y, we have to define a decision rule in which we can predic which choice will be made by the consumer.

X is weakly prefered to y if and only if:
I defined the components of the bundles. According to this decision rule, for this individual the first bundle prefered to the second if summing the quantity of the goods in the bundle I obtain a sum that is greater or equal to the sum of the components of the second bundles.

BUNDLES :

x $\quad$ y

(x_1, x_2) $\quad$ (y_1, y_2)

$x \succsim y$ IF SUMMING THE COMPONENTS OF 1° BUNDLE I GET A BIGGER VALUE THAN THE SUM OF 2° BUNDLE COMPONENTS

x $\quad$ y

$(1,2)$ $\quad$ $(2,1) \longrightarrow 2+1 \geq 2+1 \longrightarrow 3 \geq 3$? ✓ THEN WE CAN CONCLUDE $x \succsim y$

$(1,2)$ $\quad$ $(3,1) \longrightarrow 2+1 \geq 3+1 \longrightarrow 3 \geq 4$? ✗ THEN WE CAN'T CONCLUDE $x \succsim y$

Imaging the guys that we propose, choices the bundle with the highest quantity respected tot he two goods. We have to prove wherever a preference satisfy some preferences.
How can we test if this relation is complete and transitive?

Let's start with completeness. You have to be able to decide if x is weakly pref to y or y weakly pref to x of both (indifferent)?
This preference relation satisfy completeness? If i give you two bundles, i will always be able to compare this two bundle? This is the definition of completeness. Yes, it's complete because we can always compare two real number.

Transitivity: this satisfy transitivity. If i found that x is weakly pref to y, y strictly pref to z then x weakly pref to z? YES.

$$x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$$

$$\sum x_i \geq \sum y_i \wedge \sum y_i \geq z \Rightarrow \sum x_i \geq \sum z_i$$

We can always compare real number. Just compare the quantity in a bundle to verify the preference relation and verify is the preference relation is rational.

Although we did this assumption, there's a branch of economics which is called experimental eco monomials: takes individual and bring individuals to lab and test some assumption about Microeconomics theory. There are quite a lot of examples that individuals choice violate this assumptions.

There are potential **source of intransitivity** in preference:
1. Indistinguishable alternatives
2. framing effects
3. Aggregation of criteria
4. Change in preferences

This may violate the transitivity properties.

Example of framing effects.
Framing: phenomena in which your answer may depend on the order of the question.
Imaging that I bring you this and then ask you to decide this three alternative:
(Paris for $574)
We should actually say that a and b are the same because the holiday are the same. The holidays is a week in Paris for $574 so same offer. So we change the way we presenting it. If i compare a with c and b with c, if a > c also b>c. Instead, in the lab many individuals that violate this properties.

Another example:
Coffee paradigm (Paradigma del caffè). How many spoon of sugar you want? Maybe you cannot distinguished between 2 or 3 spoon maybe.
This is **also violation of rationality!**

If i giving you 70 orange and 70 apple and make you choose by majority. Then this violate transitivity.
This assumption help us to simplify the problems! But in reality is not like this.

The final goal is to define if a simple model can describe a reality and how well this can be predicted.

Ch. 2 - Utility function

Preference can be described in the way he showed before. We implicitly define a function for the preference relation that was the sum of the components of the bundle.

$(x_1, x_2) \rightarrow y = x_1 + x_2$ — THIS IS SUM OF COMPONENTS

FUNCTION

THIS PREFERENCE RELATION CAN BE DEFINE BY THIS FUNCTION:

$\underline{u(x_1, x_2) = x_1 + x_2}$ A WAY TO DESCRIBE PREFERENCE RELATION

Utility function We can generalise:
Utility function is a function that is define the consumption set. So taking as input the bundle in the consumption set it give us a real number. This function can be called utility function representing the preferences if for any two alternatives we can say that x if weakly prefered to y if and of if the utility of the x is greater or equal than the utility of y.
We assume that we know the preference of the individuals in a sense that we know the utility function of the individuals.

So preference relation can be describe by utility function. If this is true, we can say if x weakly prefered to y or viceversa.
An important thing is that for our consumption theory is that we are able to rank the alternatives.

$x \succeq y \iff u(x) \geq u(y)$

20	10
30	10

The utility of bundle is greater than utility of bundle of y. I will choose x. So we want to predict if we will choose x instead of y. We don't care about cardinality, so the number of the utility function but we just care about the rank. So we want to allowed the consumer to rank (put in order) the different alternatives.

Any strictly increasing transformation of the utility function also give a utility function that describe the same preferences. If i apply an increasing transformation to the utility function which gives another utility function that describe the same preference.

Describe the same preference since it’s a strictly increasing transformation.

$u(x) = (x_1^1, x_2^1) = 2 \qquad u(y) = (y_1^2, y_2^1) = 3$

$3 \geq 2$ so $y \succeq x$

IF I USE $3 \cdot u(x)$

$3 \cdot u(x) = 6 \qquad 3u(y) = 9 \implies y \succeq x$

In the example that we take we assume that all goods are desiderable. We will speak about monotonicity, strong monotonicity, satiation and non-satiation. We can include different goods in combination of n goods in which a set of vector with n component in which every components is a real number.

Advanced Microeconomics (EPS)

Chapter 1: Preferences

Outline

- Preference and Choice
- Preference-Based Approach
- Utility Function
- Indifference Sets, Convexity, and Quasiconcavity
- Special and Continuous Preference Relations
- Social and Reference-Dependent Preferences
- Hyperbolic and Quasi-Hyperbolic Discounting
- Choice-Based Approach
- Weak Axiom of Revealed Preference (WARP)
- Consumption Sets and Constraints

Preference and Choice

Preference and Choice

- We begin our analysis of individual decision-making in an abstract setting.
- Let $X \in \mathbb{R}_+^N$ be a set of possible alternatives for a particular decision maker.
  - It might include the consumption bundles that an individual is considering to buy.
  - *Example*:

$$X = \{x, y, z, \dots\}$$

$$X = \{\text{Apple}, \text{Orange}, \text{Banana}, \dots\}$$

Preference and Choice

- Two ways to approach the decision making process:
  1) ***Preference-based approach***: analyzing how the individual uses his preferences to choose an element(s) from the set of alternatives X.
  2) ***Choice-based approach***: analyzing the actual choices the individual makes when he is called to choose element(s) from the set of possible alternatives.

Preference and Choice

- Advantages of the Choice-based approach:
  - It is based on observables (actual choices) rather than on unobservables (individual preferences)
- Advantages of Preference-based approach:
  - More tractable when the set of alternatives X has many elements.

Preference and Choice

- After describing both approaches, and the assumptions on each approach, we want to understand:

Rational Preferences $\Longrightarrow$ Consistent Choice behavior

Rational Preferences $\Longleftarrow$ Consistent Choice behavior

Preference-Based Approach

Preference-Based Approach

- ***Preferences***: "attitudes" of the decision-maker towards a set of possible alternatives X.

- For any $x, y \in X$, how do you compare x and y?
  - I prefer x to y $(x \succ y)$
  - I prefer y to x $(y \succ x)$
  - I am indifferent $(x \sim y)$

Preference-Based Approach

By asking:	We impose the assumption:
Tick one box (i.e., not refrain from answering)	*Completeness*: individuals must compare any two alternatives, even the ones they don't know.

Preference-Based Approach

- ***Completeness***:
  - For any pair of alternatives $x, y \in X$, the individual decision maker:
    - ❑ $x \succ y$, or
    - ❑ $y \succ x$, or
    - ❑ both, i.e., $x \sim y$
    - (The decision maker is allowed to choose one, and only one, of the above boxes).

Preference-Based Approach

- *A **binary relation** is a collection of ordered pairs (x,y) from a set $x,y \in X$.*

- ***Not all binary relations satisfy Completeness***.

Preference-Based Approach

- ***Weak preferences***:
  - Consider the following questionnaire:
  - For all $x, y \in X$, where x and y are not necessarily distinct, is x **at least as preferred** to y?
    - ❑ Yes $(x \succsim y)$
    - ❑ No $(y \succsim x)$
  - Respondents must answer yes, no, or both
    - Checking both boxes reveals that the individual is indifferent between x and y.
    - Note that the above statement relates to completeness, but in the context of weak preference $\succsim$ rather than strict preference $\succ$.

Preference-Based Approach

- ***Reflexivity***: every alternative x is weakly preferred to, at least, one alternative: itself.
- A preference relation satisfies reflexivity if for any alternative $x \in X$, we have that:
  1) $x \sim x$: any bundle is indifferent to itself.
  2) $x \succsim x$: any bundle is preferred or indifferent to itself.
  3) $x \nsucc x$: any bundle belongs to at least one **indifference set** (i.e. set of alternatives over which the consumer is indifferent), namely, the set containing itself if nothing else.

Preference-Based Approach

- The preference relation $\succsim$ is ***rational*** if it possesses the following two properties:

  a) *Completeness*: for all $x, y \in X$,
  either $x \succsim y$, or $y \succsim x$, or both.

  b) *Transitivity*: for all $x, y, z \in X$,
  if $x \succsim y$ and $y \succsim z$, then it must be that $x \succsim z$.

Preference-Based Approach

- ***Example 1.1*.**

Consider the preference relation

$$x \succsim y \quad \text{if and only if} \quad \sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$$

In words, the consumer prefers bundle x to y if the total number of goods in bundle x is larger than in bundle y.

In case of two goods $x_1 + x_2 \geq y_1 + y_2$

Preference-Based Approach

- ***Example 1.1*** (continues).
- *Completeness*:
  - either $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$ (which implies $x \succsim y$), or
  - $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} x_i$ (which implies $y \succsim x$), or
  - both, $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$ (which implies $x \sim y$).

- *Transitivity:*
  - If $x \succsim y$, $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$, and
  - $y \succsim z$, $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} z_i$,
  - Then it must be that $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} z_i$ (which implies $x \succsim z$, as required).

Preference-Based Approach

- The assumption of transitivity is understood as that preferences should not cycle.
- Example violating transitivity:

$$\underbrace{apple \succsim banana \quad banana \succsim orange}_{apple \succsim orange \text{ (by transitivity)}}$$

but $orange \succ apple$.

- Otherwise, we could start the cycle all over again, and extract infinite amount of money from individuals with intransitive preferences.

Preference-Based Approach

- Sources of intransitivity:
  a) Indistinguishable alternatives
  b) Framing effects
  c) Aggregation of criteria
  d) Change in preferences

Preference-Based Approach

- ***Example 1.2*** (Indistinguishable alternatives):
  - Take $X = \mathbb{R}$ as a share of pie and $x \succ y$ if $x \geq y - 1$ $(x + 1 \geq y)$ but $x \sim y$ if $|x - y| < 1$ (indistinguishable).
  - Then,

$$1.5 \sim 0.8 \text{ since } 1.5 - 0.8 = 0.7 < 1$$

$$0.8 \sim 0.3 \text{ since } 0.8 - 0.3 = 0.5 < 1$$

  - By transitivity, we would have $1.5 \sim 0.3$, but in fact $1.5 \succ 0.3$ (intransitive preference relation).

Preference-Based Approach

- *Other examples*:
  - similar shades of gray paint
  - milligrams of sugar in your coffee

Preference-Based Approach

- ***Example 1.3*** (Framing effects):
  - Transitivity might be violated because of the way in which alternatives are presented to the individual decision-maker.
  - What holiday package do you prefer?
    a) A weekend in Paris for \$574 at a four star hotel.
    b) A weekend in Paris at the four star hotel for \$574.
    c) A weekend in Rome at the five star hotel for \$612.
  - By transitivity, we should expect that if $a \sim b$ and $b \succ c$, then $a \succ c$.

Preference-Based Approach

- ***Example 1.3*** (continued):
  - However, this did not happen!
  - More than 50% of the students responded $c \succ a$.
  - Such intransitive preference relation is induced by the framing of the options.

Preference-Based Approach

- ***Example 1.4*** (Aggregation of criteria):
  - Aggregation of several individual preferences might violate transitivity.
  - Consider $X = \{MIT, WSU, Home\ University\}$
  - When considering which university to attend, you might compare:
    a) Academic prestige (criterion #1)
    $$\succ_1: \ MIT \succ_1 WSU \succ_1 Home\ Univ.$$
    b) City size/congestion (criterion #2)
    $$\succ_2: \ WSU \succ_2 Home\ Univ. \succ_2 MIT$$
    c) Proximity to family and friends (criterion #3)
    $$\succ_3: \ Home\ Univ. \succ_3 MIT \succ_3 WSU$$

Preference-Based Approach

- ***Example 1.4*** (continued):
  - By majority of these considerations:

$$MIT \underbrace{\succsim}_{\text{criteria 1 \& 3}} WSU \underbrace{\succsim}_{\text{criteria 1 \& 2}} Home\ Univ \underbrace{\succsim}_{\text{criteria 2 \& 3}} MIT$$

  - Transitivity is violated due to a cycle.
  - A similar argument can be used for the aggregation of individual preferences in *group decision-making*:
    - Every person in the group has a different (transitive) preference relation but the group preferences are not necessarily transitive ("***Condorcet paradox***").

Preference-Based Approach

- Intransitivity due to a *change in preferences*
  - When you start smoking

    $$\text{One cigarette} \succsim \text{No smoking} \succsim \text{Smoking heavily}$$

    By transitivity,

    $$\text{One cigarette} \succsim \text{Smoking heavily}$$

  - Once you started

    $$\text{Smoking heavily} \succsim \text{One cigarette} \succsim \text{No smoking}$$

    By transitivity,

    $$\text{Smoking heavily} \succsim \text{One cigarette}$$

  - But this contradicts the individual's past preferences when he started to smoke.

Desirability

- monotonicity
- Strong monotonicity
- Non-satiation
- Local non-satiation

All x1,x2,x3 are defined on the set of real numbers.

Now we are going to define the first property.

Monotonicity

If i take any two bundles x and y and x != y.
If xk >=yk (quantity of good k in bundle x and y) then implies that x pref y.
If xk > yk then implies that x strictly pref to y.

1. So increasing the amount of some commodores cannot hurt x>=y.
2. $x = (x_1, x_2)$ $y = (x_1 + \varepsilon, x_2)$ $y \succeq x$ y PREF TO x

   $\varepsilon > 0$

   $y = (x_1 + \varepsilon, x_2 + \varepsilon)$ $y \succ x$

   THEN PREF RELATION SATISFY MONOTONICITY

Strong monotonicity

Two bundles in consumption set if xk >= yk for every good k then we conclude that x strictly pref to y (before was weakly preferred).

In the last example are monotone in which add eps only to x1 then we obtain a bundle strictly preference to x.
==> this means that is strong because we got a stronger condition. Even increasing the quantity of 1 good then you obtain a bundle that is strictly pref.
Also for the second in which we add eps to x1 and x2 then it hold because y>x comprende y>=x

Now we wander how this monotonicity can be translated in the characteristic of the utility function?
Monotonicity in preference implies that utility function is weakly monotonicity in its arguments.
If we increase all arguments we obtain a value that it is strictly increases its value.

If i have x1 and x2 and if ai multiply by a scalar alpha > 1.
Alpha x1 > x1 so is greater or equal than the initial utility of the bundle u(x1,x2).
Increasing quantity of x1 i get a greater utility so if weakly pref to the original one.

If alpha x1 and alpha x2 then the utility is strictly preferred than the original one.

If we change and we want to see what strong monotonicity imply in the utility function.
In case you increase only one good you obtain a strictly greater than the original one. U(ax1, x2) > u(x1,x2).

Examples

PREF CAN BE REPRESENTED BY THIS FUNCTION

$$u(x_1, x_2) = \min\{x_1, x_2\}$$

MONOTONE?

1. INCREASE QUANTITY OF 1 OF THE 2 GOODS

WE HAVE TO CHECK THE TWO MINIMUM

$$u(\alpha x_1, x_2) = \min\{\alpha x_1, x_2\} \overset{?}{\geq} \min\{x_1, x_2\}$$

$\underline{\alpha > 1}$

ALWAYS TRUE?

$$u(x_1, x_2) = \min\{x_1, x_2\} \to x_1, \; x_2$$

$(1, 2)$

$(2, 1)$

IF MIN == x_2
EQUAL

IF MIN == x_1
GREATER

$x_1 \leq x_2$?

$\alpha x_1 \leq x_2$?

IF $x_1 \leq x_2$ → x_1, x_2 → MIN STILL INCREASE SO IT'S FINE

IF 1° CONDITION OF MONOTONICITY, THE SECOND CONDITION HOLDS FOR SURE

$$u(\alpha x_1, \alpha x_2) = \min\{\alpha x_1, \alpha x_2\}$$

THIS IS FOR SURE $\geq$ THAN MIN IN THE ORIGINAL BUNDLE

STRONG MONOTONICITY?

IF WE JUST CHANGE ONE OF TWO ARG WE OBTAIN A STRICLY PREFERENCE RELATION

1. ONE IS MONOTONICITY
2. STRONG MONOTONICITY

$u(\alpha x_1, x_2) > u(x_1, x_2)$

$\min\{\alpha x_1, x_2\} > \min\{x_1, x_2\}$

WE CAN OBTAIN MINIMUM AS x_1 OR x_2

IF x_1 MEANS x_2 LESS THAN x_1.
IF INCREASE THE x_1 AND MIN IS x_2

$x_2 > x_2$ IMPOSSIBLE SO STRONG MONOTONICITY IS NOT SATISFIED

STRONG IS STRICTER THAN MONOTONICITY

MONOTONICITY $\Rightarrow$ STRONG MONO
$\not\Leftarrow$ NOT THE VICEVERSA

THEN WE WILL SPEAK ABOUT COMPLEMENTS
(PARADOSSO DEL CAFFÈ)

(1,3) (1,2)
C S C S

Example2 —> linear utility function

$u(x_1, x_2) = x_1 + x_2$ → OR ANY SUM COMBINATION LIKE $2x_1 + 3x_2$ ...

- MONOTONICITY $\quad \alpha > 1$

$u(\alpha x_1, x_2) = \alpha x_1 + x_2$

LARGER THAN 1 SO IS STRICTLY

$u(\alpha x_1, \alpha x_2) = \alpha x_1 + \alpha x_2$

UTI IS STRICTLY LARGER SO RELATION IS STRICTLY MONOTONIC

SO SATISFY STRONG MONOTONICITY

WE SHOULD DO THE SAME FOR x_2 TO BE SURE

BUT IT IS CLEAR WE GET A STRONG MONOTONICITY

$u(x_1, \alpha x_2) = x_1 + \alpha x_2$

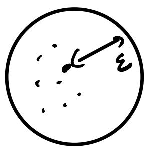
rational —> complete reflexives and transitivity. Transitivity assume completeness ??

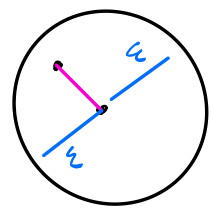
Non-satiation

You are never happy. You always find a bundle that is strictly pref than the original one. So this is not very usable. We will use more frequently local non-satiation

local non-satiation

We always find a bundle that is close to the original one, but we pref the original one.





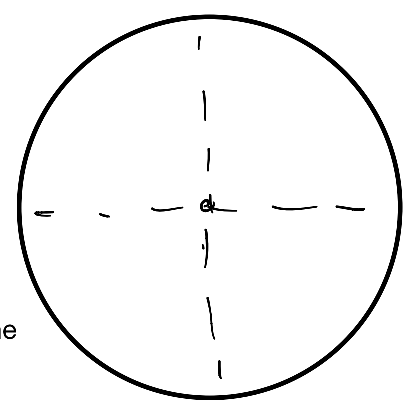
We always have an Euclidean distance < eps.
Euclidean distance is computed as
x = (x1,x2) Y(x1, x2)
take difference power of two and then rad.
So we compute the distance we got a point the in circle by increase for a small quantity. This must happen for any distance eps.

For instance you can compare very close alternatives that differ for a very small amount.

Application of definition of local association.
Two goods.
[slide cerchio]
In x1 we have quantity of first good in bundle x. In y we have the second quantity of bundle x (which is x2). The bundle (2,2) can be represented by a point, also for y.
Y2 contain a small quantity of x2 and y1 larger than x1. So the distance

In case we have two bad good [called bads](pollutions of water and air)

The more we are close to the origin
The more we are happy.



(0,0) can we find another bundle close to this and
Preferred to the original one?
We can’t have negative pollution.
Drawing small circle x we don’t find any bundle pref to the original one
So this violate the LNS.

Another situation is the thick indifference sets (or curve).
An indifference set is the set of all bundle that are indifferent to the consumer (same level of utility)
Imagine now we have an area then, so this mean we cannot draw arbitrarialy small circle, because all

circle in this area of the indifference curve are indifference. So we will not consider this case.

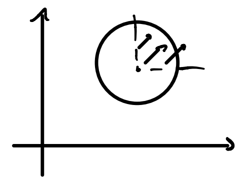
IF PREF $\succeq$ SATISFY MONOTONICITY

$\Rightarrow$ IT ALSO SATISFY LNS

LNS $\rightarrow$ DRAW CIRCLE TO FIND BUNDLE $\succ$ TO THE ORIGINAL ONE

a) $u(x_1, x_2)$ $\quad$ $u(\underset{a>1}{a x_1}, a x_2) \rightarrow u(x_1 + a, x_2 + \underset{a>0}{a})$

THIS $>$ $u(x_1, x_2)$



WE WILL ALWAYS FIND SOMETHING PREF TO THE ORIGINAL ONE

SO IF WE PROVE MONOTONICITY IMPLY WE ALSO HAVE NLS

Indifference set

A bundle x and the indifference bundle in the consumption sets are indifference to the respect to x.
Y ind to X
IND(x)

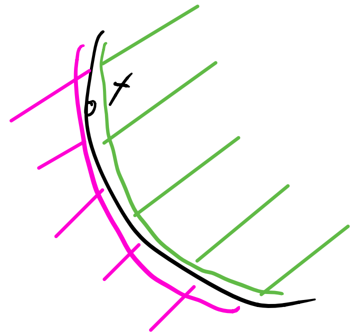
The upper-counter set

The set of all bundle in the consumption set such that bundle are strictly preferred to x
UCS(x)

Lower-counter set

The set of all bundle in the consumption set such that bundle are strictly preferred to x Such that x is strictly pref to y
LCS(x)

Graphically we can show it in this example in the following way.



ALL BUNDLES IN LINE ARE IND TO X

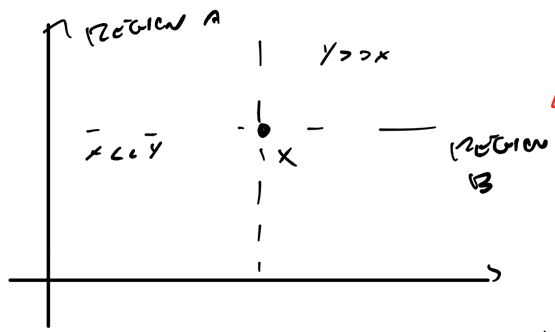
• UCS IMPLY ALL THE AREA AND THE LINE

• LCS AREA IN BASSO AND CONSUMPTION SET

$LCS \cap UCS = IND$

We saw properties of preference relation. Now we will see properties in indifference set (or curves)

Strong monotonicity



POSITIVE QUADRANT DIVIDED BY 4

ALL BUNDLES IN $y >> x$ CONTAINS LARGER QT OF THE GOODS. THEY ARE STRICTLY PREFE

SO IND CANNOT BE IN THIS REGION

IN $x << y$ IND CANNOT BE IN THIS AREA

THE ONLY AREA IS REGION A AND B, WHICH MEANS CURVE ARE NEGATIVE SLOPES

We will have curve that decrease???

Convexity of preferences

A preference relation is convex if for every two bundle in consumption set such that
X weak pref y ==> ax + (1-a) y >= y —> like a weighted average a+(1-a) = 1

$a \in [0,1]$

$a = 1/2$ $\qquad (x_1, x_2) \succeq (y_1, y_2)$

$\left(\frac{1}{2}x_1, \frac{1}{2}x_2, \frac{1}{2}x_2, \frac{1}{2}y_2\right) \succ (y_1, y_2)$

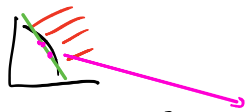
IF TRUE FOR ANY α

CONVEXITY OF PREFERENCE

TASTE FOR DIVERSITY / CORE ASSUMPTION

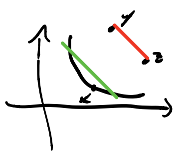
You can say another property of convexity with upper counter set (UCS).
So UCS(x) = {y app X: y>= x}

CONVEX -> ALL POINTS IN STRAIGHT LINE ARE IN THE SET

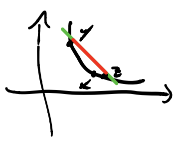


CONCAVE -> POINT NOT IN THE SET

Y in the UCS and if i have another bundle and if i have z also, then convex combination of the two good. So any bundle in this line is strictly pref to the original bundle. Not only weak but also strictly pref.



All bundle in the strictly line are preferred.



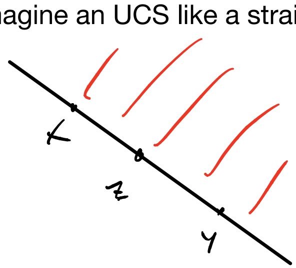
ALSO LIKE THIS - SO STRICTLY PREF OR IND

Convexity 1 we need just 2 bundles. For convexity 2 we need 3 bundles.

Strict convexity if you take x,y,z app X
If x weak pref z
If y weak pref to z
Then convex combination is strictly preferred.

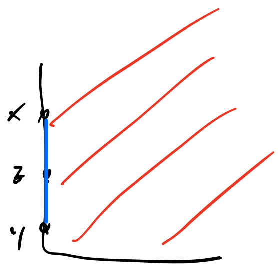
The only example strictly convex (have a shape like a curve)

Imagine an UCS like a straigh line



UCS

Taking two point x and y that are weak pref to z. Which mean z is in the indifference set.
Any points are indifferent to z and not strictly pref to z. So straight indifference curve (rette) represent preference that are not strictly convex but weakly convex. This correspond with linear utilty function which is the example of perfect substitutes goods.



UCS

> ALL THIS POINT IND TO Z DOES NOT SATISFY STRICT CONVEX

In this case pref relation is not strictly convex. But in most of our example the curve will no have this shape.

Try to do example 1.7 as an exercise applying the definition that this u satisfy both convexity and strict convexity.

Interpretation of convexity

You consume a lot of good 1 and a small quantity of good2. The coordinate of y is high and the second is low. You don't like the bundles unbalance to the two good. We pref to consume a little bit of everything. Are weakly preferred.

Advanced Microeconomics (EPS)

Chapter 1: Utility functions, indifference sets, quasi-concavity

Utility Function

- A function u: $X \rightarrow \mathbb{R}$ is a ***utility function*** representing preference relations $\succsim$ if, for every pair of alternatives $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y)$$

Utility Function

- Two points:
  1) Only the ranking of alternatives matters.
     - That is, it does not matter if

$$u(x) = 14 \text{ or if } u(x) = 2000$$

$$u(y) = 10 \text{ or if } u(y) = 3$$

     - We do not care about *cardinality* (the number that the utility function associates with each alternative) but instead care about *ordinality* (ranking of utility values among alternatives).

Utility Function

2) If we apply any **strictly increasing function** $f(\cdot)$ on $u(x)$, i.e.,

$$f\colon \mathbb{R} \to \mathbb{R} \text{ such that } v(x) = f(u(x))$$

the new function keeps the ranking of alternatives intact and, therefore, the new function still represents the same preference relation.

– *Example*:

$$v(x) = 3u(x)$$

$$v(x) = 5u(x) + 8$$

Desirability

- We can express desirability in different ways.
  - Monotonicity
  - Strong monotonicity
  - Non-satiation
  - Local non-satiation

- In all the above definitions, consider that x is an n-dimensional bundle

  $$x \in \mathbb{R}^n, \text{ i.e., } x = (x_1, x_2, \dots, x_N)$$

  where its k^{th} component represents the amount of good (or service) k, $x_k \in \mathbb{R}$.

Desirability

- ***Monotonicity***:
  - A preference relations satisfies monotonicity if, for all $x, y \in X$, where $x \neq y$,
    - $a)$ $x_k \geq y_k$ for every good k implies $x \succsim y$
    - $b)$ $x_k > y_k$ for every good k implies $x \succ y$
  - That is,
    - increasing the amounts of some commodities (without reducing the amount of any other commodity) **cannot hurt**, $x \succsim y$; and
    - increasing the amounts of all commodities is strictly preferred, $x \succ y$.

Desirability

- ***Strong Monotonicity***:
  - A preference relation satisfies strong monotonicity if, for all $x, y \in X$, where $x \neq y$,

$$x_k \geq y_k \text{ for every good } k \text{ implies } x \succ y$$

  - That is, even if we increase the amounts of only one of the commodities, we make the consumer strictly better off.

Desirability

- Relationship between **monotonicity** and utility function:
  - Monotonicity in preferences implies that the utility function **is weakly monotonic (weakly increasing) in its arguments**
    - That is, increasing some of its arguments weakly increases the value of the utility function, and increasing all its arguments strictly increases its value.
  - For any scalar $\alpha > 1$,

$$u(\alpha x_1, x_2) \geq u(x_1, x_2)$$
$$u(\alpha x_1, \alpha x_2) > u(x_1, x_2)$$

Desirability

- Relationship between **strong monotonicity** and utility function:
  - Strong monotonicity in preferences implies that the utility function **is strictly monotonic (strictly increasing) in all its arguments.**
    - That is, increasing some of its arguments strictly increases the value of the utility function.
  - For any scalar $\alpha > 1$,
$$u(\alpha x_1, x_2) > u(x_1, x_2)$$

Desirability

- ***Example 1.5***: $u(x_1, x_2) = \min\{x_1, x_2\}$
  - Monotone, since
    $$\min\{x_1 + \delta, x_2 + \delta\} > \min\{x_1, x_2\}$$
    for all $\delta > 0$.
  - Not strongly monotone, since
    $$\min\{x_1 + \delta, x_2\} \not> \min\{x_1, x_2\}$$
    if $\min\{x_1, x_2\} = x_2$.

Desirability

- ***Example 1.6***: $u(x_1, x_2) = x_1 + x_2$
  - Monotone, since
    $$(x_1 + \delta) + (x_2 + \delta) > x_1 + x_2$$
    for all $\delta > 0$.
  - Strongly monotone, since
    $$(x_1 + \delta) + x_2 > x_1 + x_2$$
- **Hence, strong monotonicity implies monotonicity, but the converse is not necessarily true.**

Desirability

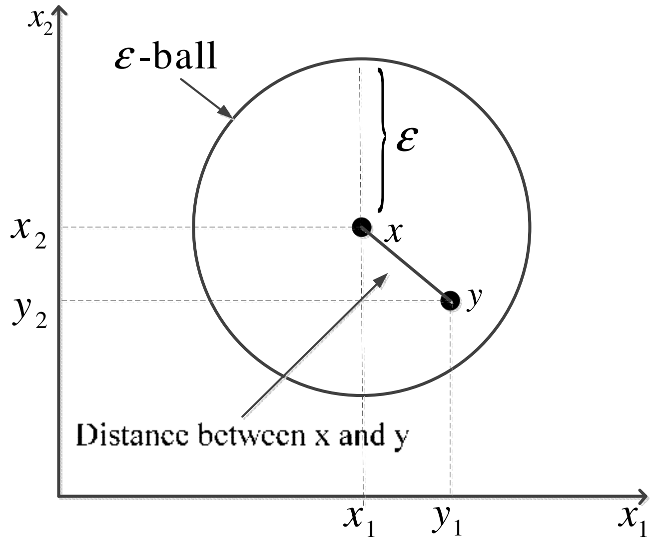
- ***Non-satiation*** (NS):
  - A preference relation satisfies NS if, for every $x \in X$, there is another bundle in set X, $y \in X$, which is strictly preferred to x, i.e., $y \succ x$.
    - NS is too general, since we could think about a bundle y containing extremely larger amounts of some goods than x.
    - How far away are y and x?

Desirability

- ***Local non-satiation*** (LNS):
  - A preference relation satisfies LNS if, for every bundle $x \in X$ and every $\varepsilon > 0$, there is another bundle $y \in X$ which is less than ε-away from x, $||y - x|| < \varepsilon$, and for which $y \succ x$.
    - $||y - x|| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the Euclidean distance between x and y, where $x, y \in \mathbb{R}^2_+$.
    - In words, for every bundle x, and for ***every*** distance ε from x, we can find a more preferred bundle y.

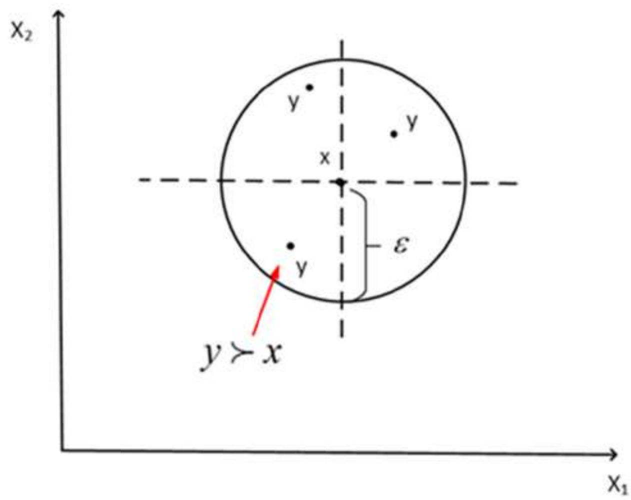
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of good 2 (but more of good 1) than bundle x.



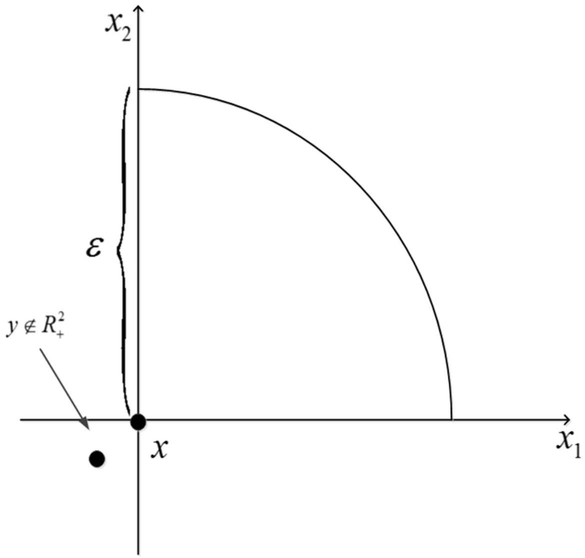
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of *both* goods than bundle x.



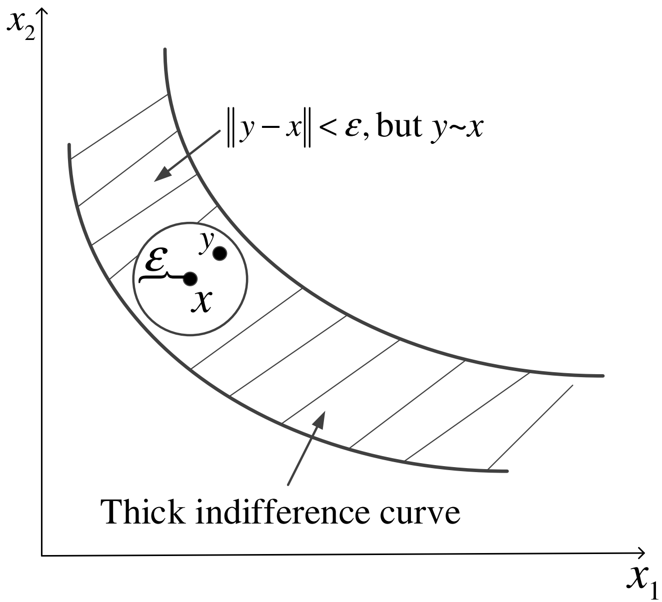
Desirability

- *Violation of LNS*
  - LNS rules out the case in which the decision-maker regards all goods as bads.
  - Although $y \succ x$, y is unfeasible given that it lies away from the consumption set, i.e., $y \notin \mathbb{R}^2_+$.



Desirability

- *Violation of LNS*
  - LNS also rules out "thick" indifference sets.
  - Bundles y and x lie on the same indifference curve.
  - Hence, decision maker is indifferent between x and y, i.e., $y \sim x$.



Desirability

- *Note*:
  - If a preference relation satisfies monotonicity, it must also satisfy LNS.
    - Given a bundle $x = (x_1, x_2)$, increasing all of its components yields a bundle $(x_1 + \delta, x_2 + \delta)$, which is strictly preferred to bundle (x_1, x_2) by monotonicity.
    - Hence, there is a bundle $y = (x_1 + \delta, x_2 + \delta)$ such that $y \succ x$ and $\|y - x\| < \varepsilon$.

Indifference sets

Indifference sets

- The **indifference set** of a bundle $x \in X$ is the set of all bundles $y \in X$, such that $y \sim x$.

$$IND(x) = \{y \in X: y \sim x\}$$

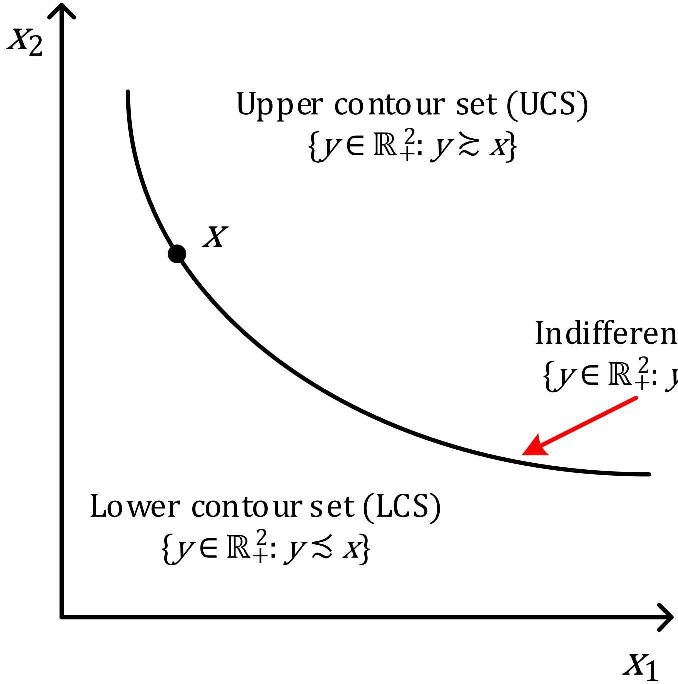
- The **upper-contour set** of bundle x is the set of all bundles $y \in X$, such that $y \succsim x$.

$$UCS(x) = \{y \in X: y \succsim x\}$$

- The **lower-contour set** of bundle x is the set of all bundles $y \in X$, such that $x \succsim y$.

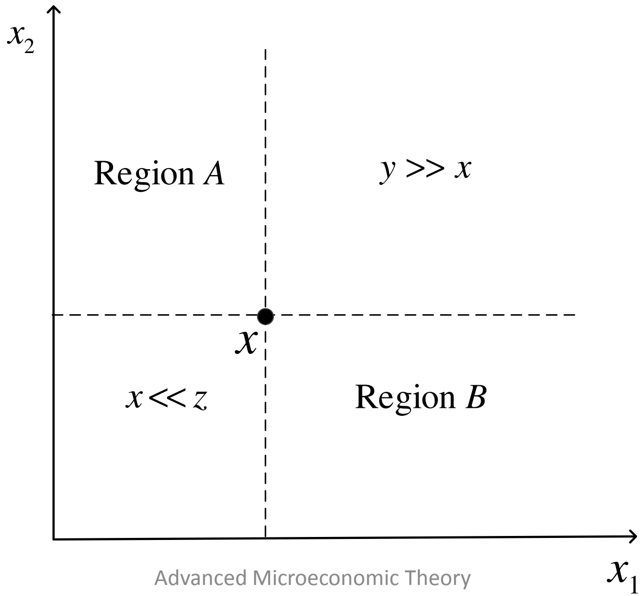
$$LCS(x) = \{y \in X: x \succsim y\}$$

Indifference sets



Indifference sets

- **Strong monotonicity** implies that indifference curves must be negatively sloped.



Indifference sets

- *Note*:
  - Strong monotonicity implies that indifference curves must be negatively sloped.
  - In contrast, if an individual preference relation satisfies LNS, indifference curves can be upward sloping.
    - This can happen if, for instance, the individual regards good 2 as desirable but good 1 as a bad.

Convexity of Preferences

Convexity of Preferences

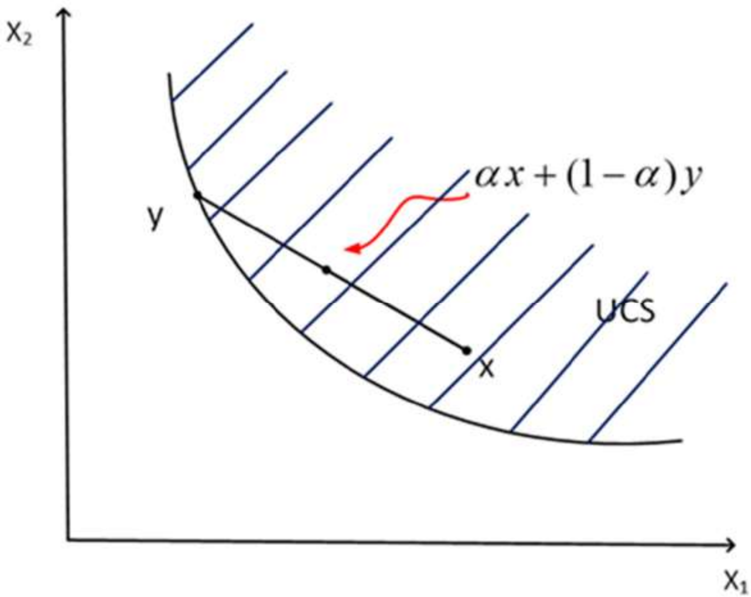
- ***Convexity 1***: A preference relation satisfies convexity if, for all $x, y \in X$,

$$x \succsim y \implies \alpha x + (1-\alpha)y \succsim y$$

for all $\alpha \in (0,1)$.

Convexity of Preferences

- Convexity 1



Convexity of Preferences

- ***Convexity 2***: A preference relation satisfies convexity if, for every bundle x, its upper contour set is convex.

$$UCS(x) = \{y \in X: y \succsim x\} \text{ is convex}$$

- That is, for every two bundles y and z,

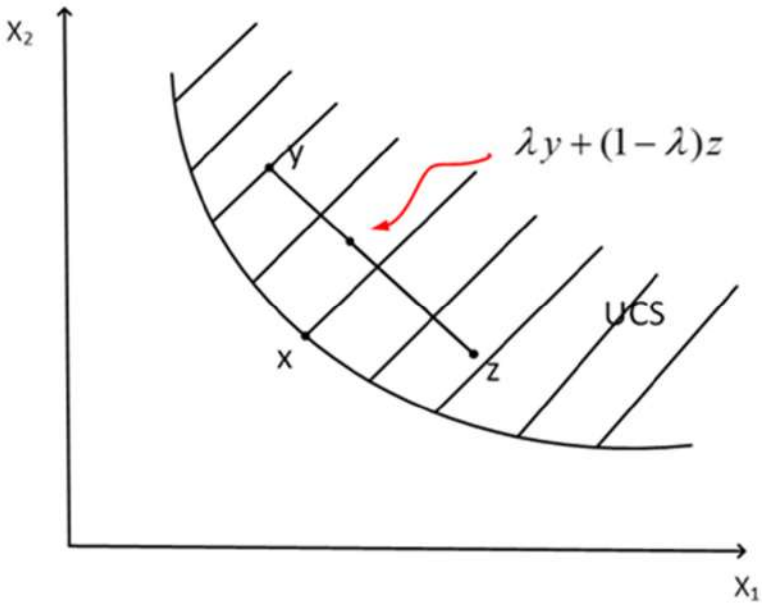
$$\begin{cases} y \succsim x \\ z \succsim x \end{cases} \Longrightarrow \lambda y + (1-\lambda)z \succsim x$$

for any $\lambda \in [0,1]$.

- Hence, points y, z, and their convex combination belongs to the UCS of x.

Convexity of Preferences

- Convexity 2



Convexity of Preferences

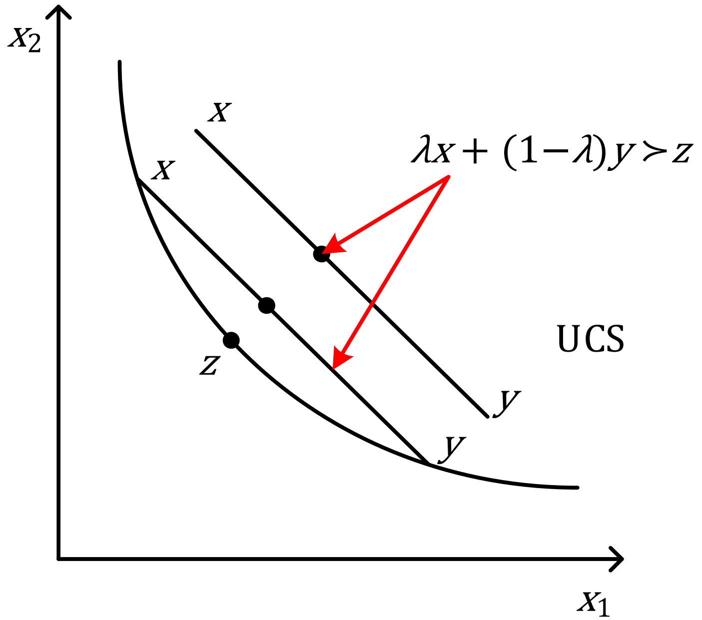
- ***Strict convexity***: A preference relation satisfies strict convexity if, for every $x, y \in X$ where $x \neq y$,

$$\begin{cases} x \succsim z \\ y \succsim z \end{cases} \Longrightarrow \lambda x + (1-\lambda)y \succ z$$

for all $\lambda \in [0,1]$.

Convexity of Preferences

- Strictly convex preferences

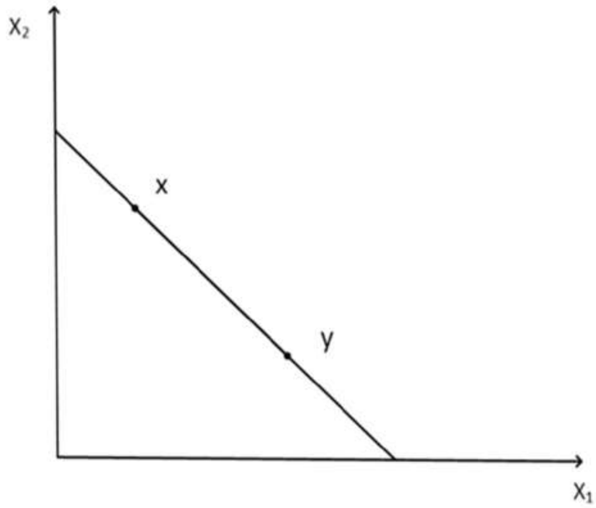


Convexity of Preferences

- **Convex but not strict convex preferences**
  - $\lambda x + (1-\lambda)y \sim z$
  - This type of preference relation is represented by linear utility functions such as

    $$u(x_1, x_2) = ax_1 + bx_2$$

    where x_1 and x_2 are regarded as substitutes.

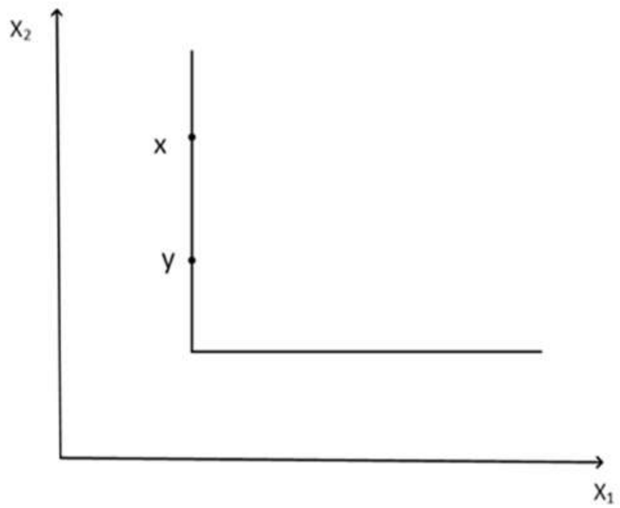


Convexity of Preferences

- **Convex but not strict convex preferences**
  - *Other example*: If a preference relation is represented by utility functions such as

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

where $a, b > 0$, then the pref. relation satisfies convexity, but not strict convexity.



Convexity of Preferences

- ***Example 1.7***

$u(x_1, x_2)$	**Satisfies convexity**	**Satisfies strict convexity**
$ax_1 + bx_2$	√	X
$\min\{ax_1, bx_2\}$	√	X
$ax_1^{\frac{1}{2}} bx_2^{\frac{1}{2}}$	√	√
$ax_1^2 + bx_2^2$	X	X

Do the last two for exercise

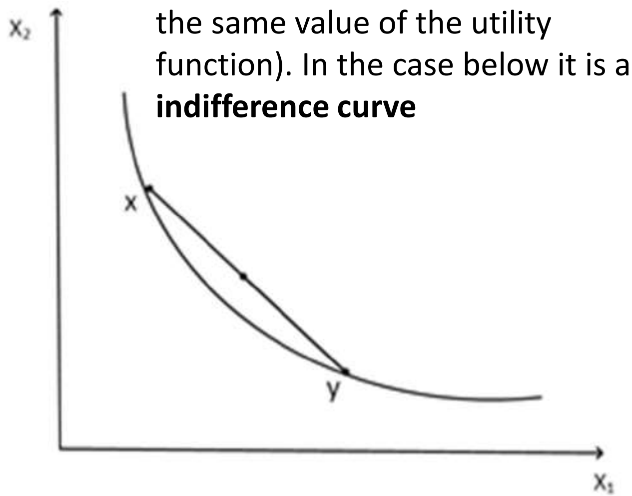
Convexity of Preferences

- ***Interpretation of convexity***

  1) Taste for diversification:

  – An individual with convex preferences prefers the convex combination of bundles x and y, than either of those bundles alone.

Indifference sets can be interpreted as the bundles that give the same level of utility (i.e. the same value of the utility function). In the case below it is an **indifference curve**



MRS is the slope of this indifference curve. Now we will see how to compute the marginal rate of substitution.

SUPPOSE WE HAVE A UTILITY FUNCTION

$u(x_1, x_2, \dots x_m)$

$x_1 \dots x_m$ QUANTITY OF GOODS

MARGINAL UTILITY : $\frac{\partial u}{\partial x_1}$ → PARTIAL DER OF UTILITY WITH RESPECT TO x_1

INCR UTILITY GAIN OBTAINED BY A SMALL INCR OF x_1.

TOTAL DIFFERENTIAL → INCREASING THE VALUE IF WE INCREASE ALL ARGUMENTS OF THE FUNCTION

$$du = \frac{\partial u}{\partial x_1} \cdot dx_1 + \frac{\partial u}{\partial x_2} \cdot dx_2 + \dots + \frac{\partial u}{\partial x_m} dx_m$$

AMOUNT IN WHICH x_1 INCREASES

TOT VARIATION DEPENDS ON ... [MIN 8; 29]

WHAT IS THE MAIN PROPERTIES OF IND CURVE?

IF WE GO FROM POINT TO ANOTHER THEN VARIATION IS Ø

ALONG AN IND. CURVE $du = 0$

$0 =$

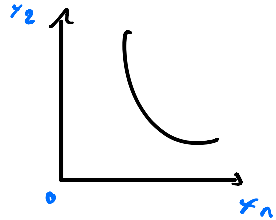
Marginal Rate of Substitution (MRS)

- *Remark*:
  - Let us show that the slope of the indifference curve is given by the MRS.
  - Consider a continuous and differentiable utility function $u(x_1, x_2, \dots, x_n)$.
  - Totally differentiating, we obtain

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \cdots + \frac{\partial u}{\partial x_n} dx_n$$

  - But since we move along the same indifference curve, $du = 0$. $\frac{\partial u}{\partial x_i}$ is called the **marginal utility** of x_i.

Convexity of Preferences



– Inserting $du = 0$, and taking any two goods

$$0 = \frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} dx_j$$

u(xi, xj)

or $$-\frac{\partial u}{\partial x_i} dx_i = \frac{\partial u}{\partial x_j} dx_j$$

Slope => $\boxed{\frac{dxj}{dxi}}$

– If we want to analyze the **rate at which the consumer substitutes units of good *i* for good *j***, we must solve for $\frac{dx_j}{dx_i}$, to obtain

$$\boxed{-\frac{dx_j}{dx_i} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} \equiv MRS_{i,j}}$$

Marg UT is POSITIVE
(If I consume more, is better)

For instance if $-\frac{dx_j}{dx_i}$ = 2/1 = 2 you have to replace 2 units of good *j* for one unit of good *i* to remain in the same indifference curve.

MU xi = 2 · xj MU
because ration is 2

$U(i) = 2U(j)$ IF I GIVE YOU x_1, YOU HAVE TO GIVE AT LEAST TWO x_2

MRS → RATIO OF MARG UTILITY OF GOODS

↳ SLOPE OF IND. CURVE!

Convexity of Preferences

- ***Interpretation of convexity***

  2) *Diminishing marginal rate of substitution:*

$$MRS_{1,2} \equiv -\frac{dx_2}{dx_1} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

  – *MRS* describes the additional amount of good 2 that the consumer needs to receive in order to keep her utility level unaffected, when the amount of good 1 is reduced by one unit.

  – Hence, a *diminishing MRS* implies that the consumer needs to receive increasingly larger amounts of good 2 in order to accept further reductions of good 1.

One properties of MRS:
Since we are using convex Ind curve.

IND set or ind set is decreasing. So IND curve decreasing, slope is negative and then the slope is decreasing. What does it mean?

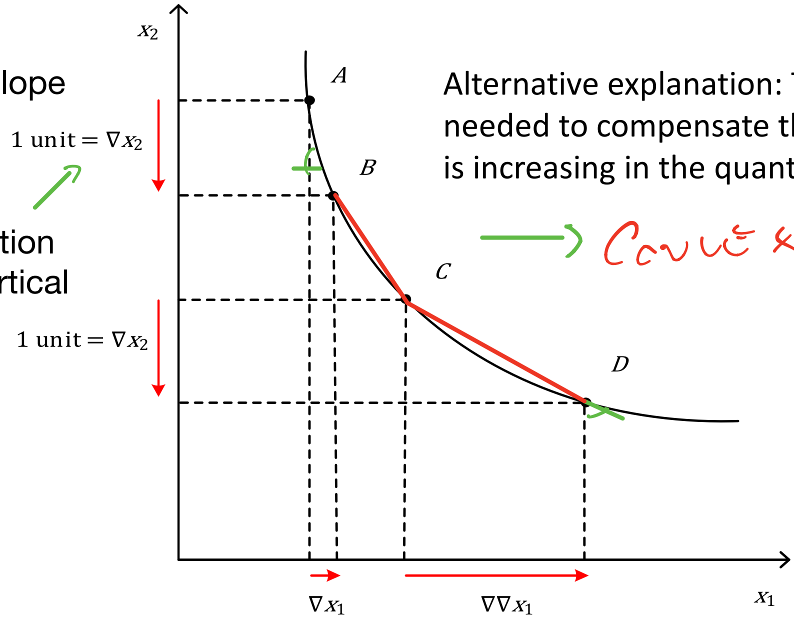
Slope of a curve in one point, if the slope of the angle in this point.

Convexity of Preferences

- Diminishing marginal rate of substitution

Small slope means slope is ..

Imagine having variation from x1 to x2 this vertical jump.



Alternative explanation: The increase in x1 needed to compensate the same decrease in x2, is increasing in the quantity consumed of x1

→ Convex → line above ([illegible]) curve

Amount of x1 you need to mantain(mantenere) utilty invariance is larger.

Implication of marginal rate of substitution...
[25:]

Indifferent curve decreasing mean slope < 0 and the slope is decreasing. The slope is the Marginal rate of substitution.
So this are all thing we are using in the next lectures.

Quasiconcavity

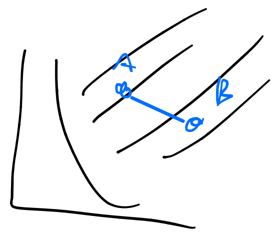
A utility function is concave if UCS is convex.

Quasiconcavity

- A utility function $u(\cdot)$ is ***quasiconcave*** if, for every bundle $y \in X$, the set of all bundles for which the consumer experiences a higher utility, i.e., the $UCS(x) = \{y \in X \mid u(y) \geq u(x)\}$ is convex.
- The following three properties are equivalent:

$$\text{Convexity of preferences} \Leftrightarrow UCS(x) \text{ is convex} \Leftrightarrow u(\cdot) \text{ is quasiconcave}$$

In the example before the UCS is convex. SO if we take two point in the set and link it with a straight line then they depends on the set.



Function convex, UCS convex ==> u(°) is quasiconcave.

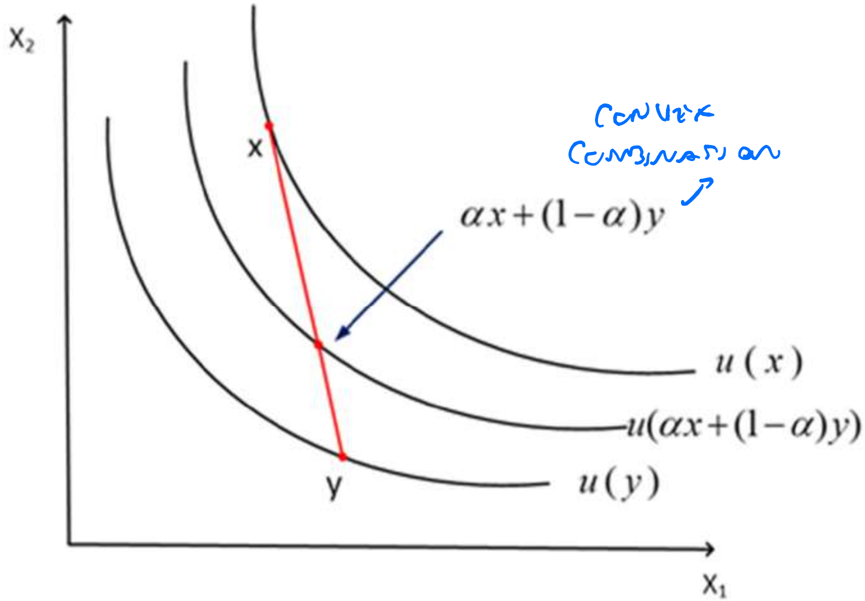
Quasiconcavity

- ***Alternative definition of quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *weakly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$$

Quasiconcavity

- Quasiconcavity (second definition)



CONVEX COMBINATION

IND CURV DIFFR BUNDLE

PREF REPRESENTED BY A MAP OF IND CURVE

IN THIS MAP COULD BE INFINITE IND CURVE

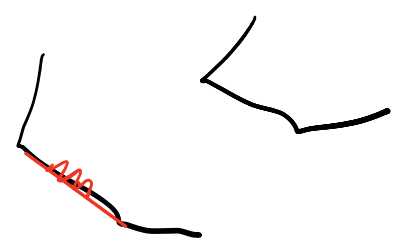
STRICT QUASICONCAVE IS THE SAME

BUT THE VALUE OF CONVEX COMBINATION

IS NO MORE ≥ BUT ONLY > THAN

THE MIN UTILITY BETWEEN THE TWO

BUNDLES



QUASI

CONCAVE BUT

NOT STRICT

QUASI CONC

EACH OF THIS POINT WILL

BE WEAKLY PREFERED

IN THIS CASE THEY ARE

STRICT

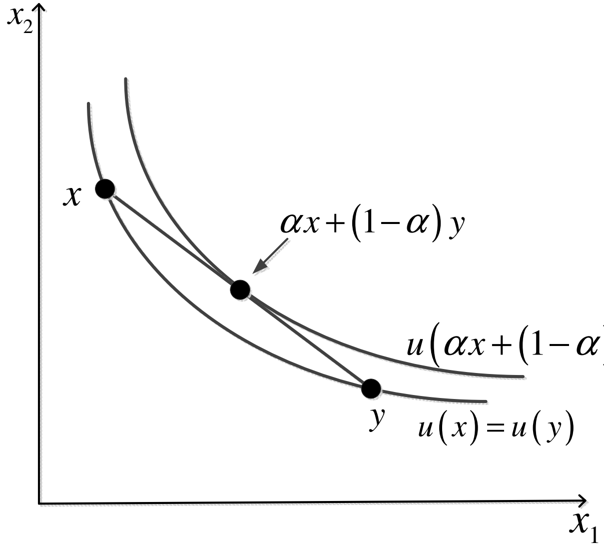
Quasiconcavity

- ***Strict quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *strict quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *strictly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$$

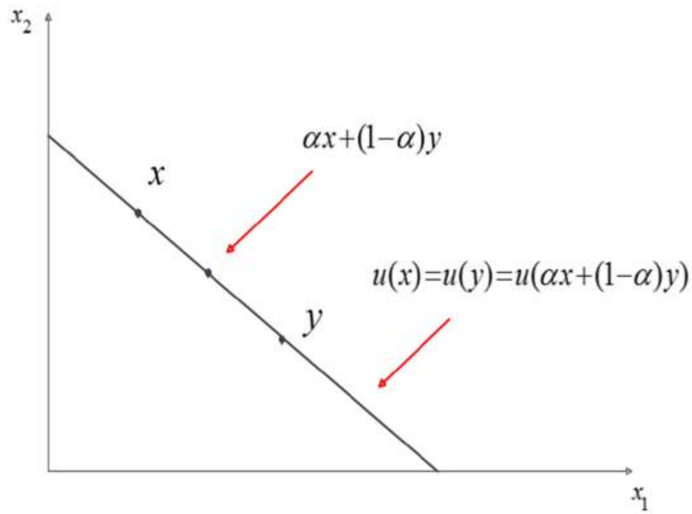
Quasiconcavity

- *What if bundles x and y lie on the same indifference curve?*
- Then, $u(x) = u(y)$.
- Since indifference curves are strictly convex, $u(\cdot)$ satisfies quasiconcavity.



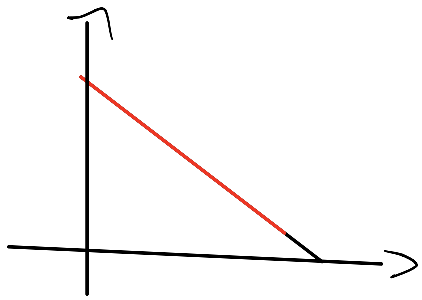
Quasiconcavity

- *What if indifference curves are linear?*
- $u(\cdot)$ satisfies the definition of a quasiconcavity since
$$u(\alpha x + (1-\alpha)y) = \min\{u(x), u(y)\}$$
- But $u(\cdot)$ does not satisfy *strict* quasiconcavity.



NOT STRICT!

IF I CONNECT THIS POINT ARE NOT POINT



STRICTLY CONCAVE
BUT IS THE SAME

Quasiconcavity

- ***Relationship between concavity and quasiconcavity***:

$$\text{Concavity} \underset{\nLeftarrow}{\Longrightarrow} \text{Quasiconcavity}$$

  – If a function $f(\cdot)$ is *concave*, then for any two points $x, y \in X$,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$$
$$\geq \min\{f(x), f(y)\}$$

Since it is a weighted average of the two

  for all $\alpha \in (0,1)$.

    - The first inequality follows from the definition of concavity, while the second holds true for all concave functions.
    - Hence, **quasiconcavity is a weaker condition than concavity**.

IF FUNCTION IS CONCAVE
IS QUASI CONCAVE BUT NOT
THE OPPOSITE

WE DO NOT REQUIRE CONCAVITY IN OUR
STUDY

YOU HAVE TO LOOK AT MATRIX

HESSIAN

FIRST DERIVATIVES $\frac{\partial u}{\partial x_1}$ $\frac{\partial u}{\partial x_2}$

$u(x_1, x_2)$

$$H = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1 \partial x_1} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_2 \partial x_1} & \frac{\partial^2 u}{\partial x_2 \partial x_2} \end{bmatrix}$$

A_{11} A_{22}

SYMMETRIC!

2,3 ARE EQUAL

H IS POSITIVE DEFINITE $\Rightarrow u(\cdot)$ CONVEX

HESSIAN MATRIX
(SECOND DERIVATIVE)

H IS NEG DEFINITE $\Rightarrow u(\cdot)$ IS CONCAVE

$|A_{11}| > 0$ $|A_{22}| > 0$

$$\frac{\partial^2 u}{\partial x_1 \partial x_1} > 0 \qquad \frac{\partial^2 u}{\partial x_1 \partial x_1} \cdot \frac{\partial^2 u}{\partial x_2 \partial x_2} - \overbrace{\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2}^{\text{POSITIVE}} > 0$$

Π

THIS IS VERIFIED BEFORE

THIS SHOULD BE POSITIVE THEN

PROVE MATRIX DEFINITE NEGATIVE

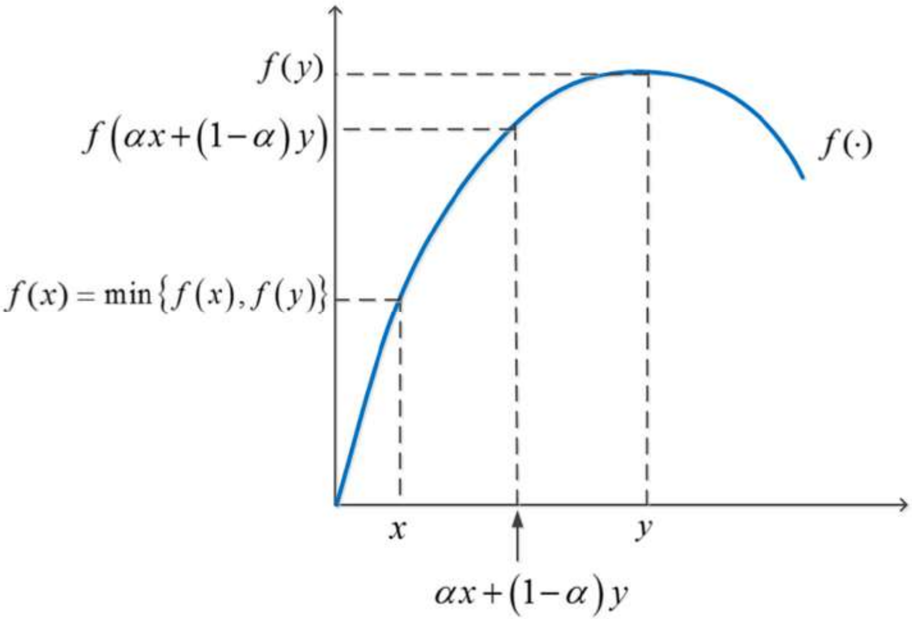
CONCAVE IF

$|A_{11}| < 0 \quad |A_{22}| > 0$

SECOND DERIVATIVE OF FUNCTION CONCAVE

Quasiconcavity

- Concavity implies quasiconcavity



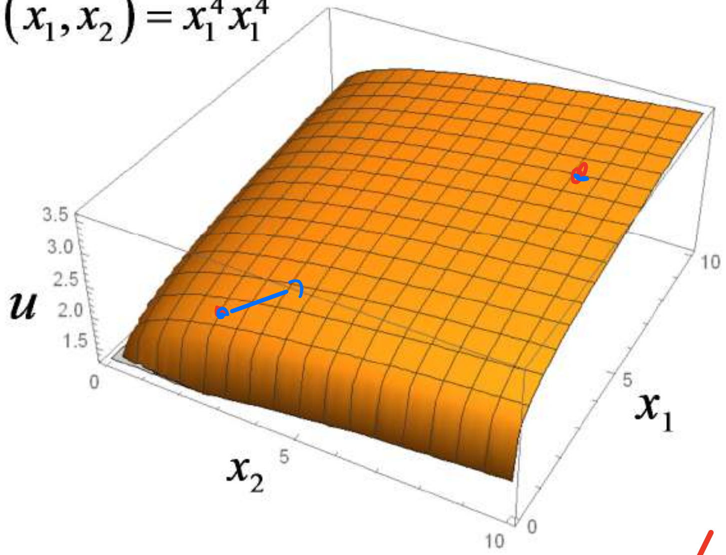
Quasiconcavity

- A concave $u(\cdot)$ exhibits diminishing marginal utility.
  – That is, **for an increase in the consumption bundle, the increase in utility is *smaller* as we move away from the origin**.
- The "jump" from one indifference curve to another requires:
  – a slight increase in the amount of x_1 and x_2 when we are close to the origin
  – a large increase in the amount of x_1 and x_2 as we get further away from the origin

Quasiconcavity

- Concave and quasiconcave utility function (3D)

$$u(x_1, x_2) = x_1^{\frac{1}{4}} x_1^{\frac{1}{4}}$$

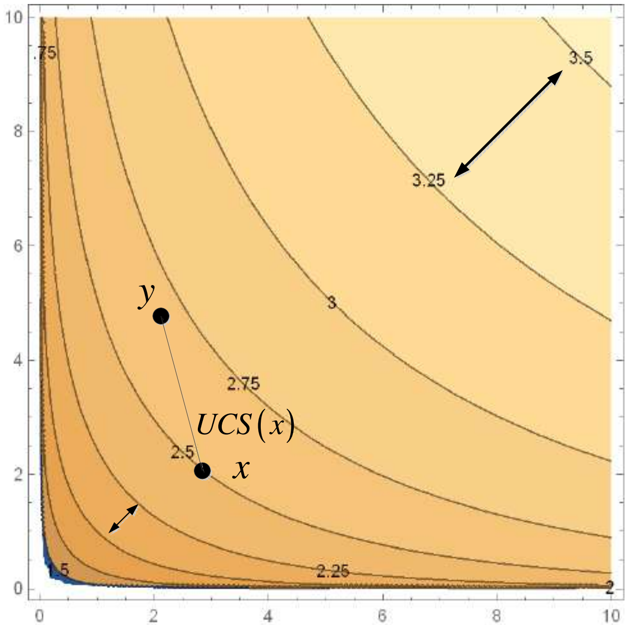


ALTITUDE?

CONTOUR MAP

Quasiconcavity

- Concave and quasiconcave utility function (2D)



PALE COLOUR

MEAN HIGHER

UTILITY

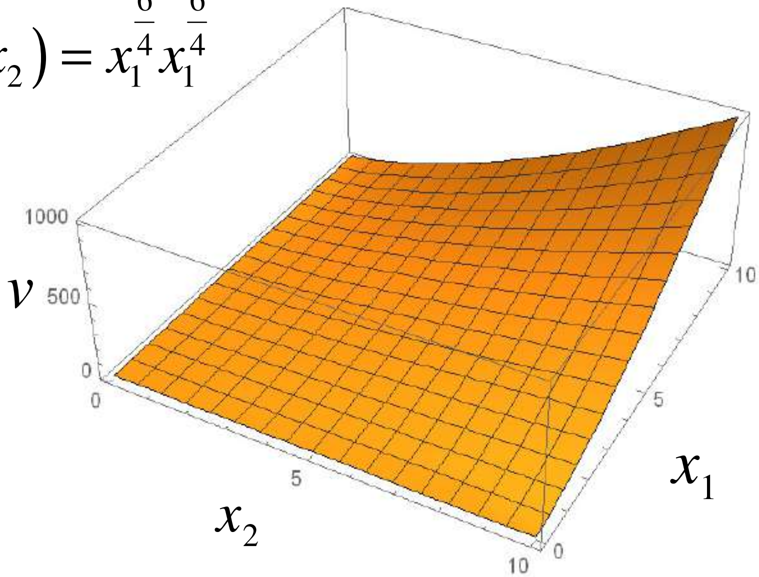
Quasiconcavity

- A convex $u(\cdot)$ exhibits increasing marginal utility.
  - That is, for an increase in the consumption bundle, the increase in utility is *larger* as we move away from the origin.
- The "jump" from one indifference curve to another requires:
  - a large increase in the amount of x_1 and x_2 when we are close to the origin, but…
  - a small increase in the amount of x_1 and x_2 as we get further away from the origin

Quasiconcavity

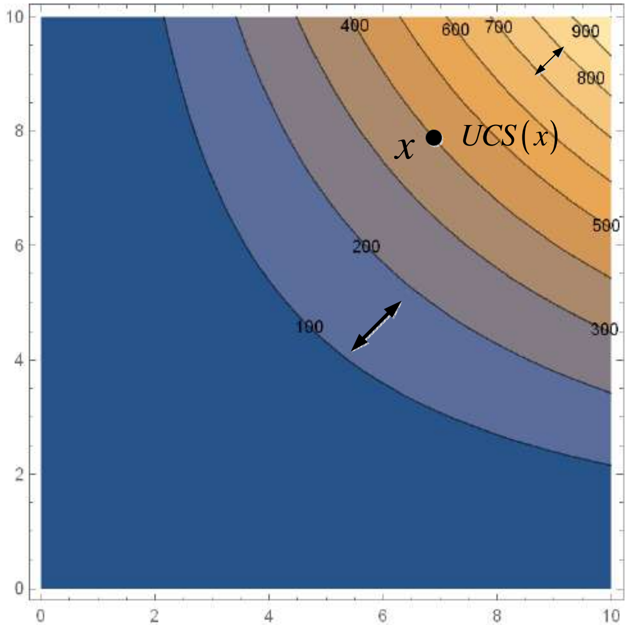
- Convex but quasiconcave utility function (3D)

$$v(x_1, x_2) = x_1^{\frac{6}{4}} x_1^{\frac{6}{4}}$$



Quasiconcavity

- Convex but quasiconcave utility function (2D)



Cobb-Douglas utility function

Quasiconcavity

- *Note*:
  - Utility function $v(x_1, x_2) = x_1^{\frac{6}{4}} x_2^{\frac{6}{4}}$ is a strictly monotonic transformation of $u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$,
    - That is, $v(x_1, x_2) = f(u(x_1, x_2))$, where $f(u) = u^6$.
  - Therefore, utility functions $u(x_1, x_2)$ and $v(x_1, x_2)$ represent the same preference relation.
  - **Both utility functions are quasiconcave although one of them is concave and the other is convex**.
  - Hence, **we normally require utility functions to satisfy quasiconcavity alone**.

A1

Diapositiva 54

A1 Show quasi-concavity with the hessian?
Administrator; 04/01/2019

Quasiconcavity

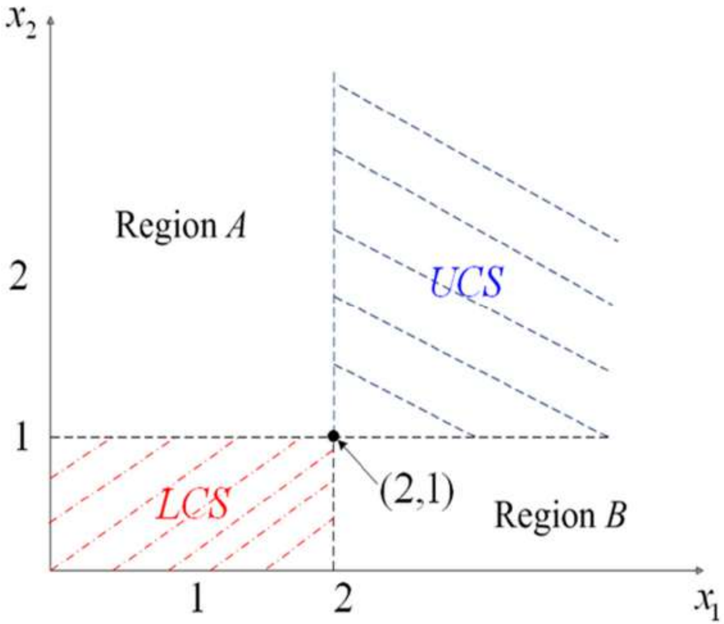
- ***Example 1.8*** (Testing properties of preference relations):
  - Consider an individual decision maker who consumes bundles in $\mathbb{R}_{+}^{L}$.
  - Informally, he "prefers more of everything"
  - Formally, for two bundles $x, y \in \mathbb{R}_{+}^{L}$, bundle x is weakly preferred to bundle y, $x \gtrsim y$, iff bundle x contains more units of every good than bundle y does, i.e., $x_k \geq y_k$ for every good k.
  - Let us check if this preference relation satisfies: (a) completeness, (b) transitivity, (c) strong monotonicity, (d) strict convexity, and (e) local non-satiation.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us consider the case of only two goods, $L = 2$.
  - Then, an individual prefers a bundle $x = (x_1, x_2)$ to another bundle $y = (y_1, y_2)$ iff x contains more units of both goods than bundle y, i.e., $x_1 \geq y_1$ and $x_2 \geq y_2$.
  - For illustration purposes, let us take bundle such as $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):



Quasiconcavity

- ***Example 1.8*** (continued):

  1) UCS:

  – The upper contour set of bundle $(2,1)$ contains bundles (x_1, x_2) with weakly more than 2 units of good 1 and/or weakly more than 1 unit of good 2:

  $$UCS(2,1) = \{(x_1, x_2) \succsim (2,1) \Leftrightarrow x_1 \geq 2, x_2 \geq 1\}$$

  – The frontiers of the UCS region also represent bundles preferred to $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):

  2) LCS:

  – The bundles in the lower contour set of bundle $(2,1)$ contain fewer units of both goods:

  $$LCS(2,1) = \{(2,1) \gtrsim (x_1, x_2) \ \Leftrightarrow x_1 \leq 2, x_2 \leq 1\}$$

  – The frontiers of the LCS region also represent bundles with fewer unis of either good 1 or good 2.

Quasiconcavity

- ***Example 1.8*** (continued):

  3) IND:

  – The indifference set comprising bundles (x_1, x_2) for which the consumer is indifferent between (x_1, x_2) and $(2,1)$ is empty:

  $$IND(2,1) = \{(2,1) \sim (x_1, x_2)\} = \emptyset$$

  – No region for which the upper contour set and the lower contour set overlap.

Quasiconcavity

- ***Example 1.8*** (continued):

  4) Regions A and B:

  – Region A contains bundles with more units of good 2 but fewer units of good 1 (the opposite argument applies to region B).

  – The consumer cannot compare bundles in either of these regions against bundle $(2,1)$.

  – For him to be able to rank one bundle against another, one of the bundles must contain the same or more units of all goods.

Quasiconcavity

- ***Example 1.8*** (continued):

  5) Preference relation is not complete:

  – Completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both).

  – Consider two bundles $x, y \in \mathbb{R}_+^2$ with bundle x containing more units of good 1 than bundle y but fewer units of good 2, i.e., $x_1 > y_1$ and $x_2 < y_2$ (as in Region B)

  – Then, we have neither $x \succsim y$ (UCS) nor $y \succsim x$ (LCS).

Quasiconcavity

- ***Example 1.8*** (continued):

  6) Preference relation is transitive:

  – Transitivity requires that, for any three bundles x, y and z, if $x \gtrsim y$ and $y \gtrsim z$ then $x \gtrsim z$.

  – Now $x \gtrsim y$ and $y \gtrsim z$ means $x_k \geq y_k$ and $y_k \geq z_k$ for all k goods.

  – Then, $x_k \geq z_k$ implies $x \gtrsim z$.

Quasiconcavity

- ***Example 1.8*** (continued):

  7) Preference relation is strongly monotone:

  – Strong monotonicity requires that if we increase one of the goods in a given bundle y, then the newly created bundle x must be strictly preferred to the original bundle.

  – Now $x \geq y$ and $x \neq y$ implies that $x_l \geq y_l$ for all good l and $x_k > y_k$ for at least one good k.

  – Thus, $x \geq y$ and $x \neq y$ implies $x \succsim y$ and not $y \succsim x$.

  – Thus, we can conclude that $x \succ y$.

Quasiconcavity

- ***Example 1.8*** (continued):

  8) Preference relation is strictly convex:

  – Strict convexity requires that if $x \gtrsim z$ and $y \gtrsim z$ and $x \neq y$, then $\alpha x + (1 - \alpha)y \succ z$ for all $\alpha \in (0,1)$.

  – Now $x \gtrsim z$ and $y \gtrsim z$ implies that $x_l \geq y_l$ and $y_l \geq z_l$ for all good l.

  – $x \neq z$ implies, for some good k, we must have $x_k > z_k$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Hence, for any $\alpha \in (0,1)$, we must have that

    $$\alpha x_l + (1-\alpha)y_l \geq z_l \text{ for every good } l$$

    $$\alpha x_k + (1-\alpha)y_k > z_k \text{ for some } k$$

  - Thus, we have that $\alpha x + (1-\alpha)y \geq z$ and $\alpha x + (1-\alpha)y \neq z$, and so

    $$\alpha x + (1-\alpha)y \succsim z$$

    $$\text{and not } z \succsim \alpha x + (1-\alpha)y$$

  - Therefore, $\alpha x + (1-\alpha)y \succ z$.

Quasiconcavity

- ***Example 1.8*** (continued):
  9) ***Preference relation satisfies LNS***:
     – Take any bundle (x_1, x_2) and a scalar $\varepsilon > 0$.
     – Let us define a new bundle (y_1, y_2) where
     $$(y_1, y_2) \equiv \left(x_1 + \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}\right)$$
     so that $y_1 > x_1$ and $y_2 > x_2$.
     – Hence, $y \gtrsim x$ but not $x \gtrsim y$, which implies $y \succ x$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us know check if bundle y is within an ε-ball around x.
  - The Cartesian distance between x and y is

$$\|x - y\| = \sqrt{\left[x_1 - \left(x_1 + \frac{\varepsilon}{2}\right)\right]^2 + \left[x_1 - \left(x_1 + \frac{\varepsilon}{2}\right)\right]^2} = \frac{\varepsilon}{\sqrt{2}}$$

which is smaller than ε for all $\varepsilon > 0$.

Advanced Microeconomics (EPS)

Chapter 1: Common utility functions

Common Utility Functions

- ***Cobb-Douglas utility functions***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$$
  where $A, \alpha, \beta > 0$.
  - Applying logs on both sides
  $$\log u = \log A + \alpha \log x_1 + \beta \log x_2$$
  - Hence, the exponents in the original $u(\cdot)$ can be interpreted as *elasticities*:

$$\varepsilon_{u,x_1} = \frac{\partial u(x_1,x_2)}{\partial x_1} \cdot \frac{x_1}{u(x_1,x_2)} = \alpha A x_1^{\alpha-1} x_2^{\beta} \cdot \frac{x_1}{A x_1^{\alpha} x_2^{\beta}} = \alpha$$

Compute marginal derivative.

$a x^2 b$

$$A x_1^a x_2^b \qquad \frac{\partial u}{\partial x_1} = a A x_2^\beta x_1^{a-1}$$

$K x_1^a$

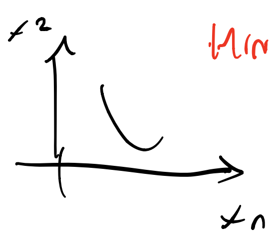
$K x_1^{a-1}$

$$\frac{a x_2}{b x_1} \leftarrow \frac{a A x_2^\beta x_1^{a-1}}{A b x_2^{b-1} x_1^a}$$

$$A x_1^a x_2^b \qquad \frac{\partial u}{\partial x_2} = A x_1^a b x_2^{b-1}$$

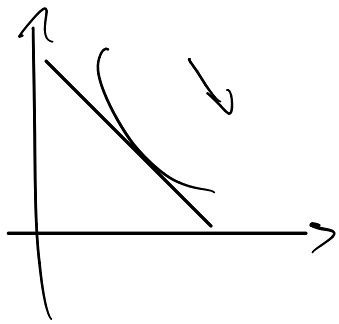
IF $A, \alpha, \beta > 0$ SO MARG UTILITY IS POSITIVE!

$$\underbrace{MRS}_{\text{SLOPE OF IND. CURVE}} = -\frac{dx_2}{dx_1} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$$



HINT TO REMEMBER

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\cancel{A}\,\cancel{x_2^{\beta}}\,\cancel{x_1^{\alpha-1}}\,\alpha}{A x_1^{\alpha}\,\beta\,\cancel{x_2^{\beta-1}}\,\cancel{A}} = \frac{\alpha x_2}{\beta x_1}$$



IF $x_1 \uparrow$

MRS $\downarrow$

$\boxed{\text{ELASTICITY}}$ OF UTILITY IN THIS CASE

$$y = f(x) \Rightarrow \varepsilon_{y,x} = \frac{\partial f}{\partial x} \cdot \frac{x}{y}$$

$$\downarrow$$

$$\frac{\frac{\partial y}{\partial x}}{\frac{y}{x}}$$

% CHANGE IN Y
PRODUCE BY CHANGING X

$$\frac{\frac{\partial y}{y}}{\frac{\partial x}{x}} \rightarrow \frac{\partial y}{y} \cdot \frac{x}{\partial x} \rightarrow \frac{\partial y}{\partial x} \cdot \frac{x}{y}$$

DER Y WITH RESPECT TO X MULTIPLY BY X/Y

APPLY THIS TO UTILITY FUNCTION

$$\varepsilon_{u,x_1} = \frac{\partial u}{\partial x_1} \cdot \frac{x_1}{u}$$

ELASTICITY OF UTILITY WITH RESPECT TO X_1

MARGINAL UTILITY

INITIAL FUNCTION

$$\varepsilon_{u,x_1} = (\cancel{A} \cdot \cancel{x_2^{\beta}})\,\alpha \cancel{x_1^{\alpha-1}} \cdot \frac{\cancel{x_1}}{\cancel{A}\cancel{x_1^{\alpha}}\cancel{x_2^{\beta}}} = \boxed{\alpha}$$

SO ELASTICITY IS CONSTANT

If we have utility function and we apply

log of the product is the sum of the log of the product

$$\log x_1^{\alpha} = \alpha \log x_1$$

$$\log x_2^{\beta} = \beta \log x_2$$

SO IS JUST α AND IT'S SIMPLER

$$\varepsilon_{u,x_1} = \frac{d \log u}{d \log x_1} =$$

$(x_1^{\alpha} + x_2^{\beta})$ we cannot separate x_1, x_2 $\rightarrow$ we did before compute calculus

with log transformation we obtain something linear

Common Utility Functions

- Intuitively, a one-percent increase in the amount of good x_1 increases individual utility by α percent.

- Similarly, $\varepsilon_{u,x_2} = \beta$.

- Special cases:
  - $\alpha + \beta = 1$: $u(x_1, x_2) = Ax_1^{\alpha} x_2^{1-\alpha}$
  - $A = 1$: $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$
  - $A = \alpha = \beta = 1$: $u(x_1, x_2) = x_1 x_2$

Common Utility Functions

– Marginal utilities:

$$\frac{\partial u}{\partial x_1} > 0 \text{ and } \frac{\partial u}{\partial x_2} > 0$$

– Diminishing MRS, since

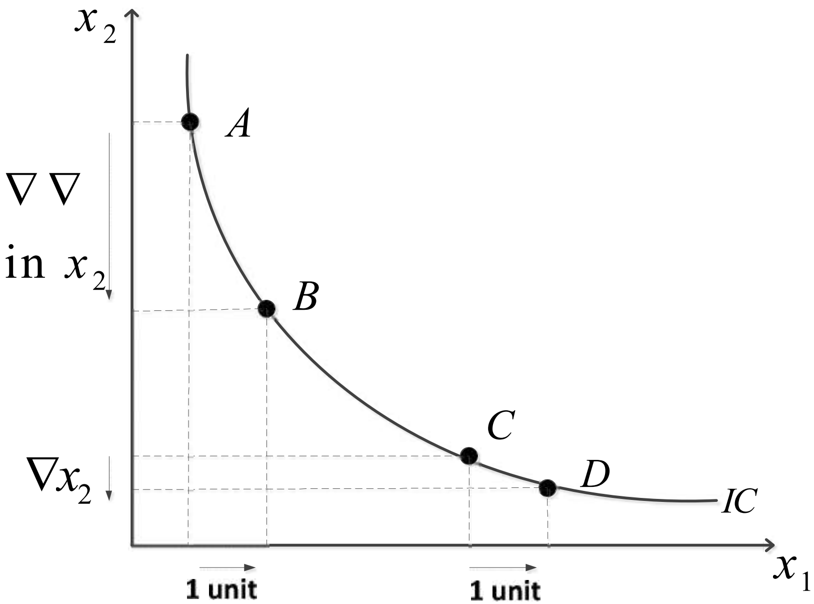
$$MRS_{x_1,x_2} = \frac{\alpha A x_1^{\alpha-1} x_2^{\beta}}{\beta A x_1^{\alpha} x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}$$

which is decreasing in x_1.

- Hence, indifference curves become flatter as x_1 increases.

Common Utility Functions

- Cobb-Douglas preference



Common Utility Functions

- ***Perfect substitutes***:
  - In the case of two goods, x_1 and x_2,
    $$u(x_1, x_2) = Ax_1 + Bx_2$$
    where $A, B > 0$.
  - Hence, the marginal utility of every good is constant:
    $$\frac{\partial u}{\partial x_1} = A \text{ and } \frac{\partial u}{\partial x_2} = B$$
  - MRS is also constant, i.e., $MRS_{x_1,x_2} = \frac{A}{B}$
    - Therefore, indifference curves are straight lines with a slope of $-\frac{A}{B}$.

Utility depends on x1 and x2 but they enter separately in the utility function. A and B must be greater than 0.

$$A = 1 \quad \Rightarrow \quad x_1 + x_2$$
$$B = 1$$

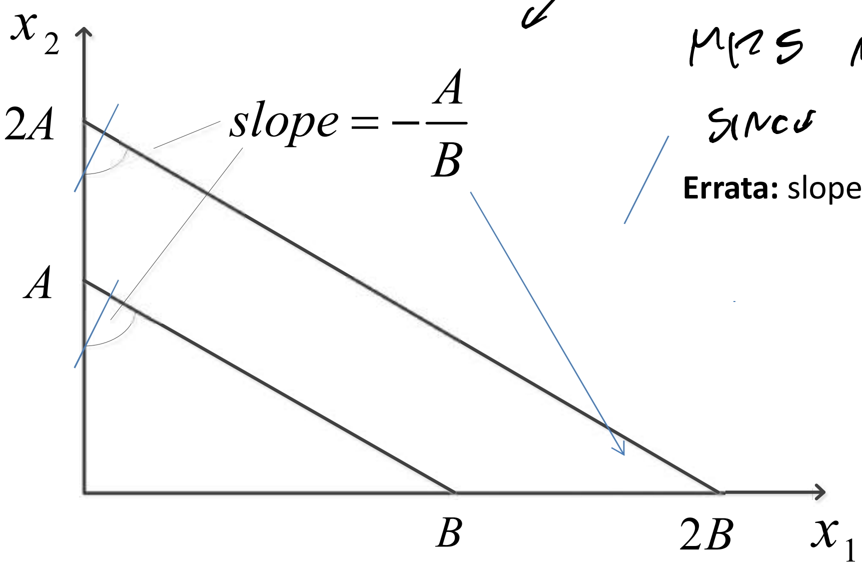
Marginal utility of this ?
Marginal of x1 is A and marginal of x2 is B.
In this case marginal utility is a constant and don't depend on x1 and x2. In the linear utilty function, MU is constant. What does this imply for MRS (rateo of MU)? If the MU are constant then MRS is constant.

$$MRS = \frac{A}{B}$$

Common Utility Functions

- Perfect substitutes

CONSTANT SLOPE



MRS NOT DECREASE SINCE IS A CONSTANT!

Errata: slope is with x_1 axes

Common Utility Functions

– Intuitively, the individual is willing to give up $\frac{A}{B}$ units of x_2 to obtain one more unit of x_1 and keep his utility level unaffected.

– Unlike in the Cobb-Douglas case, such willingness is independent in the relative abundance of the two goods.

– *Examples*: butter and margarine, coffee and black tea, or two brands of unflavored mineral water

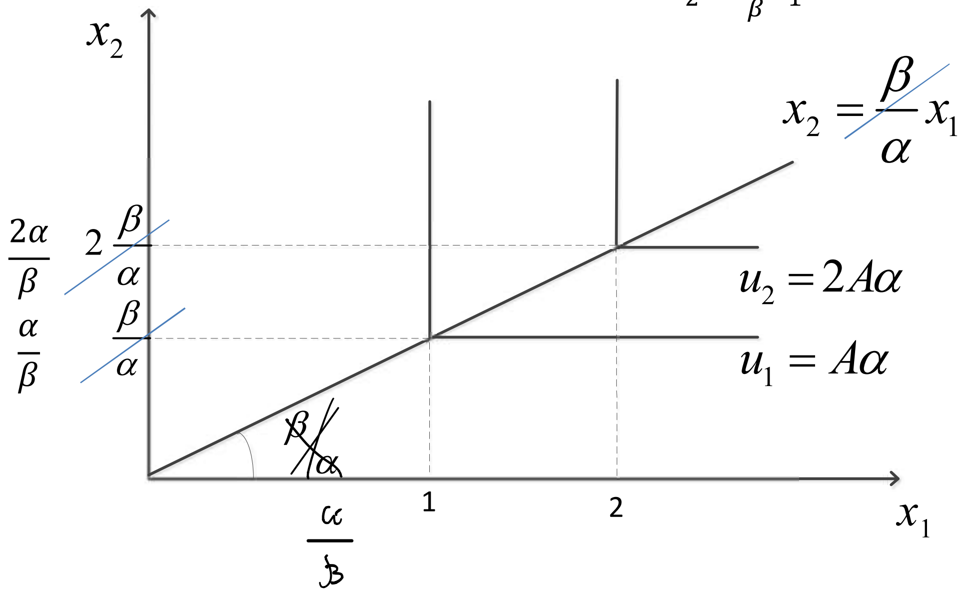
Common Utility Functions

- ***Perfect Complements***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = A \cdot \min\{\alpha x_1, \beta x_2\}$$
  where $A, \alpha, B > 0$.
  - Intuitively, increasing one of the goods without increasing the amount of the other good entails *no* increase in utility.
    - The amounts of *both* goods must increase for the utility to go up.
  - The indifference curve is right angle with a kink at $\alpha x_1 = \beta x_2$ that is $x_2 = (\alpha/\beta)\, x_1$

Common Utility Functions

- Perfect complements

Errata: $x_2 = \frac{\alpha}{\beta} x_1$



Common Utility Functions

– The slope of a ray $x_2 = \frac{\alpha}{\beta} x_1$, $\frac{\alpha}{\beta}$, indicates the rate at which goods x_1 and x_2 must be consumed in order to achieve utility gains.

– Special case: $\alpha = \beta$

$$\begin{aligned} u(x_1, x_2) &= A \cdot \min\{\alpha x_1, \alpha x_2\} \\ &= A\alpha \cdot \min\{x_1, x_2\} \\ &= B \cdot \min\{x_1, x_2\} \text{ if } B \equiv A\alpha \end{aligned}$$

– *Examples*: cars and gasoline, or peanut butter and jelly.

Common Utility Functions

- ***CES utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = \left[a x_1^{\frac{\sigma-1}{\sigma}} + b x_2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

    where σ measures the elasticity of substitution between goods x_1 and x_2.

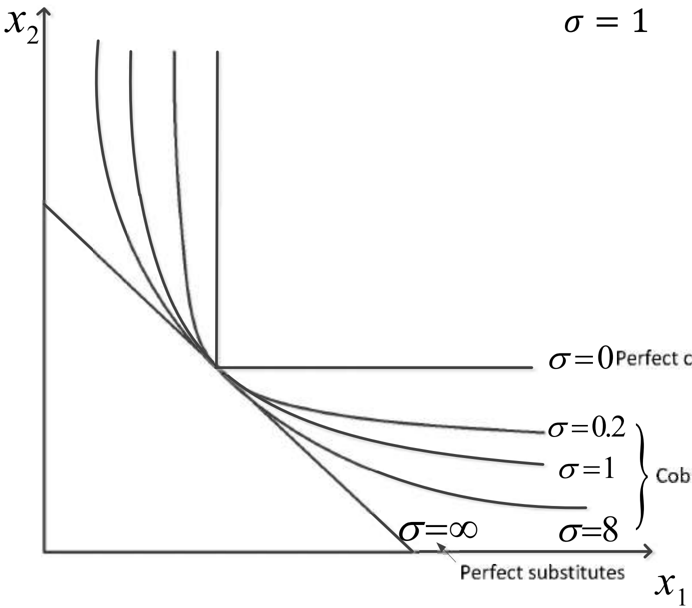
  - In particular,

$$\sigma = \frac{\partial\left(\frac{x_2}{x_1}\right)}{\partial MRS_{1,2}} \cdot \frac{MRS_{1,2}}{\frac{x_2}{x_1}}$$

Common Utility Functions

- CES preferences

Errata: Cobb Douglas if $\sigma = 1$



Common Utility Functions

– CES utility function is often presented as

$$u(x_1, x_2) = \left[a x_1^{\rho} + b x_2^{\rho}\right]^{\frac{1}{\rho}}$$

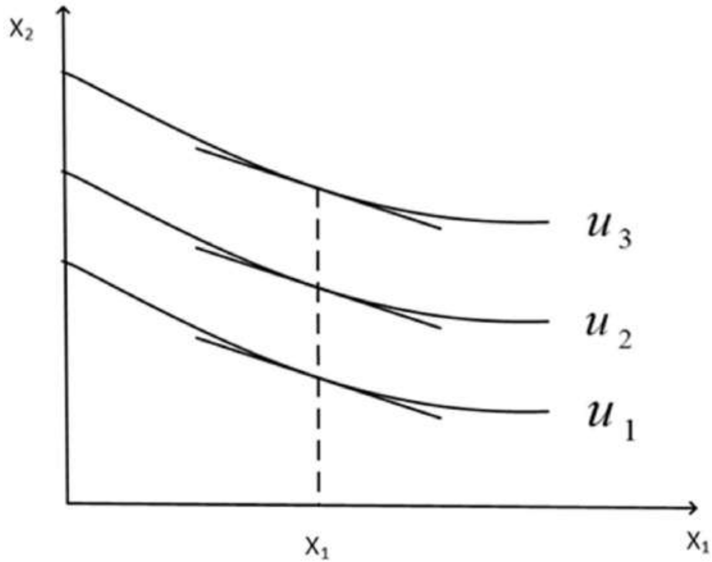
where $\rho \equiv \frac{\sigma - 1}{\sigma}$.

Common Utility Functions

- ***Quasilinear utility function***:
  - In the case of two goods, x_1 and x_2,
    $$u(x_1, x_2) = v(x_1) + bx_2$$
    where x_2 enters *linearly*, $b > 0$, and $v(x_1)$ is a *nonlinear* function of x_1.
    - For example, $v(x_1) = a \ln x_1$ or $v(x_1) = ax_1^{\alpha}$, where $a > 0$ and $\alpha \neq 1$.
  - The MRS is constant in the good that enters linearly in the utility function (x_2 in our case).

Common Utility Functions

- MRS of quasilinear preferences



Common Utility Functions

– For $u(x_1, x_2) = v(x_1) + bx_2$, the marginal utilities are

$$\frac{\partial u}{\partial x_2} = b \text{ and } \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$$

which implies

$$MRS_{x_1,x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

which is constant in the good entering linearly, x_2

– Quasilinear preferences are often used to represent the consumption of goods that are relatively insensitive to income.

– *Examples*: garlic, toothpaste, etc.

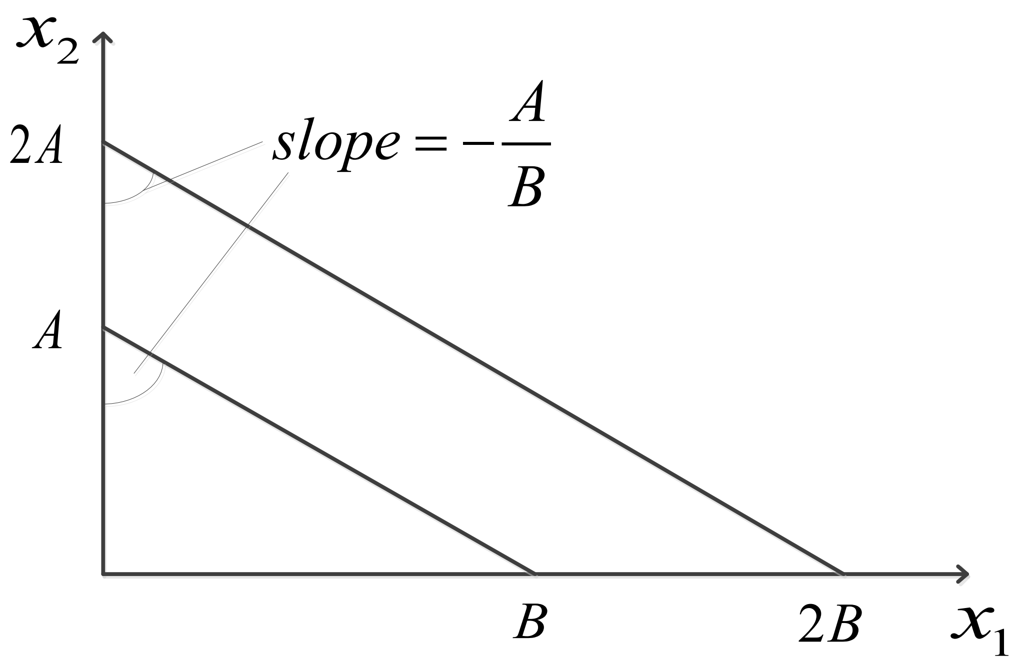
Summary
Perfect substitutes. A and B positive.
Last time introduce the concept of marginal utility = increase in the utility derived in infinitesima of x_1.
MU of first good is der of u / der of x_1 = A.
MRS is the slope of the indifference curve. In mathematics how do we compute? Ratio of the two MU.
If MU are constant also the ratio is constant. This mean that the slope is constant.

Common Utility Functions

- ***Perfect substitutes***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = Ax_1 + Bx_2$$
  where $A, B > 0$.
  - Hence, the marginal utility of every good is constant:
  $$\frac{\partial u}{\partial x_1} = A \text{ and } \frac{\partial u}{\partial x_2} = B$$
  - MRS is also constant, i.e., $MRS_{x_1,x_2} = \frac{A}{B}$
    - Therefore, indifference curves are straight lines with a slope of $-\frac{A}{B}$.

Common Utility Functions

- Perfect substitutes



How can you draw IC in a graph giving the utility function? (FOR PERFECT SUBSTITUTES)

$u(x_1, x_2) = x_1 + x_2$

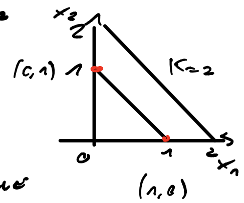
IS LINEAR? YES $A=1$ $B=1$

IF YOU WANT TO COMPUTE MU I HAVE TO COMPUTE DERIVATIVES

$$\frac{\partial u}{\partial x_1} = 1 \qquad \frac{\partial u}{\partial x_2} = 1$$

$$MRS = -\frac{dx_2}{dx_1} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = 1$$ HOW TO DRAW IT?

DEFINITION (MRS IS SLOPE)



∀ IND CURVE WHICH WILL GIVE SAME UTILITY

$K = x_1 + x_2$ → IND. CURVE WITH UTILITY LEVEL K

$K=1$ → $1 = x_1 + x_2$ $x_2 = 1 - x_1$

(1,0) (0,1)

$K=2$? DO THE SAME GET STRAIGHT LINE CROSSING 2 AND 2

u → LINEAR IND ARE STRAIGHT LINE

(MRS DO NOT DEP ON TWO GOODS)

Common Utility Functions

- Intuitively, the individual is willing to give up $\frac{A}{B}$ units of x_2 to obtain one more unit of x_1 and keep his utility level unaffected.
- Unlike in the Cobb-Douglas case, such willingness is independent in the relative abundance of the two goods.
- *Examples*: butter and margarine, coffee and black tea, or two brands of unflavored mineral water

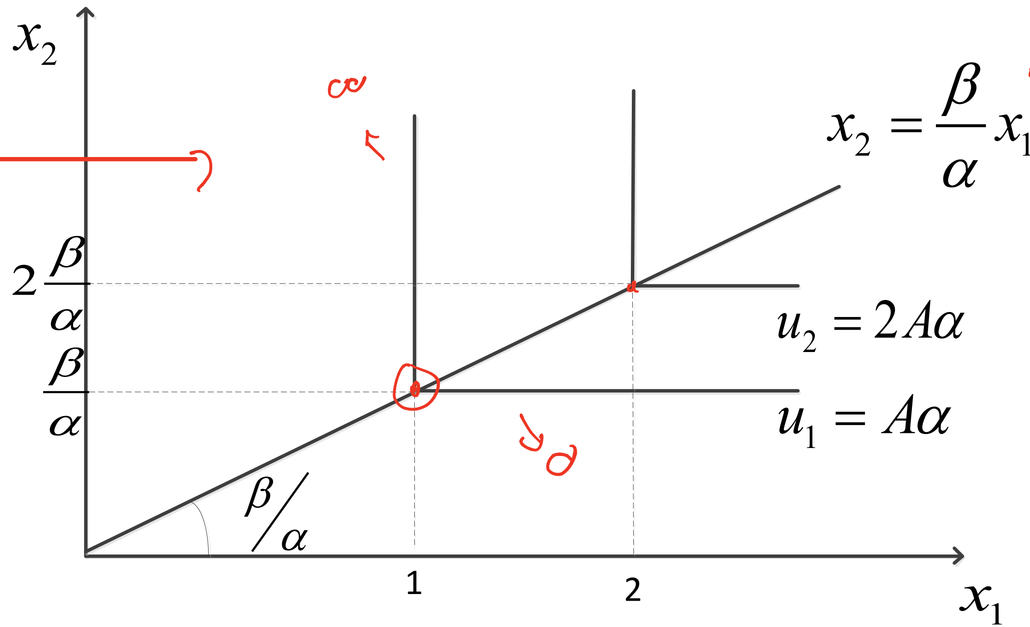
Common Utility Functions

- ***Perfect Complements***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = A \cdot \min\{\alpha x_1, \beta x_2\}$$
  where $A, \alpha, B > 0$.
  - Intuitively, increasing one of the goods without increasing the amount of the other good entails *no* increase in utility.
    - The amounts of *both* goods must increase for the utility to go up.
  - The indifference curve is right angle with a kink at $\alpha x_1 = \beta x_2$. $\longrightarrow \left(\frac{\alpha}{\beta}\right) x_1$

Common Utility Functions

- Perfect complements

ALL CORNERS LINE ON STRAIGHT LINE



$\alpha x_1 = \beta x_2$ CUT LINE OF CORNERS IS LINE (??)

$x_2 = \frac{\alpha}{\beta} x_1$ SLOPE IS α / β

IMAGINE I WANT TO DRAW $A \cdot \min(\alpha x_1, \beta x_2)$

$A \alpha$

$2 \alpha A$

$\alpha = 1$

In the case the slope is not decreasing. Slope is infinite in a vertical line, in orizzontal line slope is 0. In the point of corners the slope is not define.

Another function more complex that is called the constant elasticity of substitution.

Common Utility Functions

– The slope of a ray $x_2 = \frac{\beta}{\alpha} x_1$, $\frac{\beta}{\alpha}$, indicates the rate at which goods x_1 and x_2 must be consumed in order to achieve utility gains.

– Special case: $\alpha = \beta$

$$\begin{aligned} u(x_1, x_2) &= A \cdot \min\{\alpha x_1, \alpha x_2\} \\ &= A\alpha \cdot \min\{x_1, x_2\} \\ &= B \cdot \min\{x_1, x_2\} \text{ if } B \equiv A\alpha \end{aligned}$$

– *Examples*: cars and gasoline, or peanut butter and jelly.

Common Utility Functions

- ***CES utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = \left[a x_1^{\frac{\sigma-1}{\sigma}} + b x_2^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

  where σ measures the elasticity of substitution between goods x_1 and x_2.

  - In particular,

$$\boxed{\sigma = \frac{\partial\left(\frac{x_2}{x_1}\right)}{\partial MRS_{1,2}} \cdot \frac{MRS_{1,2}}{\frac{x_2}{x_1}}}$$

This form of the utility function that is called CES. A combination of cobddouglas function with only one good. We get a constant elasticity substitution.
This elasticity is define d in this way.

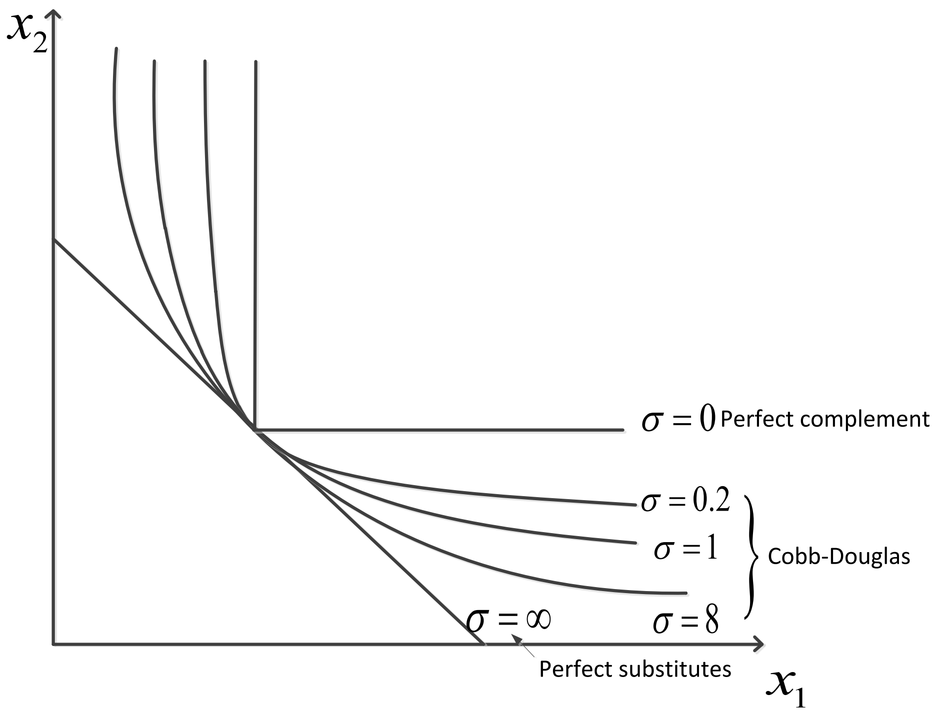
Elasticity is percentage change of one variable of the percentage change in the other
.

$$\frac{\dfrac{\partial\left(\frac{x_2}{x_1}\right)}{\frac{x_2}{x_1}}}{\dfrac{\partial MRS}{MRS}}$$

Graphical representation in the next slide.
Depending on value of sigma you can get all the utility functions that we have already introduce. If elasticity 1 we get cob Douglas, if 0 we obtain leonthief, if infinity you get a linear function.
Elasticity of substitution is infinity is that for me the two good are totally indifferent. So I don't care which of the two good consume.
On Ariel he will put the proof (not necessary).

Common Utility Functions

- CES preferences



Common Utility Functions

– CES utility function is often presented as

$$u(x_1, x_2) = \left[a x_1^{\rho} + b x_2^{\rho}\right]^{\frac{1}{\rho}}$$

where $\rho \equiv \frac{\sigma - 1}{\sigma}$.

Sometimes the CES is indicated in this way, using rho that is a function of sigma. Still remain constant.

Last utility function we consider is the quasi linear utilty function. This depend on the quantity consumed of two good. Quasilinear mean that one of the two good enter linearly in the utility function. X2 is linear.
Log function and cob double.

MRS of substitution (ratio of the two MU).

$$MU_{x_1} = \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1} \qquad \frac{\partial u}{\partial x_2} = b \qquad MRS_{x_1 x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

SO MRS DOES NOT DEPEND ON X_2!

IN CASE, LINEAR FUNCTION

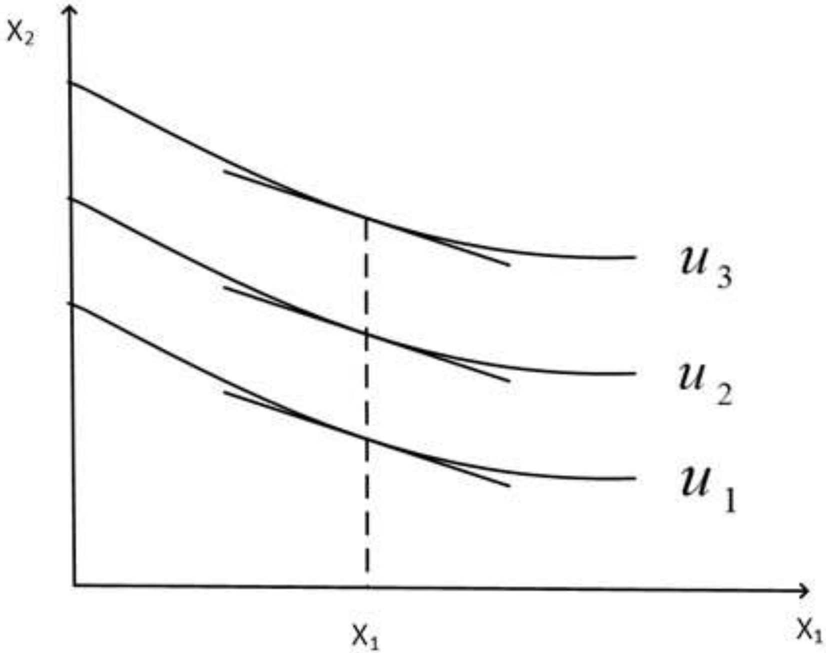
So this is the overview of all utility function we will consider.

Common Utility Functions

- ***Quasilinear utility function***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = v(x_1) + bx_2$$
  where x_2 enters *linearly*, $b > 0$, and $v(x_1)$ is a *nonlinear* function of x_1.
    - For example, $v(x_1) = a \ln x_1$ or $v(x_1) = ax_1^{\alpha}$, where $a > 0$ and $\alpha \neq 1$.
  - The MRS is constant in the good that enters linearly in the utility function (x_2 in our case).

Common Utility Functions

- MRS of quasilinear preferences



Common Utility Functions

– For $u(x_1, x_2) = v(x_1) + bx_2$, the marginal utilities are

$$\frac{\partial u}{\partial x_2} = b \text{ and } \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$$

which implies

$$MRS_{x_1,x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

which is constant in the good entering linearly, x_2

– Quasilinear preferences are often used to represent the consumption of goods that are relatively insensitive to income.

– *Examples*: garlic, toothpaste, etc.

Properties of Preference Relations

We go on with another section of the chapter and we will introduce other properties of preference relation.
Rational preference: completeness, transitivity.
Completeness: DM can define for any two bundle you are able to compare each goods in the bundle
Transitivity: 1° > 2° and 2°>3° then 1°> 3°

Now we define a bundle that is a combination of good in a given points.
We define other feature in the preference relation.

One is the definition of ***homogeneity***: utility function is homogeneous, if you take utility and multiply each argument by alpha then utility function is equal to utility multiply by a^k (with alpha > 0)
If 0<a<1 we are decreasing quantity of the original bundle.

If this happen we define the utility function as homogenous of degree k.

$$\frac{\alpha x_1}{x_1} = \alpha$$ RATIO BETWEEN GOODS

$$\frac{\alpha x_2}{x_2} = \alpha \qquad \frac{u(\alpha x_1, \alpha x_2)}{u(x_1, x_2)} = \frac{\alpha^k (u(x_1, x_2))}{u(x_1, x_2)} = \alpha^k$$

u CHANGES TO PROPORTION α TO POWER OF k

If a function if utilty of degree k, then the first derivative is homogeneous of degree k-1.
How to prove?

$$u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$$

TO PROVE!

$$\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1} = \alpha^{k-1} \frac{\partial u(x_1, x_2)}{\partial x_1}$$

HINT → DERIVE EACH SIDE

$$\frac{\partial u(a x_1, a x_2)}{\partial x_1} a = a^k \frac{\partial u(x_1, x_2)}{\partial x_1}$$

$$u'(ax_1, ax_2) = \frac{a^k}{a} u'(x_1, x_2) \qquad u'(ax_1, ax_2) = a^{k-1} u(x_1, x_2)$$

Properties of Preference Relations

- ***Homogeneity***:
  - A utility function is *homogeneous of degree k* if varying the amounts of all goods by a common factor $\alpha > 0$ produces an increase in the utility level by α^k.
  - That is, for the case of two goods,
  $$u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$$
  where $\alpha > 0$. This allows for:
    - $\alpha > 1$ in the case of a common increase
    - $0 < \alpha < 1$ in the case of a common decrease

Properties of Preference Relations

– Three properties:

*1) The first-order derivative of a function $u(x_1, x_2)$ which is **homogeneous of degree k** is homogeneous of degree $k - 1$.*

- Given $u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$, we can take derivatives of both sides with respect to x_i that is

$$\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial(\alpha x_i)} \cdot \alpha = \alpha^k \cdot \frac{\partial u(x_1, x_2)}{\partial x_i}$$

and re-arranging

$$u_i'(\alpha x_1, \alpha x_2) = \alpha^{k-1} u_i'(x_1, x_2)$$

Where u_i' denotes partial derivative w.r.t. *i* argument.

1. If function is homogeneous the IC has a specific shape. In particular is radial expansion of one another. If we increase with the same quantity the two bundle then they lie on the same indifference curve. Radial expaction because with increase the value by the same proportion of alpha.
2. If we compute the MRS along radial expansion the slope of first IC is equal to the slope of the second IC. So marginal rate of substitution is constant then IC are parallel curve.

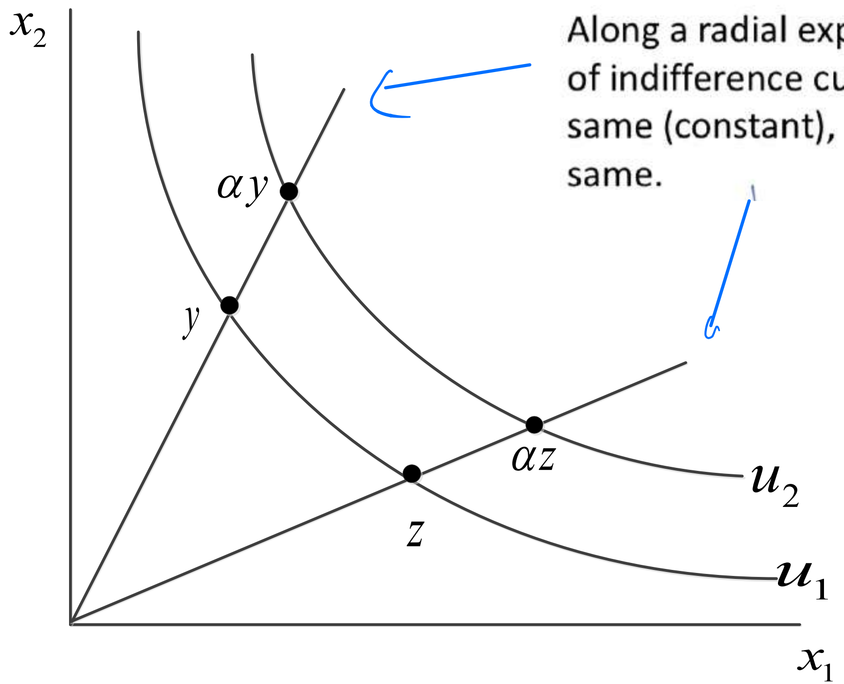
Properties of Preference Relations

2) The indifference curves of homogeneous functions are radial expansions of one another.

- That is, if two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$.

Properties of Preference Relations

- Homogenous preference



Along a radial expansion slope of indifference curve is the same (constant), i.e. MRS is the same.

Properties of Preference Relations

3) *The MRS of a homogeneous function is constant for all points along each ray from the origin.*

- That is, the slope of the indifference curve at point y coincides with the slope at a "scaled-up version" of point y, αy, where $\alpha > 1$. TRY PROVE IT.
- The MRS at bundle $x = (x_1, x_2)$ is

$$MRS_{1,2}(x_1, x_2) = -\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

- The MRS at $(\alpha x_1, \alpha x_2)$ is

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = -\frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

Since is homogeneous the derivative is equal to u time a^k-1 degree

$$= -\frac{\alpha^{k-1}\frac{\partial u(x_1, x_2)}{\partial x_1}}{\alpha^{k-1}\frac{\partial u(x_1, x_2)}{\partial x_2}} = -\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

where the second equality uses the first property.

- Hence, the MRS is unaffected along all the points crossed by a ray from the origin.

Along radial expansion we prove MRS is the same since has degree k

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
  - Proof: if $u(y) = u(z) \Longrightarrow g\big(v(y)\big) = g\big(v(z)\big)$ and being $g(.)$ monotonic then $v(y) \ = v(z)$ (two arguments cannot have the same value of the function. From homogeneity of degree k of v (.) we know that

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k \, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k \, v(z))$$

Hence, $\alpha^k \, v(y)$= $\alpha^k \, v(z)$ and $u(\alpha y)$= $u(\alpha z)$.

Increasing transformation of (similar to say monotonic transformation) this new utilty function is called homothetic.

Monotonic preserve the ordering of the arguments.

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
  - Proof: if $u(y) = u(z) \Longrightarrow g\big(v(y)\big) = g\big(v(z)\big)$ and being $g(.)$ monotonic then $v(y) \; = v(z)$ (two arguments cannot have the same value of the function. From homogeneity of degree k of v (.) we know that

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k \, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k \, v(z))$$

Hence, $\alpha^k \, v(y)$= $\alpha^k \, v(z)$ and $u(\alpha y)$= $u(\alpha z)$.

$u(\alpha x_1, \alpha x_2) \cdot g(v(\alpha x_1, \alpha x_2))$

$u(\alpha z_1, \alpha z_2) \cdot g(v(\alpha z_1, \alpha z_2))$

GARCIA

Properties of Preference Relations

– The MRS of a homothetic function is homogeneous of degree zero.

– In particular,

Slope of IC will be equal. So along expaction, MRS is equal.

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\cancel{\frac{\partial g}{\partial u}} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1} \cancel{a}}{\cancel{\frac{\partial g}{\partial u}} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2} \cancel{a}}$$

$\to \partial(\alpha x_1)$

$\to \partial(\alpha x_2)$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

– Canceling the $\frac{\partial g}{\partial u}$ terms yields (V is homogeneous of degree k)

Since v hom. of degree k

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\cancel{\alpha^{k-1}} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\cancel{\alpha^{k-1}} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

– **The MRS of a homothetic function is homogeneous of degree zero.**

Proof.

$$|MRS_{1,2}(\alpha x_1, \alpha x_2)| = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial v} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial(\alpha x_1)} \alpha}{\frac{\partial g}{\partial v} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial(\alpha x_2)} \alpha}$$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

– Canceling the $\frac{\partial g}{\partial v} \alpha$ terms yields (v is homogeneous of degree k)

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial(\alpha x_1)}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial(\alpha x_2)}} = \frac{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Vera

GARCIA

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \left| \quad \frac{\alpha^k \frac{\partial u(x_1,x_2)}{\partial x_1}}{\alpha^k \frac{\partial u(x_1,x_2)}{\partial x_2}} \quad \right| =$$

PROVE THIS!

$$= \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = MRS_{1,2}(x_1, x_2)$$

Graph 1. If we increase by 2 both arguments also the value of the function doblued. So this IC will correspond with twice the level of utilty.
In homothetic actually the level of utilty does not doble in some case. So all the thing i notice graphically are summarised in the slide (homogeneous function are homothetic.. but homothetic function are not necessary homogeneous).

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial(\alpha x_1)}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial(\alpha x_2)}} = \frac{\frac{\partial v(x_1, x_2)}{\partial x_1}}{\frac{\partial v(x_1, x_2)}{\partial x_2}}$$

– In summary,

$$|MRS_{1,2}(\alpha x_1, \alpha x_2)| = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\frac{\partial v(x_1, x_2)}{\partial x_1}}{\frac{\partial v(x_1, x_2)}{\partial x_2}}$$

$= \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} \equiv |MRS_{1,2}(x_1, x_2)| = \frac{\frac{\partial v(x_1, x_2)}{\partial x_1}}{\frac{\partial v(x_1, x_2)}{\partial x_2}}$ (do the proof in this line for exercise, proof in the following slide)

VERA

Properties of Preference Relations

- But we also have

$$|MRS_{1,2}(x_1, x_2)| = \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial u} \cdot \frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial g}{\partial u} \cdot \frac{\partial v(x_1,x_2)}{\partial x_2}} = \frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}.$$

Hence $|MRS_{1,2}(\alpha x_1, \alpha x_2)| = |MRS_{1,2}(x_1, x_2)|$.
i.e. MRS is the same along radial expansions.

Properties of Preference Relations

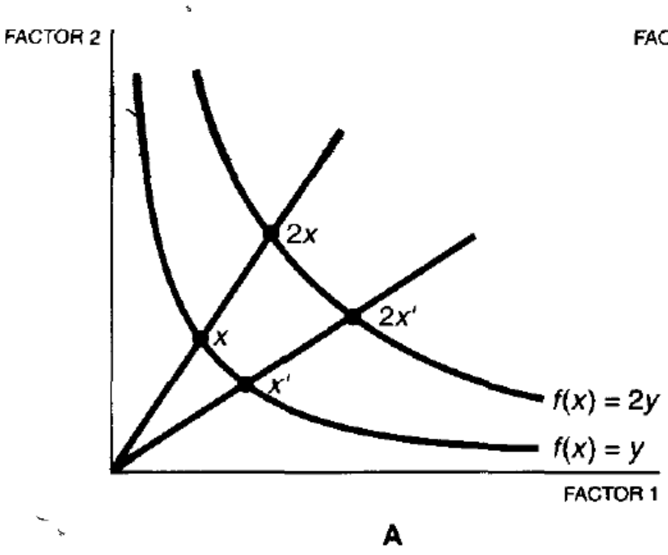
- ***Homotheticity (graphical interpretation)***
  - A preference relation on $X = \mathbb{R}^L_+$ is homothetic if all indifference sets are related to proportional expansions along the rays.
  - That is, if the consumer is indifferent between bundles x and y, i.e., $x \sim y$, he must also be indifferent between a common scaling in these two bundles, i.e., $\alpha x \sim \alpha y$, for every scalar $\alpha \geq 0$.

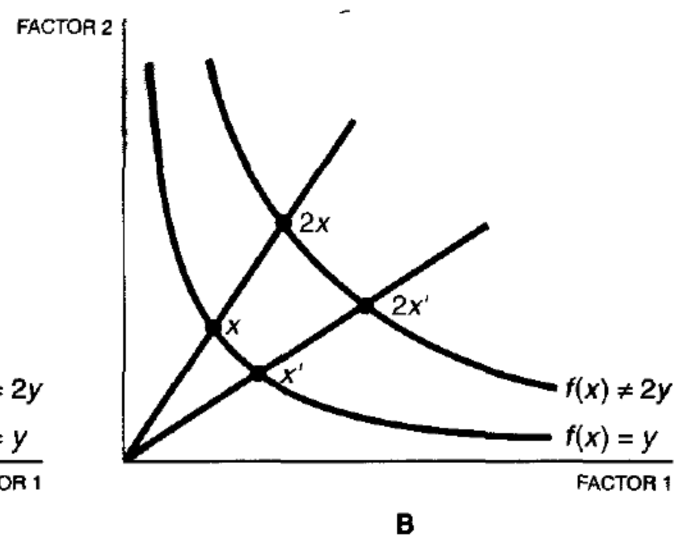
Properties of Preference Relations

- For a given ray from the origin, the slope of the indifference curves (i.e., the MRS) that the ray crosses coincides.
  - The ratio between the two goods x_1/x_2 remains constant along all points in the ray.

- Intuitively, the rate at which a consumer is willing to substitute one good for another (his MRS) only depends on:
  - the rate at which he consumes the two goods, i.e., x_1/x_2, but does not depend on the utility level he obtains.
- But it is independent in the volume of goods he consumes, and in the utility he achieves.

Properties of Preference Relations





Homogeneous of degree k=1

Homothethic

Properties of Preference Relations

- ***Homogeneity and homotheticity***:
  - Homogeneous functions are homothetic.
    - We only need to apply a monotonic transformation $g(\cdot)$ on $v(x_1, x_2)$, i.e., $u(x_1, x_2) = g(v(x_1, x_2))$.
  - But homothetic functions are not necessarily homogeneous.
    - Take a homogeneous (of degree one) function $v(x_1, x_2) = x_1 x_2$.
    - Apply a monotonic transformation $g(y) = y + a$, where $a > 0$, to obtain homothetic function
$$u(x_1, x_2) = x_1 x_2 + a$$

$u(x_1, x_2) = x_1 x_2$ ORIGINAL UTILITY FUNCTION

APPLY DEFINITION (INC BOTH QUANTITY BY α)

$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) = \alpha^2 (x_1 x_2) = \underline{\alpha^2 \cdot u(x_1, x_2)}$

$\alpha > 1$

THIS IS HOMOGENEOUS OF DEGREE 2

STRICTLY INCR TRANSFORMATION EXAMPLE

PRESERVE ORDERING

$u(x_1, x_2) = x_1 \cdot x_2 + \alpha$

SO THIS IS HOMOTHETIC → CHECK HOMOGENEOUS

$u(\alpha x_1, \alpha x_2) = \alpha^2 (x_1 \cdot x_2) + \alpha$ NOT HOMOGENEOUS

NOT WAY DEFINE VALUE ... [1 : 48]

$\neq \alpha^K u(x_1, x_2)$??

Homogeneous with strictly incr transformation we get homothetic function.
If we get hothetic function is not implied that we also get it homogeneous.

Properties of Preference Relations

- This function is not homogeneous, since increasing all arguments by α yields

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) + a$$
$$= \alpha^2 v(x_1, x_2) + a$$
$$\neq \alpha^k u(x_1, x_2)$$

- Other monotonic transformations yielding non-homogeneous utility functions are

$$g(y) = (ay^\gamma) + by, \quad \text{where } a, b, \gamma > 0, \quad \text{or}$$
$$g(y) = a(\ln y), \quad \text{where } a > 0$$

NOT HOMOGENEOUS → ORDINAL FUNCTION WORKS

Do as an exercise: prove that the two function are homogeneous.

V

2.

$$g(y) = a(y^3) + by$$

$$g(ay) = a^2 y^3 + aby \neq a^k u(x_1, x_2)$$

$$g(y) = a \ln y \qquad g(ay) = a^2 \ln y$$

$$\neq a^k u(y)$$

Properties of Preference Relations

$u(ax_1, ax_2) = a(ax_1) + b(ax_2) = [ax_1 + bx_2]a$

$u(x_1, x_2)$

So degree 1

- Utility functions that satisfy homotheticity:
  - Linear utility function $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$

    $u(tx_1, tx_2) = a(tx_1) + b(tx_2)$

    derivative resp. $x_1 \to at$ der. re $x_2 \to bt$

    - Goods x_1 and x_2 are perfect substitutes
    - $MRS(x_1, x_2) = \frac{a}{b}$ and $MRS(tx_1, tx_2) = \frac{at}{bt} = \frac{a}{b}$

    $\frac{at}{bt} = \frac{a}{b}$

    MRS is same of the original bundle and array, it's homothetic

  - The Leontief utility function $u(x_1, x_2) = A \cdot \min\{ax_1, bx_2\}$, where $A > 0$

    $ax_1 = \beta x_2$

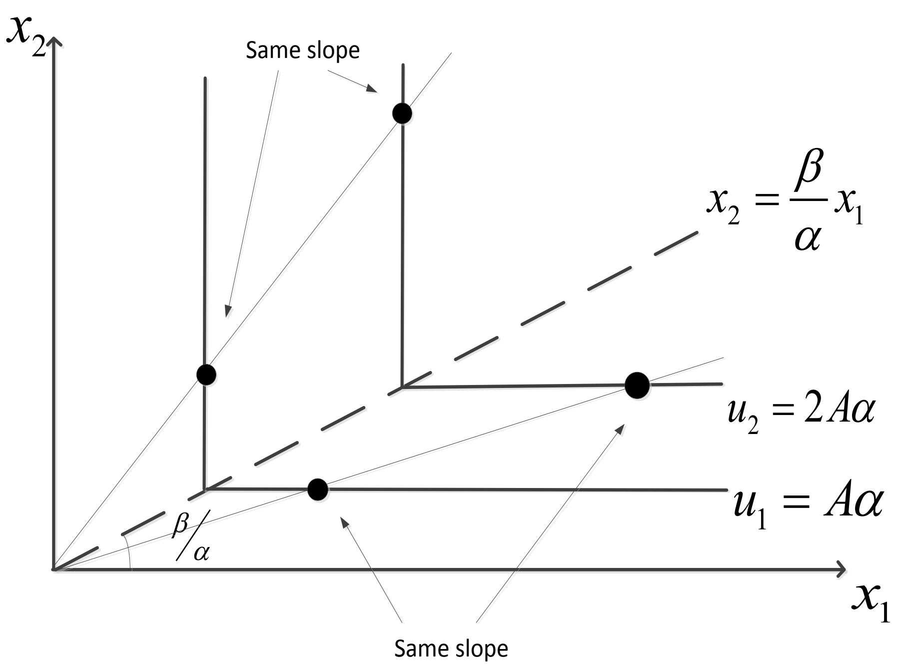
    $x_2 = \frac{a}{b} x_1$

    Manca roba qua!!

    - Goods x_1 and x_2 are perfect complements
    - We cannot define the MRS along all the points of the indifference curves
    - However, the slope of the indifference curves coincide for those points where these curves are crossed by a ray from the origin.

Properties of Preference Relations

- Perfect complements and homotheticity



Properties of Preference Relations

- ***Homotheticity***:
  - A utility function $u(x)$ is homothetic if it is a monotonic transformation of a homogeneous function.
  - That is, $u(x) = g(v(x))$, where
    - $g: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, and
    - $v: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k.

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
    - In particular,

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k\, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k\, v(z))$$

Properties of Preference Relations

– The MRS of a homothetic function is homogeneous of degree zero.

– In particular,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial u} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial g}{\partial u} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

– Canceling the $\frac{\partial g}{\partial u}$ terms yields

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} =$$

$$= \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = MRS_{1,2}(x_1, x_2)$$

Properties of Preference Relations

- ***Homotheticity (graphical interpretation)***
  - A preference relation on $X = \mathbb{R}^L_+$ is homothetic if all indifference sets are related to proportional expansions along the rays.
  - That is, if the consumer is indifferent between bundles x and y, i.e., $x \sim y$, he must also be indifferent between a common scaling in these two bundles, i.e., $\alpha x \sim \alpha y$, for every scalar $\alpha \geq 0$.

Properties of Preference Relations

- For a given ray from the origin, the slope of the indifference curves (i.e., the MRS) that the ray crosses coincides.
  - The ratio between the two goods x_1/x_2 remains constant along all points in the ray.
- Intuitively, the rate at which a consumer is willing to substitute one good for another (his MRS) only depends on:
  - the rate at which he consumes the two goods, i.e., x_1/x_2, but does not depend on the utility level he obtains.
- But it is independent in the volume of goods he consumes, and in the utility he achieves.

Properties of Preference Relations

- ***Homogeneity and homotheticity***:
  - Homogeneous functions are homothetic.
    - We only need to apply a monotonic transformation $g(\cdot)$ on $v(x_1, x_2)$, i.e., $u(x_1, x_2) = g(v(x_1, x_2))$.
  - But homothetic functions are not necessarily homogeneous.
    - Take a homogeneous (of degree one) function $v(x_1, x_2) = x_1 x_2$.
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$$u(x_1, x_2) = x_1 x_2 + a$$

Properties of Preference Relations

- This function is not homogeneous, since increasing all arguments by α yields

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) + a$$
$$= \alpha^2 v(x_1, x_2) + a$$

- Other monotonic transformations yielding non-homogeneous utility functions are

$$g(y) = ay^\gamma + by, \text{ where } a, b, \gamma > 0, \quad \text{or}$$
$$g(y) = a \ln y, \text{ where } a > 0$$

Properties of Preference Relations

- Utility functions that satisfy homotheticity:
  - Linear utility function $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$
    - Goods x_1 and x_2 are perfect substitutes
    - $MRS(x_1, x_2) = \frac{a}{b}$ and $MRS(tx_1, tx_2) = \frac{at}{bt} = \frac{a}{b}$
  - The Leontief utility function $u(x_1, x_2) = A \cdot \min\{ax_1, bx_2\}$, where $A > 0$
    - Goods x_1 and x_2 are perfect complements
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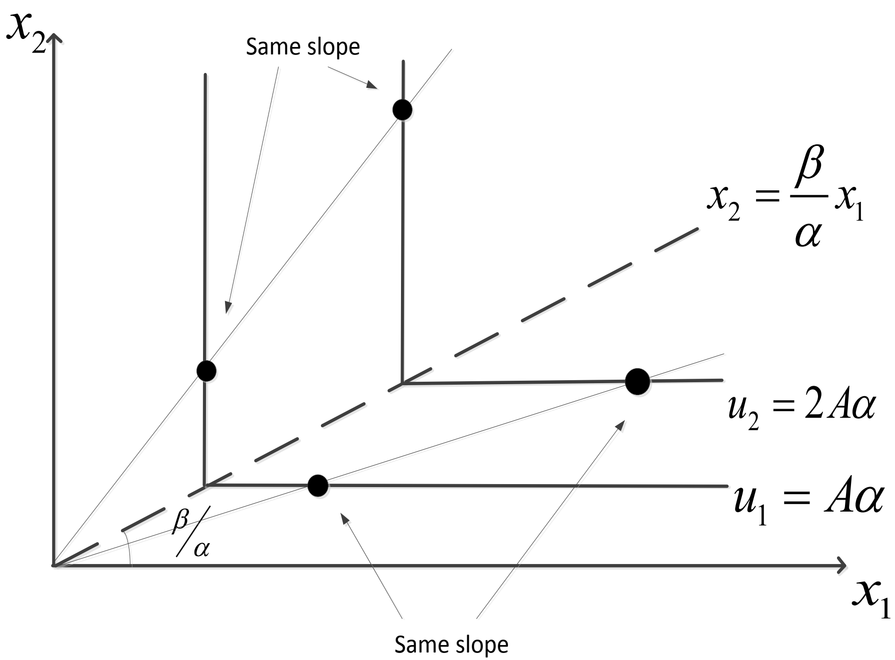
Properties of Preference Relations

- Perfect complements and homotheticity



Properties of Preference Relations

- ***Example 1.9*** (Testing for quasiconcavity and homotheticity):
  - Let us determine if $u(x_1, x_2) = \ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave, homothetic, both or neither.
  - *Quasiconcavity*:
    - Note that $\ln(x_1^{0.3} x_2^{0.6})$ is a monotonic transformation of the Cobb-Douglas function $x_1^{0.3} x_2^{0.6}$.
    - Since $x_1^{0.3} x_2^{0.6}$ is a Cobb-Douglas function, where $\alpha + \beta = 0.3 + 0.6 < 1$, it must be a concave function.
    - Hence, $x_1^{0.3} x_2^{0.6}$ is also quasiconcave, which implies $\ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave (as quasiconcavity is preserved through a monotonic transformation).

Properties of Preference Relations

- ***Example 1.9*** (continued):
  - *Homogeneity*:
    - Increasing all arguments by a common factor α,

$$(\alpha x_1)^{0.3}(\alpha x_2)^{0.6} = \alpha^{0.3}x_1^{0.3}\alpha^{0.6}x_2^{0.6} = \alpha^{0.9}x_1^{0.3}x_2^{0.6}$$

    - Hence, $x_1^{0.3}x_2^{0.6}$ is homogeneous of degree 0.9
  - *Homotheticity:*
    - Therefore, $x_1^{0.3}x_2^{0.6}$ is also homothetic.
    - As a consequence, its transformation, $\ln(x_1^{0.3}x_2^{0.6})$, is also homothetic (as homotheticity is preserved through a monotonic transformation).

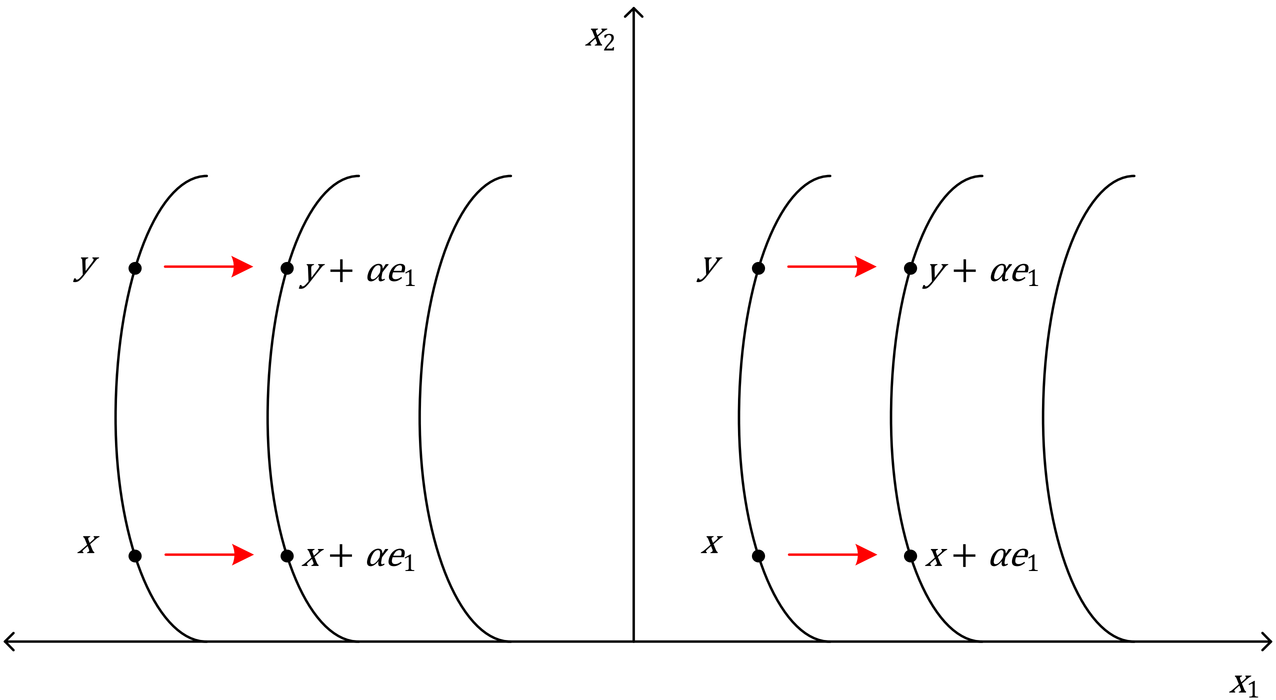
Properties of Preference Relations

- ***Quasilinear preference relations***:
  - The preference relation on $X = (-\infty, \infty)$ $x \in \mathbb{R}_{+}^{L-1}$ is *quasilinear* with respect to good 1 if:

    1) All indifference sets are parallel displacements of each other along the axis of good 1.
       - That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$, where $e_1 = (1,0, \dots, 0)$.
    2) Good 1 is desirable.
       - That is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Properties of Preference Relations

- Quasilinear preference-I

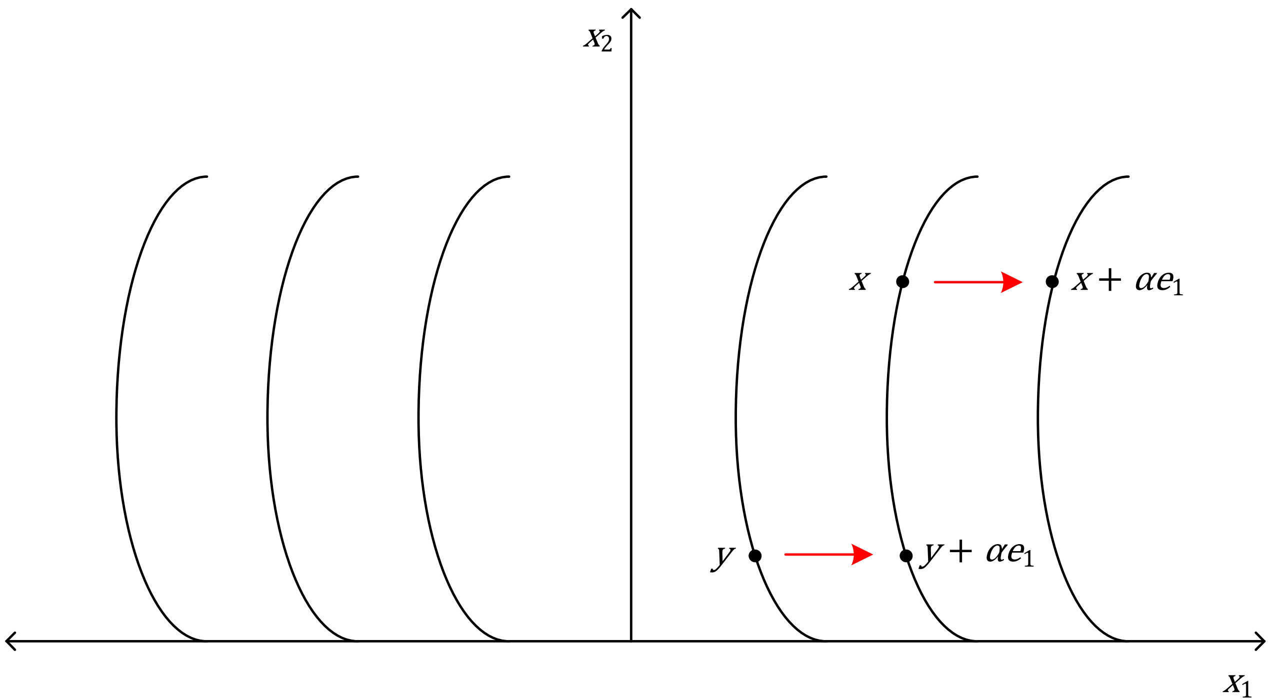


Properties of Preference Relations

- *Notes*:
  - No lower bound on the consumption of good 1, i.e., $x_1 \in (-\infty, \infty)$.
  - If $x \succ y$, then $(x + \alpha e_1) \succ (y + \alpha e_1)$.

Properties of Preference Relations

- Quasilinear preference-II



Properties of Preference Relations

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    - Hence, $x_1^{0.3} x_2^{0.6}$ is also quasiconcave, which implies $\ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave (as quasiconcavity is preserved through a monotonic transformation).

EXAMPLE 1.9

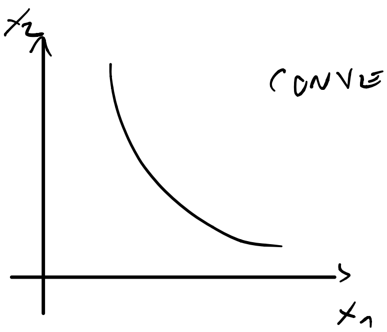
$$u(x_1, x_2) = \ln\left(x_1^{0.3}\, x_2^{0.6}\right)$$

IS QUASI CONCAVE, HOMOTETIC, BOTH, NEITHER

QUASI CONCAVITY

IND. CURVE ARE CONVEX $\Rightarrow$ UTILITY FUNCTION IS QUASI CONCAVE

$$\ln\left(x_1^{0.3}, x_2^{0.6}\right) = k$$



CONVEX?

1. ISOLATE VARIABLE

$$\left(x_1^{0.3}\, x_2^{0.6}\right) = \underbrace{\exp(k)}_{a}$$

$$x_2^{0.6} = \exp \frac{a}{x_1^{0.3}}$$

$$x_2 = \left(\frac{a}{x_1^{0.3}}\right)^{\frac{1}{0.6}} \; > 1$$

$$\frac{1}{x_1} = x_1^{-1}$$

$$a^{1/0.6} \cdot x_1^{-0.3 \cdot \frac{1}{0.6}} = a^{1/0.6} \cdot x_1^{-1/2}$$

PROVE IT'S CONVEX $\longrightarrow$ 2° DERIVATIVE!

$$\frac{\partial x_2}{\partial x_1} = a^{1/0.6}\left(-\frac{1}{2}\right) x_1^{-3/2}$$

$$\frac{\partial^2 x_2}{\partial^2 x_1} = \left(-\frac{1}{2}\, a^{1/0.6}\right) -\frac{3}{2} \cdot x_1^{-\frac{5}{2}}$$

SIGN POSITIVE TO BE CONVEX

$$\left(-\frac{1}{2}\, a^{1/0.6}\right) -\frac{3}{2} \cdot x_1^{-\frac{5}{2}} > 0$$ SO IT'S CONVEX

CONVEX $\Rightarrow$ QUASI CONCAVE $u(x_1, x_2)$

HOMOTHETICITY

MONOTONIC

STRICTLY INCREASING TRANSFORMATION OF AN HOMOGENEOUS FUNCTION (OF DEGREE K)

$\ln(x_1^{0.3}, x_2^{0.6})$ COB-DOUGLAS WITH LOG TRANSFORMATION

LINEAR HOMOGENEITY

COB-DOUGLAS $\rightarrow x_1^{a} x_2^{b}$

$$u(a x_1, a x_2) = a^k u(x_1, x_2)$$

IF WE CAN WRITE THIS, THEN IT'S HOMOGENEOUS OF DEGREE K

$a > 1$

t BECAUSE α AS EXPONENT

$$u(t x_1, t x_2) = t^k u(x_1, x_2)$$

$t > 1$

$$u(t x_1, t x_2) = (t x_1)^{\alpha} (t x_2)^{\beta} = t^{\alpha+\beta}(x_1^{\alpha} x_2^{\beta})$$

$$= t^{\alpha+\beta} u(x_1, x_2)$$

LN IS INCREASING

u IS HOMOTHETIC BECAUSE IS A LOG TRANSFORMATION OF A COB-DOUGLAS WHICH IS HOMOGENEOUS

OF DEGREE 0.5

CONVEXITY OF PREF $\neq$ CONVEXITY UTILITY

1. PREF CONVEX IF YOU TAKE UCS CONVEX

2. LINEAR QUASI CONCAVE

Social preferences

u(x1, x2) is utility function of an individual. Is not indexed by individual i.

Social and Reference-Dependent Preferences

Social Preferences

- We now examine social, as opposed to individual, preferences.
- Consider additively separable utility functions of the form
$$u_i(x_i, x) = f(x_i) + g_i(x)$$
where
  - $f(x_i)$ captures individual i's utility from the monetary amount that he receives, x_i;
  - $g_i(x)$ measures the utility/disutility he derives from the distribution of payoffs $x = (x_1, x_2, \dots, x_N)$ among all N individuals.

This is a case in which is indexed by individual. Utility of individual is define by his consumption Xi but also the consumption of all other people. So f(xi) is the egoistic part, and gi(x) is the consumption of all other people. Gi mean that can be some sort of altruism.

In this example we don't take average consumption. In x we have all bundle of consumption of all individual (kindy absurd to have all consumption so we have average). X is a vector of consumption of all the other individual. Xi could be a vector and also x2, x3 ... could be a vector.
Usually we will take much simpler utility function.

Social Preferences

- **Fehr and Schmidt (1999)**:
  - For the case of two players,

$$u_i(x_i, x_j) = x_i - \alpha_i \max\{\overset{\text{ENVY}}{x_j - x_i, 0}\} - \beta_i \max\{\overset{\text{ALTRUISM}}{x_i - x_j, 0}\}$$

    where x_i is player i's payoff and $j \neq i$.

  - Parameter α_i represents player i's disutility from envy
    - When $x_i < x_j$, $\max\{x_j - x_i, 0\} = x_j - x_i > 0$ but $\max\{x_i - x_j, 0\} = 0$.
    - Hence, $u_i(x_i, x_j) = x_i - \alpha_i(x_j - x_i)$.

Fehr and Schmidt we assume we have only two individuals, so we have only two consumption of the two individual. We also have consumption of j.
Xi is your consumption and the from this level of utility we subtract something: ai max(xk-xi, 0) if i consume less than xj i get a max of 0. Else if you consuming more than the other guy you take in the utility function Bi (xi-xj). So which between the two are altruistic consort. If you consume more Than the other guys you are not happy. a is for envy.

In this model we assume that player envy is stronger than their guilt. So alpha >= bi. You don't want to be the poor one.

Social Preferences

– Parameter $\beta_i \geq 0$ captures player i's disutility from guilt
  - When $x_i > x_j$, $\max\{x_i - x_j, 0\} = x_i - x_j > 0$ but $\max\{x_j - x_i, 0\} = 0$.
  - Hence, $u_i(x_i, x_j) = x_i - \beta_i(x_i - x_j)$.

– Players' envy is stronger than their guilt, i.e., $\alpha_i \geq \beta_i$ for $0 \leq \beta_i < 1$.
  - Intuitively, players (weakly) suffer more from inequality directed at them than inequality directed at others.

Social Preferences

– Thus players exhibit "concerns for fairness" (or "social preferences") in the distribution of payoffs.

– If $\alpha_i = \beta_i = 0$ for every player i, individuals only care about their material payoff $u_i(x_i, x_j) = x_i$.
  - Preferences coincide with the individual preferences.

Social Preferences

– Let's depict the indifference curves of this utility function.
– Fix the utility level at $u = \overline{u}$. Solving for x_j yields

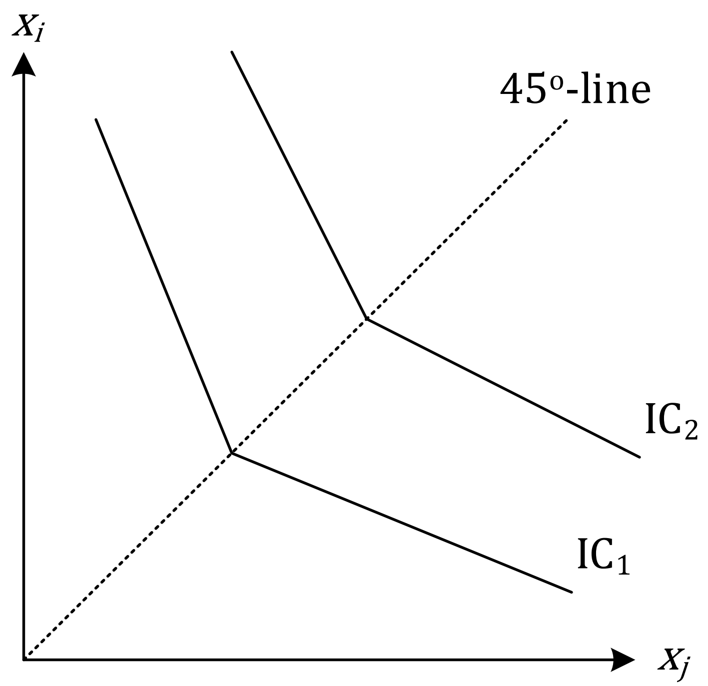
$$x_j = \frac{\overline{u}}{\beta} - \frac{1-\beta}{\beta} x_i \text{ if } x_i > x_j$$

$$x_j = \frac{\overline{u}}{\alpha} - \frac{1-\alpha}{\alpha} x_i \text{if } x_i < x_j$$

– Hence each indifference curve has two segments:
  - one with slope $\frac{1-\beta}{\beta}$ above the 45-degree line
  - another with slope $\frac{1-\alpha}{\alpha}$ below 45-degree line
– Note that (x_i, x_j)-pairs to the northeast yield larger utility levels for individual i.

Social Preferences

- Fehr and Schmidt's (1999) preferences



Advanced Microeconomic Theory

Chapter 2: Demand Theory

Consumption Sets

Consumption Sets

- ***Consumption set***: a subset of the commodity space $\mathbb{R}^L$, denoted by $x \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume, given the physical constraints imposed by his environment.

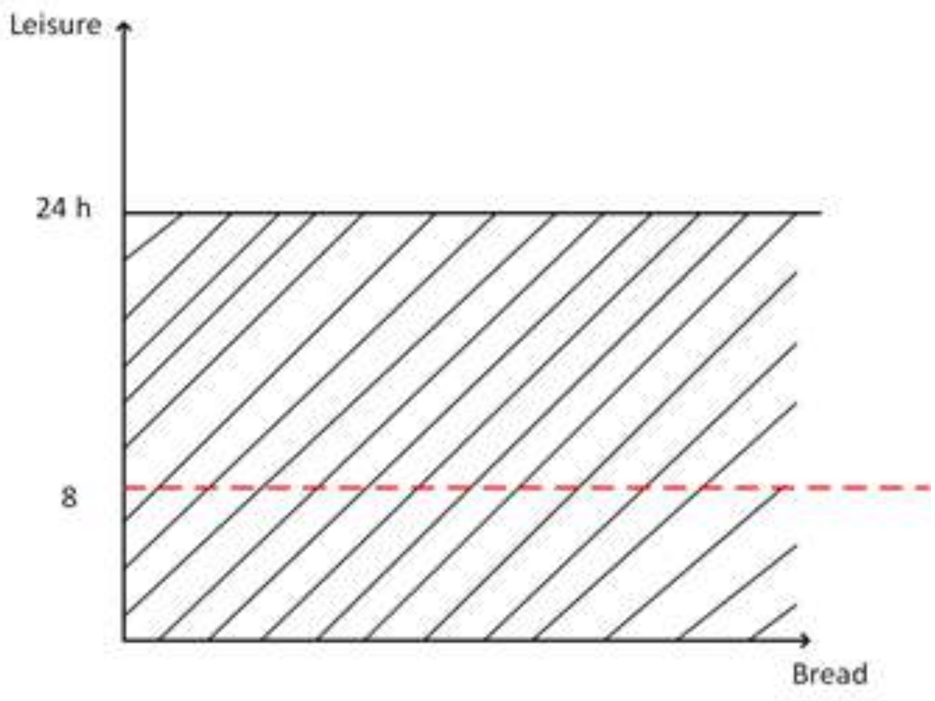
- Let us denote a commodity bundle x as a vector of L components.

Consumption Set
Set of all possible alternatives (which are bundles)

sometime some bundle are not feasible, so we cannot consume it because there are constrained imposed by his environment.
A bundle is a vector of L components.

Consumption Sets

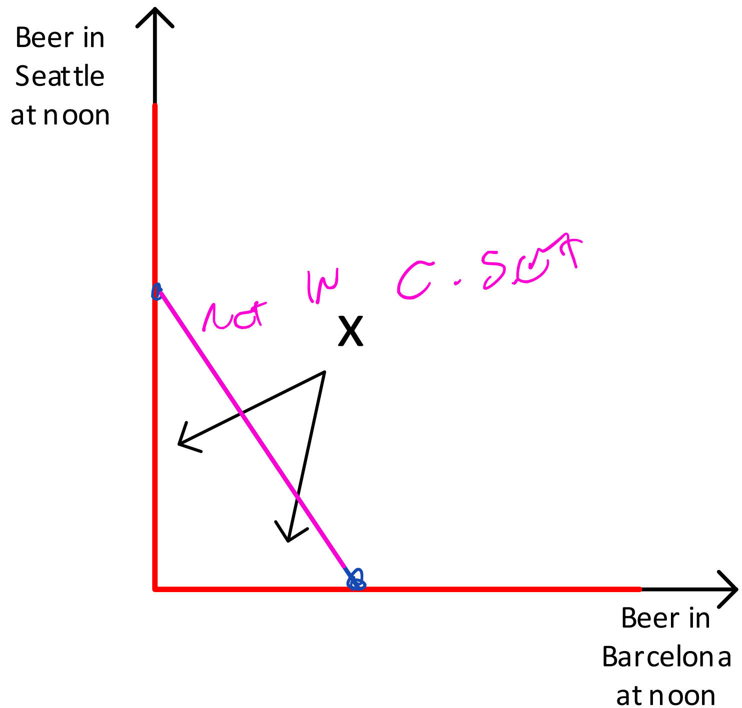
- Physical constraint on the labor market



How people decide labour supply (so how many hours they work). Leisure can also be called as house work. If we consider leisure as a good and bread. People don't want to work all day but you want to have some leisure. You have to sleep, what are the main free activity. Studying working and having fun. Even if you don't sleep any hour you do not consume any bread, the maximum amount of leisure is 24h. It's physical constraint on the environment.
If you have pleasure you don't work, if you work you have more income and more pleasure.

Consumption Sets

- Consumption at two different locations



Imagine this two goods are beer in Seattle and Barcellona at the same day. So there's a physical constraint. The consumption set is in the axes. Since Barcellona is 0 if I'm in Seattle and vice versa. Not convex, if i take point in a straight line they are not in the consumption set.

Consumption Sets

- ***Convex consumption sets***:
  - A consumption set X is convex if, for two consumption bundles $x, x' \in X$, the bundle
    $$x'' = \alpha x + (1 - \alpha)x'$$
    is also an element of X for any $\alpha \in (0,1)$.
  - Intuitively, a consumption set is convex if, for any two bundles that belong to the set, we can construct a straight line connecting them that lies completely within the set.

Consumption Sets: Economic Constraints

- Assumptions on the price vector in $\mathbb{R}^L$:
  1) All commodities can be traded in a market, at prices that are publicly observable.
     - This is the principle of completeness of markets
     - It discards the possibility that some goods cannot be traded, such as pollution.
  2) Prices are strictly positive for all L goods, i.e., $p \gg 0$ for every good k.
     - Some prices could be negative, such as pollution.

Economic constraint —> we do some additional assumption that characterise perfect competition.
All commodities can be traded in a marker and all good has a price in the market. This is called a market completeness.
For instance, we do not consider pollution because cannot be traded. Even though expert create market with pollution.
[Let’s say 100 firm, each one 100 and then sell certificate and trade the right to pollute. The reason to create a market is that if you have a cost to pollute. You sell the right to pollute.]

Also price is positive. If something is free i can ask for infinite amount of the good??

Consumption Sets: Economic Constraints

3) Price taking assumption: a consumer's demand for all L goods represents a small fraction of the total demand for the good.

Consumer cannot affect the price.
In some situation consumer can affect the price. Big enterprise in the retail distribution and you supply all shop and then you go to people working on agriculture if price is this, then i get it else i will go to another one.

Consumption Sets: Economic Constraints

FEASIBLE

- Bundle $x \in \mathbb{R}_+^L$ is affordable if
$$p_1 x_1 + p_2 x_2 + \cdots + p_L x_L \leq w$$
or, in vector notation, $p \cdot x \leq w$.

- Note that $p \cdot x$ is the total cost of buying bundle $x = (x_1, x_2, \ldots, x_L)$ at market prices $p = (p_1, p_2, \ldots, p_L)$, and w is the total wealth of the consumer.

- When $x \in \mathbb{R}_+^L$ then the set of feasible consumption bundles consists of the elements of the set:
$$B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$$

Consumer have some wealth and cannot spend more on this wealth. So consumer cannot borrow money to his consumption (???) [56.00]
Amount of goods that are consumed and income is endogenous (variables explained in the model).
Endogenous decision are about x1, x2 so the amount of consumed.
The budget inequality is saying that you expenditure must be less or equal than your income.
So this define the so called budget set.
B is a set qand then the budget set depend on price and wealth in which components are positive for which the product of price vector moltiply by good vector is less or equal of w (amount of wealth that you have, it's a scalar! Not a vector like p and x).

How is budget set represented? In the following way.

Consumption Sets: Economic Constraints

- ***Example for two goods***:

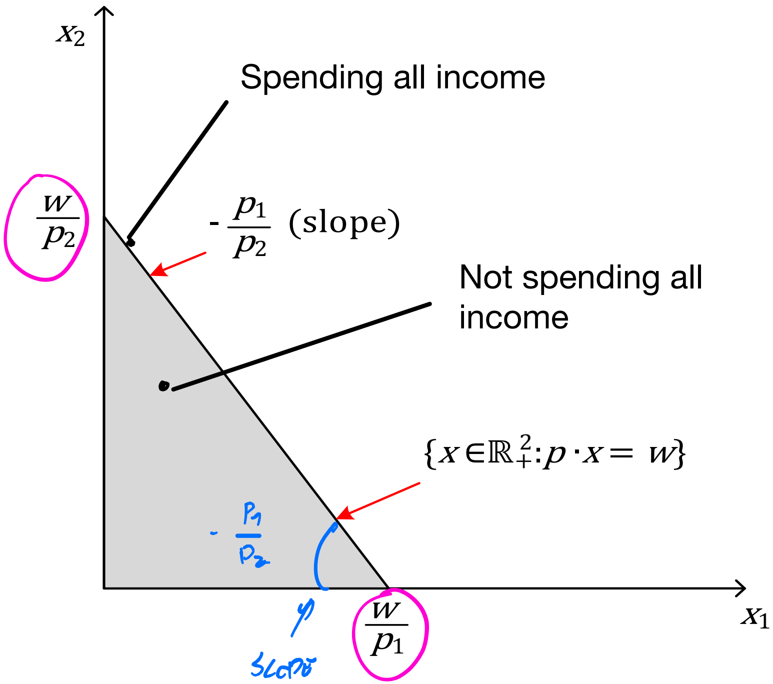
$$B_{p,w} = \{x \in \mathbb{R}^2_+ : p_1x_1 + p_2x_2 \leq w\}$$

The budget line is

$p_1x_1 + p_2x_2 = w$

Hence, solving for the good on the vertical axis, x_2, we obtain

$$x_2 = \frac{w}{p_2} - \frac{p_1}{p_2}x_1$$



Two components. How do you represent graphically a budget set?
You see you have inequality and you can take this inequality as equality and define the graph of the function of p1x1 +p2x2 = w.

$$p_1 x_1 + p_2 x_2 = w \rightarrow x_2 = \frac{w}{p_2} - \frac{p_1}{p_2} x_1 \Big\} \text{ BUDGET LINE}$$

IF $x_1 = 0$ YOU CAN CONSUME x_2 MAX AMOUNT OF $\frac{w}{p_2}$

ALSO $x_2 = 0$ x_1 MAX AMOUNT IS $\frac{w}{p_1}$

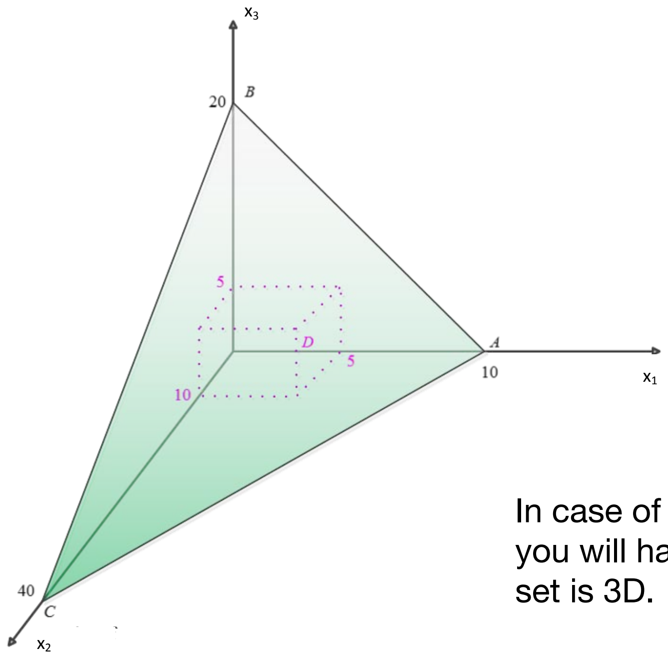
Set of all feasible bundle depending on your income and the price.

Consumption Sets: Economic Constraints

- ***Example for three goods***:

$$B_{p,w} = \{x \in \mathbb{R}^3_+ : p_1x_1 + p_2x_2 + p_3x_3 \le w\}$$

  – The surface $p_1x_1 + p_2x_2 + p_3x_3 = w$ is referred to as the "Budget hyperplane"



In case of three goods you will have the budget set is 3D.

Consumption Sets: Economic Constraints

- *Price vector p is orthogonal (perpendicular) to the budget line $B_{p,w}$.*
  - Note that $p \cdot x = w$ holds for any bundle x on the budget line.
  - Take any other bundle x' which also lies on $B_{p,w}$. Hence, $p \cdot x' = w$.
  - Then,

$$p \cdot x' = p \cdot x = w$$

$$p \cdot (x' - x) = 0 \text{ or } p \cdot \Delta x = 0$$

Consumption Sets: Economic Constraints

– Since this is valid for any two bundles on the budget line, then p must be perpendicular to Δx on $B_{p,w}$.

– This implies that the price vector is perpendicular (orthogonal) to $B_{p,w}$.

Price vector is orthogonal to the budget line Bp,w.

$(p_1, p_2)(x_1, x_2) = p_1 x_1 + p_2 x_2$

Consumption Sets: Economic Constraints

- *The budget set $B_{p,w}$ is convex*.
  - We need that, for any two bundles $x, x' \in B_{p,w}$, their convex combination
$$x'' = \alpha x + (1 - \alpha)x'$$
also belongs to the $B_{p,w}$, where $\alpha \in (0,1)$.

CHECK

  - Since $p \cdot x \leq w$ and $p \cdot x' \leq w$, then
$$p \cdot x'' = p\alpha x + p(1 - \alpha)x'$$
$$= \alpha px + (1 - \alpha)px' \leq w$$

why is $\leq w$?

$$a\,[\,px + (1-a)\,px'\,] \le w$$

<1 $\qquad$ $<w$

EXERCISES 1.7

THIS IS ABOUT PREFERENCES

$$u(x_1, x_2) = a\, x_1^{1/2}\, b\, x_2^{1/2}$$

PREFERENCES CONVEX? UCS IS CONVEX?

1. DRAW FUNCTION OF I.C. → TAKE FUNCTION AND PUT IT = TO A K VALUE

$$a x_1^{1/2}\, b\, x_2^{1/2} = k \qquad x_2 = \left(\frac{k}{ab\, x_1^{1/2}}\right)^2$$

$$x_2 = \underbrace{\left(\frac{k}{ab}\right)^2}_{\text{FUNCTION OF IC}} x_1^{-1} \quad \boxed{= A}$$

TO CHECK IF IT IS CONVEX?

$$\frac{\partial x_2}{\partial x_1} = A \cdot -x_1^{-2}$$

$$\frac{\partial^2 x_2}{\partial x_1\, \partial x_1} = A\,(2 x_1^{-3}) > 0 \quad \text{IF } x_1 > 0$$

IC. CONVEX ⇒ UTILITY IS QUASICONCAVE

PREF CURVE ARE CONVEX

$a x_1^2 + b x_2^2$

$$u(x_1, x_2) = a x_1^2 + a x_2^2$$

CONVEX PREFERENCES ?

$$a x_1^2 + b x_2^2 = K \qquad x_2 = \left[\frac{(K - a x_1^2)}{b}\right]^{1/2} = A^{1/2}$$

$$\frac{\partial x_2}{\partial x_1} = \frac{1}{2} A^{-1/2} \cdot -\left(\frac{a}{b}\right) \cdot 2x_1 = -\frac{a}{b} A^{-1/2} \cdot x_1$$

DUE OF PRODUCT!

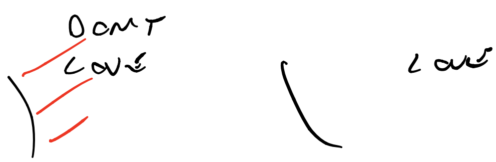
$$\frac{\partial^2 x_2}{\partial x_1 \partial x_2} = -\frac{a}{b} \cdot \left[-\frac{1}{2} A^{-3/2} \left(-\frac{a}{b}\right) 2 x_1 \cdot x_1 + A^{-1/2}\right]$$

$$< 0$$

IC. CONCAVE! $\Rightarrow$ PRIEF ARE NOT CONVEX

$$d(xy) = dx \cdot y - dy \cdot x$$

$\Rightarrow$ UTILITY IS NOT QUASICONCAVE



CHECKING PROPERTIES OF PREFERENCES RELATIONS

a) $(x_1, x_2) \succeq (y_1, y_2) \iff x_1 \geq y_1 - 1$

$\boxed{\text{COMPLETENESS}}$ → $\forall x, y \in X$ either $x \succsim y$

or $y \succsim x$

or both $x \succsim y \wedge y \succsim x \Rightarrow x \sim y$

$x_1, x_2, y_1, y_2 \in \mathbb{R}$

COMPLETENESS IS RESPECTED

$\rightarrow x_1 \geq y_1 - 1 \Rightarrow x \succsim y$

$\rightarrow x_1 < y_1 - 1 \Rightarrow y_1 \geq x_1 - 1 \Rightarrow y \succsim x$

$\rightarrow x_1 = y_1 - 1 \Rightarrow x \succsim y \wedge y \succsim x \Rightarrow x \sim y$

$\boxed{\text{TRANSITIVITY}}$ $\forall x, y \in X;\ x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$

$x_1 \geq y_1 - 1$

SUBSTITUTE OUR CONDITION IN THIS PROPERTIES

$x_1 \geq y_1 - 1$ $\qquad y_1 \geq z_1 - 1$ $\qquad x_1 \geq z_1 - 1$

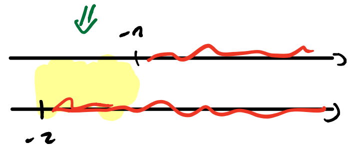
$\downarrow$ $\qquad \downarrow$ $\qquad \downarrow$

$x_1 - y_1 \geq -1$ $\qquad y_1 - z_1 \geq -1$ $\qquad |x_1 - z_1 \geq -1 \ ?|$

SUMMING THIS TWO

$|x_1 - \cancel{y_1} + \cancel{y_1} - z_1 \geq -2|$ THIS ARE DIFFERENT!

DOES
IN THIS REGION, TRANSITIVITY NOT HOLD



$(-2, -1)$ → THIS PROPERTY DOES NOT HOLD

MONOTONICITY

INCREASE OF ONE GOOD BY α AMOUNT DOES NOT HURT OUR PREFERENCE

$x \succeq y \iff x_1 \geq y_1 - 1 \longrightarrow x_1 + \alpha \geq x_1 - 1$ TRUE FOR $\alpha \geq -1$

GET VALUE OF $x - 1$ TO GET VALUE OF y

BY ASSUMPTION α HAS TO BE GREATER THAN $\alpha > 0$

CONVEXITY

IF $x \succeq y \Rightarrow \alpha x + (1-\alpha) y \succeq y$

WE HAVE TO SUBSTITUTE x AND y BY INITIAL ASSUMPTION

$(x_1 \geq y_1 - 1)$

$\alpha x_1 + (1-\alpha) y_1 \geq y_1 - 1$

I WANT THIS TO BE TRUE

I USE α TO GET A CONVEX COMBINATION (SO $0 < \alpha < 1$)

$\alpha(x_1 - y_1) + \cancel{y_1} \geq \cancel{y_1} - 1 \Rightarrow x_1 - y_1 \geq -\frac{1}{\alpha}$

SINCE $\alpha \in (0,1) \Rightarrow -\frac{1}{\alpha} < -1$

$x_1 - y_1 \geq -1$

THEREFORE $(x_1 - y_1) + 1 \geq 0$ SO IT'S A SUFFICIENT CONDITION FOR

$\alpha(x_1 - y_1) + 1 \geq 0$

$\Downarrow$

THIS PREF. RELATION IS CONVEX

ALWAYS TRUE THANKS TO ASSUMPTION

$x_1 - y_1 \geq -\frac{1}{\alpha} \Longleftarrow x_1 - y_1 \geq -1$

$\boxed{\text{CH 1 - EX 5}}$

$u(x)$ WHERE $x \in \mathbb{R}^n_+$ WITH N COMPONENTS

$v(x) = f(u(x))$ WHERE $f(\cdot)$ IS STRICTLY INCREASING AND CONCAVE

$v(x)$ MUST TO BE CONVEX

PROPERTY OF CONVEXITY

CONVEX PREFERENCES (DEF.)

$$x_1 \succsim y \wedge x_2 \succsim y \Rightarrow \bar{x} = \alpha x_1 + (1-\alpha) x_2 \succsim y$$

I CAN USE UTILITY FUNCTION

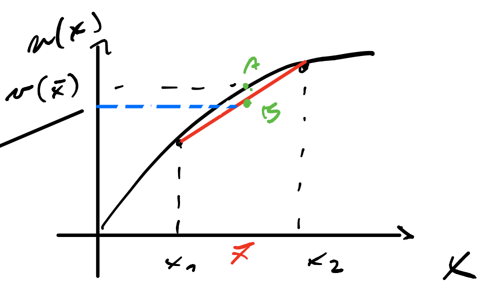
$$u(x_1) \geq u(y) \wedge u(x_2) \geq u(y) \Rightarrow u(\bar{x}) \geq u(y)$$

PROOF

IF $v(\cdot)$ IS CONCAVE (FROM INITIAL ASSUMPTION) $\Rightarrow$

$$v(\bar{x}) \geq \alpha v(x_1) + (1-\alpha) v(x_2)$$

UTILITY OF $\bar{x}$



$\alpha v(x_1) + (1-\alpha) v(x_2)$

USING $v(x) = f(u(x))$ WE CAN WRITE

$$f(u(\bar{x})) \geq \underline{\alpha f(u(x_1))} + (1-\alpha) \underline{f(u(x_2))}$$

$$\alpha f(u(x_1)) \geq f(u(y)) \qquad \alpha f(u(x_2)) \geq f(u(y))$$

$$f(u(\bar{x})) \geq \cancel{\alpha f(u(y))} + (1-\cancel{\alpha})\, f(u(y))$$

$$f(u(\bar{x})) \geq f(u(y))$$

$$f \quad u(x) \geq u(y) \Rightarrow \bar{x} \geq y \qquad \text{CONVEXITY!}$$

$\boxed{\text{EX 8 - MONOTONIC TRANSFORMATION}}$

WE WANT TO SEE IF THIS TRANSFORMATION PRESERVE THE SLOPE OF THE FUNCTION

$u(x) \geq 0 \quad \forall x \in \mathbb{R}^2_+$

(a) $f(x) = a\,u(x) + b\,[u(x)]^2$ WHERE $a, b > 0$

$u(x) = K$ TO MAKE DERIVATIVE EASIER

$$f(K) = aK + bK^2 \qquad \frac{\delta f(K)}{\delta K} = a + 2bK$$

UTILITY ≥ 0 BY ASSUMPTION

SO INCREASING TRANSFORMATION

THEREFORE $f(x)$ REPRESENT THE PREFERENCES OF THE ORIGINAL TRANSFORMATION $u(x)$

b) $f(x) = a \cdot u(x) - b\,[u(x)]^2$ WITH $a, b \geq 0$

$$f(z) = a z - b z^2$$

$$\underline{\delta f(z)} = a - 2bz \geq 0 \rightarrow z \leq \frac{a}{2b} \Rightarrow u(x) \leq \frac{a}{2b}$$

∂z

$f(x)$ NOT! MONOTONIC TRANSFORMATION OF $u(x)$

NOT REPRESENT SAME PREFERENCES AS ORIGINAL FUNCTION UTILITY $u(x)$

c) $f(x) = u(x) + \sum_{i=1}^{2} x_i$ $\quad u(x) \geq u(y) \Rightarrow f(x) < f(y)$

SO NOT MONOTONIC TRANSF.

ASSUME $x \geq y \iff x_1 \geq y_1$ $\quad (x_1, x_2) \geq (y_1, y_2)$

$(1, 2) \geq (0, 5)$

$u(x_1) \geq u(y_1) \Rightarrow u(1) > u(0)$

IF YOU TRY TO OBTAIN UTILITY OF TRANSFORMATION

$f(1,2) = \underbrace{1}_{u(x)=x} + 1 + 2 = 4$ $\quad f(0,5) = 0 + 0 + 5 = 5$

$f(1,2) < f(0,5)$ $\Rightarrow$ THIS TRANSFORMATION IS NOT MONOTONIC SINCE DOES NOT REPRESENT SAME PREFERENCE AS $u(x)$

d) $f(x) = [u(x)]^2 + b\, u(x) + c$ WITH $b, c > 0$

$u(x) = z$

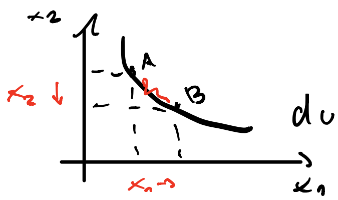
$$\frac{\partial f(z)}{\partial z} = 2z + b \Rightarrow$$ ALWAYS > 0 IF $z > 0 \rightarrow u(x) \geq 0 \in \mathbb{R}^+$

EX 10 - ADDITIVELY SEPARABLE UTILITY

(b) $u(x_1, x_2) : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}$ WHERE $u(x_1, x_2) = u_1(x_1) + u_2(x_2)$

$u(x_1)$ AND $u(x_2)$ STRICTLY INCREASING, STRICTLY CONCAVE DIFFERENTIABLE

SHOW THAT IC ARE CONVEX WHERE MU_i FOR $i = 1, 2$ IS DECREASING



$du = c$

A, B GENERATE SAME UTILITY → A AND B ALONG AN IC → BY DEFINITION

PROOF

$$du = \underbrace{\frac{\partial u_1(x_1)}{\partial x_1}}_{u_1'(x_1)} dx_1 + \underbrace{\frac{\partial u_2(x_2)}{\partial x_2}}_{u_2'(x_2)} dx_2 = 0$$

$$u_1'(x_1)\, dx_1 + u_2'(x_2)\, dx_2 = 0$$

$$-\frac{dx_2}{dx_1} = \frac{u_1'(x_1)}{u_2'(x_2)} \equiv |MRS|$$ → THIS IS EXPRESSION IS MRS

∂ MRS AND FIND DECR THEN WE CAN SAY WE HAVE CONVEX CURVE

$$\frac{\partial |MRS|}{\partial x_1} = \frac{\overbrace{u_1''(x_1)}^{\text{NEGATIVE}}}{\underbrace{u_2'(x_2)}_{\text{POSITIVE}}} < 0$$

SO IT'S NEGATIVE AND MRS IS DECREASING

$\Downarrow$

WE HAVE A CONVEX CURVE

ALSO YOU CAN USE HESSIAN MATRIX

EX 14 – CHECKING PREFERENCE OF COB - DOUGLAS FUNCTION

$$u(x) = \prod_{i=1}^{n} x_i^{\alpha i} \qquad x \in \mathbb{R}^n_+ \quad \text{AND} \quad 1 > \alpha i > 0$$

ADDITIVITY, HOM OF DEGREE K AND, HOMOTHETIC

ADDITIVITY ⟶ MARGINAL UTILITY OF GOOD x_i ONLY DEPENDS ON GOOD x_i

1. TAKE DERIVATIVE $u(x)$ WITH RESPECT TO A GOOD K

IF INCR UTILITY AND DEP ONLY ON GOOD K THEN UTILITY ADDITIVITY HOLDS

$$\frac{\partial u(x)}{\partial x_k} = \underbrace{\frac{\alpha k}{x_k}}_{\text{POSITIVE}} \cdot \underbrace{\prod_{i=1}^{m} x_i^{\alpha i}}_{u(x)\ \text{POSITIVE}} > 0$$

NOT ONLY CONSUMPTION OF x_k BUT ALSO FOR ALL OTHER GOODS

ONLY TWO GOODS

$$i = 1, 2 \Rightarrow u(x_1, x_2) = x_1^{\alpha 1} \cdot x_2^{\alpha 2}$$

$$\frac{\partial u}{\partial x_1} = \alpha 1 \cdot x_1^{\alpha 1 - 1} \cdot x_2^{\alpha 2} = \frac{\alpha 1}{x_1} \cdot x_1^{\alpha 1} \cdot x_2^{\alpha 2} = \frac{\alpha i}{x_i} u(x_1, x_2)$$

ADDITIVITY MEANS CONSUMPTION x_k DEP ONLY ON x_k IN THIS CASE DEPENDS ON CONSUMPTION OF ALL THE OTHER GOOD.

$$u(x) = x_1^2 + 2x_2 \qquad \frac{\partial u_1(x)}{\partial x_1} = 2x_1$$

HOMOGENEITY

$$u(tx) = \prod_{i=1}^{n} (tx_i)^{a_i} = \prod_{i=1}^{n} t^{a_i} \cdot x^{a_i} = t^{\sum_{i=1}^{n} a_i} \circ \prod_{i=1}^{n} x_i^{a_i} =$$

$$= t^{\sum a_i} \cdot u(x)$$

HOMOGENEITY HOLDS $\to$ DEGREE $\sum a_i$

$$i = 1,2 \to \sum_{i=1}^{n} a_i = a_1 + a_2 \begin{cases} > 1 & \text{MORE THAN PROPORTIONAL} \\ = 1 & \text{PROPORTIONAL} \\ < 1 & \text{LESS THAN PROPORTIONAL} \end{cases}$$

HOMOTHETIC $\to$ IS ALWAYS IMPLIED IN HOMOGENEITY

TO CHECK IT BY USING MRS

$$|MRS| = \frac{\frac{\partial u(x)}{\partial x_e}}{\frac{\partial u(x)}{\partial x_\ell}} = \frac{\frac{a_i}{x_k} \cdot \prod x_i^{a_i}}{\frac{a_k}{x_k} \cdot \prod x_i^{a_i}} = \frac{a_\ell}{a_k} \cdot \frac{x_k}{x_\ell}$$

$$|MRS'| = \frac{t^{\sum a_i}}{t^{\sum a_\ell}} \circ \frac{a_\ell}{a_k} \cdot \frac{x_k}{x_\ell} = |MRS|$$

Advanced Microeconomic Theory

Chapter 2: Utility Maximization Problem (UMP), Walrasian demand, indirect utility function

Outline

- Utility maximization problem (UMP)
- Walrasian demand and indirect utility function
- WARP and Walrasian demand (no, skip)
- Income and substitution effects (Slutsky equation)
- Duality between UMP and expenditure minimization problem (EMP)
- Hicksian demand and expenditure function
- Connections

Utility Maximization Problem

Utility Maximization Problem

- Consumer maximizes his utility level by selecting a bundle x (where x can be a vector) subject to his budget constraint:

$$\max_{x\geq 0} u(x)$$

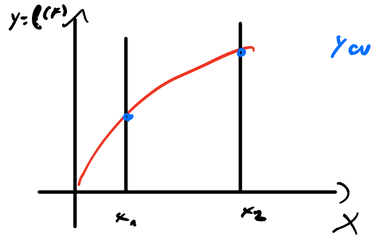
$$\text{s.t.} \quad p \cdot x \leq w$$

BUDGET CONSTRAINT

- ***Weierstrass Theorem***: for optimization problems defined on the reals, if the objective function is continuous and constraints define a closed and bounded set, then the solution to such optimization problem exists.

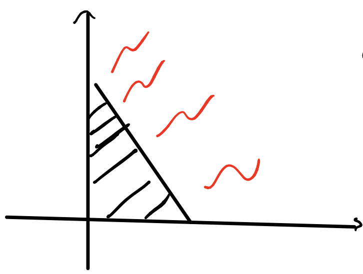
Vector is the quantity of goods. Max u(x) is a vector. Quantity must be positive. This is a constraint that we see last time.

P1x1 +p2 x2 ... is what you spend for good one and w is the total wealth.



YOU CAN ALWAYS FIND THE MAXIMUM

THIS AREA CONTAINS THE BOUNDARY

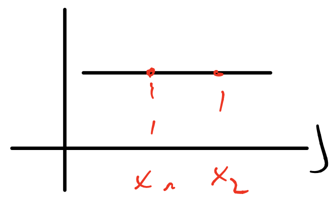


CLOSED & BOUNDED SET

Utility Maximization Problem

- ***Existence***: if $p \gg 0$ and $w > 0$ (i.e., if $B_{p,w}$ is closed and bounded), and if $u(\cdot)$ is continuous, then there exists at least one solution to the UMP.
  - If, in addition, preferences are strictly convex, then the solution to the UMP is unique. only one x maximize the function
- We denote the solution of the UMP as the **argmax** of the UMP (the argument, x, that solves the optimization problem), and we denote it as $x(p, w)$. $x^* = \{x_1^*, x_2^*, \ldots\}$
  - $x(p, w)$ is the ***Walrasian demand*** correspondence, which specifies a demand of every good in $\mathbb{R}^L_+$ for every possible price vector, p, and every possible wealth level, w.

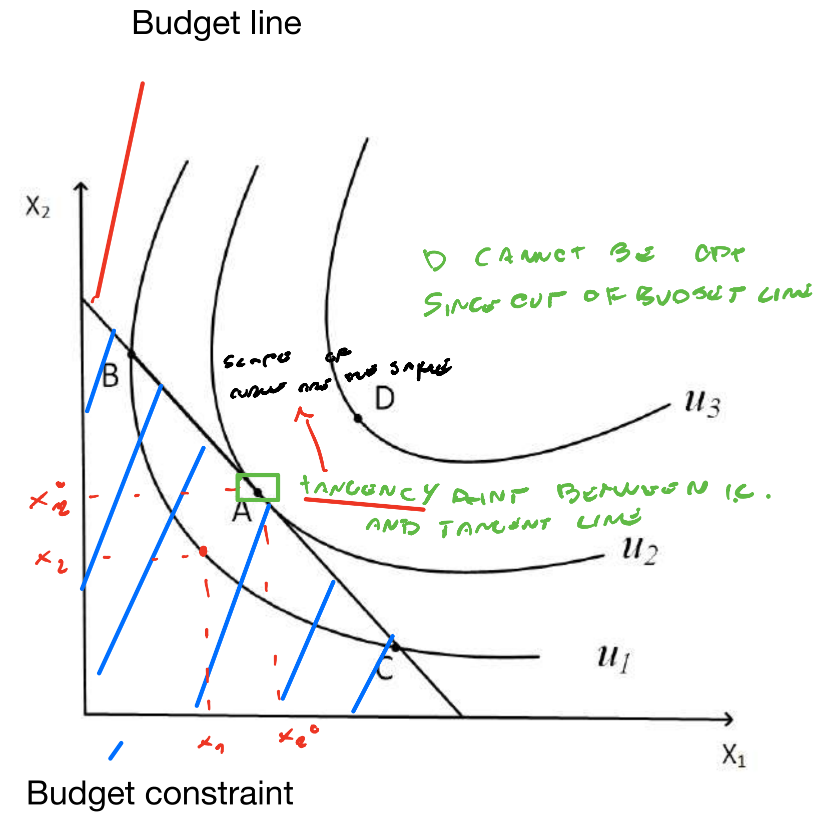
We can show that solution is unique if preferences are strictly convex and u(°) continuous.



Depends on prices and wealth!
So it’s why opt solution depend on w and p.

Utility Maximization Problem

- Walrasian demand $x(p, w)$ at bundle *A* is optimal, as the consumer reaches a utility level of u_2 by exhausting all his wealth.
- Bundles *B* and *C* are not optimal, despite exhausting the consumer's wealth. They yield a lower utility level u_1, where $u_1 < u_2$.
- Bundle *D* is unaffordable and, hence, it cannot be the argmax of the UMP given a wealth level of w.



$x^* = (x_1^*, x_2^*)$

ARG MAX

x° PREFERABLE TO x

 MONOTONE MORE IS BETTER !

IN A, WE WILL HAVE $MRS = -\frac{P_1}{P_2}$

ANSWER IS A

Properties of Walrasian Demand

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}_+^L$, then the Walrasian demand $x(p, w)$ satisfies:

1) Homogeneity of degree zero:

$x(p, w) = x(\alpha p, \alpha w)$ for all p, w, and for all $\alpha > \cancel{0}$ 1

That is, the budget set is unchanged!

$$\{x \in \mathbb{R}_+^L\colon\ p \cdot x \leq w\} = \{x \in \mathbb{R}_+^L\colon\ \alpha p \cdot x \leq \alpha w\}$$

Note that we don't need any assumption on the preference relation to show this. We only rely on the budget set being affected.

We will assume this properties for any problem of utility maximisation problem.

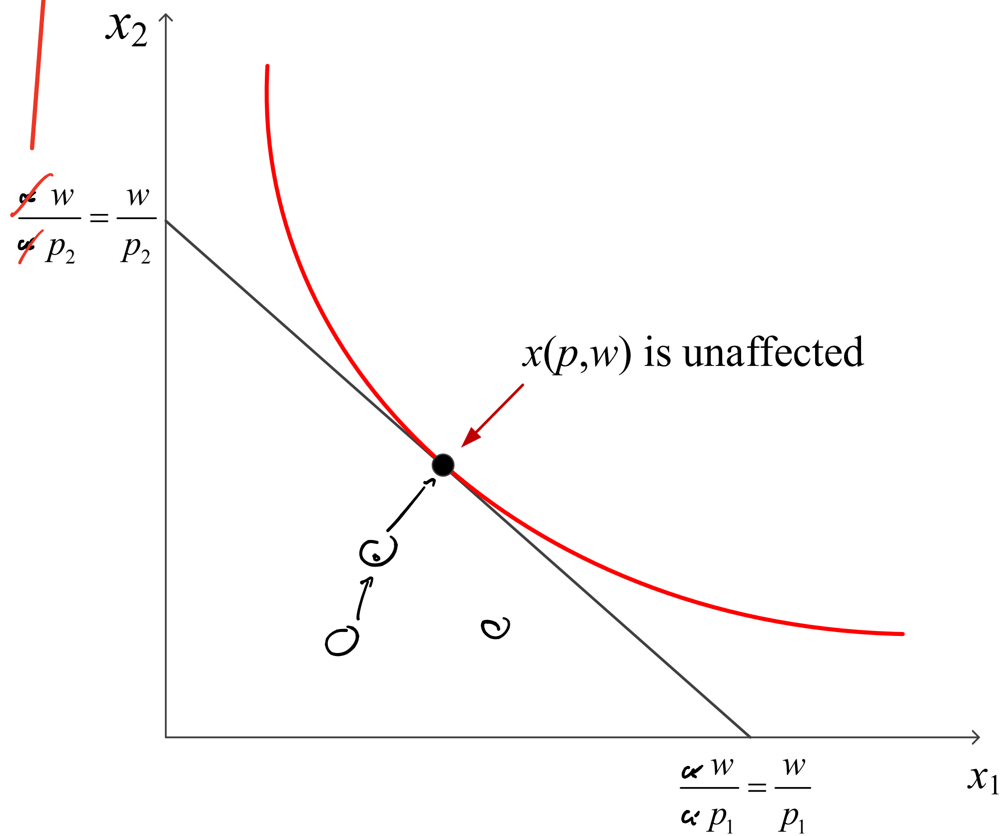
1. Homogeneity —> moltiply by alpha doesn’t change the value of the function.

Why increasing prices and wealth by same alpha we obtain a solution that is the same also for the MUP? Is easy to demonstrate with the graphical solution before.

If we increase everything by alpha.

If i multiply for alpha i obtain the same solution.

Properties of Walrasian Demand



– Note that the preference relation can be linear, and homog(0) would still hold.

x^* is arg max

$u(x^*)$ max value of utility

(utility computed in arg max)

Properties of Walrasian Demand

2) ***Walras' Law:***

$$p \cdot x = w \quad \text{for all } x = x(p, w)$$

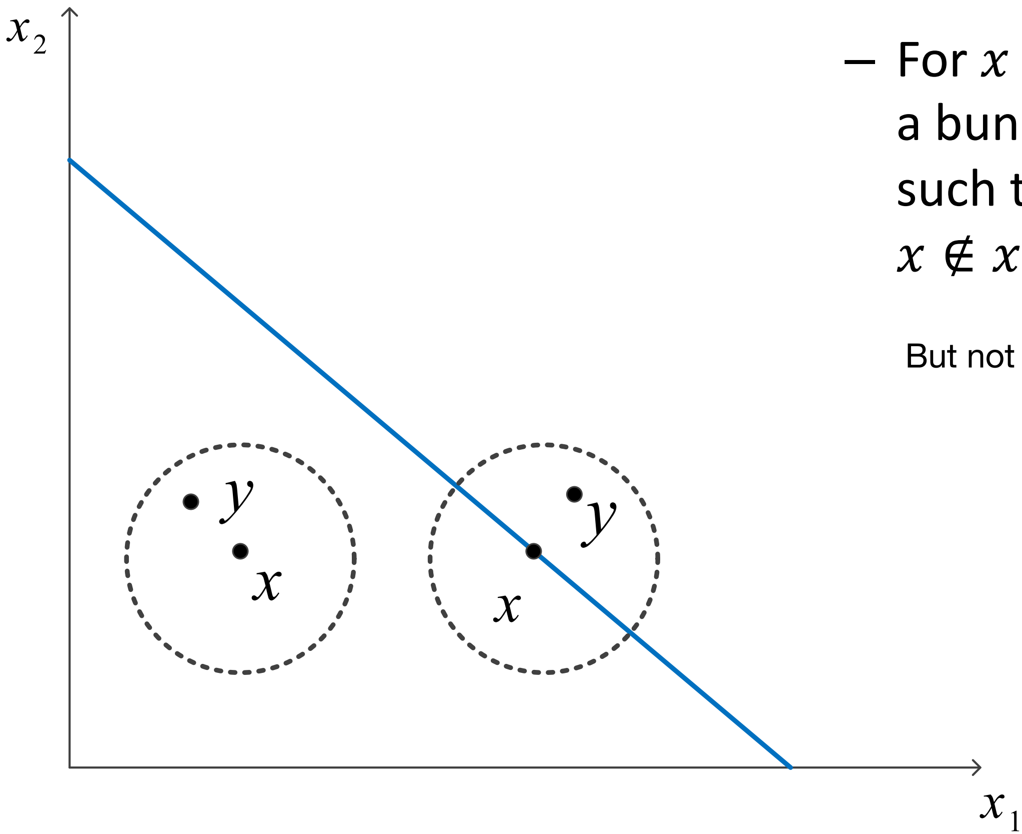
It follows from LNS: if the consumer selects a Walrasian demand $x \in x(p, w)$, where $p \cdot x < w$, then it means we can still find other bundle y, which is ε–close to x, where consumer can improve his utility level.

If the bundle the consumer chooses lies on the budget line, i.e., $p \cdot x' = w$, we could then identify bundles that are *strictly* preferred to x', but these bundles would be unaffordable to the consumer.

Walras' law. In the opt solution the consumer spends all income. Consume cannot remain with income not spent. It's irrational. In graphical term is intuitive because we must be in the budget line. In the opt solution you are in the tangency point and this define the walras law. In opt you don't have any unspent income. This depend on the fact that the utilty function satisfy LNS: you can find very close point that give you the same utility.

a) If Preferences are weakly convex then walrasian demand correspondence deifines a convex set.
b) if preference are strictly convex, then walrasian demand correspondence cointain a single element.

Properties of Walrasian Demand



– For $x \in x(p, w)$, there is a bundle y, ε–close to x, such that $y \succ x$. Then, $x \notin x(p, w)$.

But not affordable

Properties of Walrasian Demand

3) Convexity/Uniqueness:

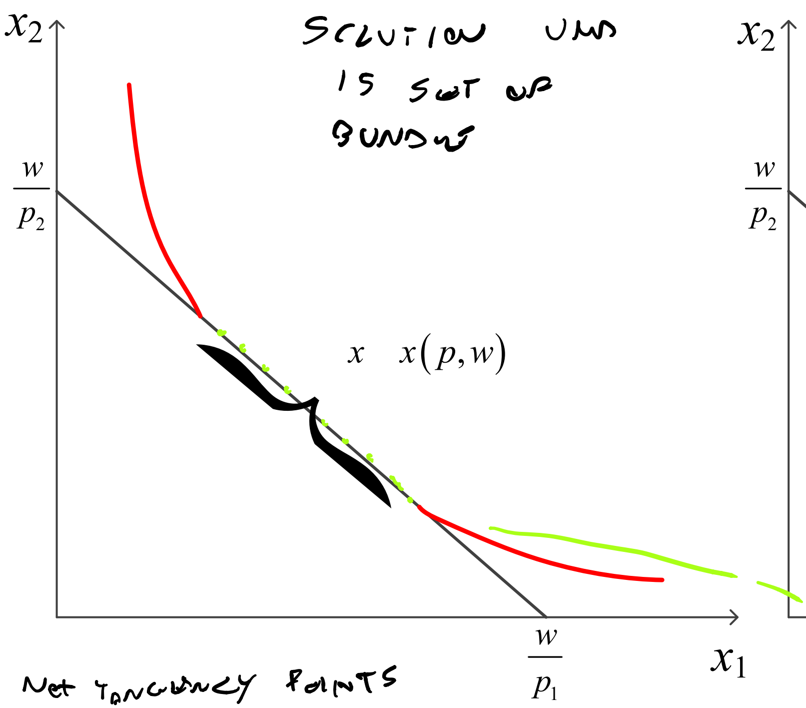
a) If the preferences are convex, then the Walrasian demand correspondence $x(p, w)$ defines a convex set, i.e., a continuum of bundles are utility maximizing. (For a given p and a given w)

b) If the preferences are strictly convex, then the Walrasian demand correspondence $x(p, w)$ contains a single element. (For a given p and a given w)

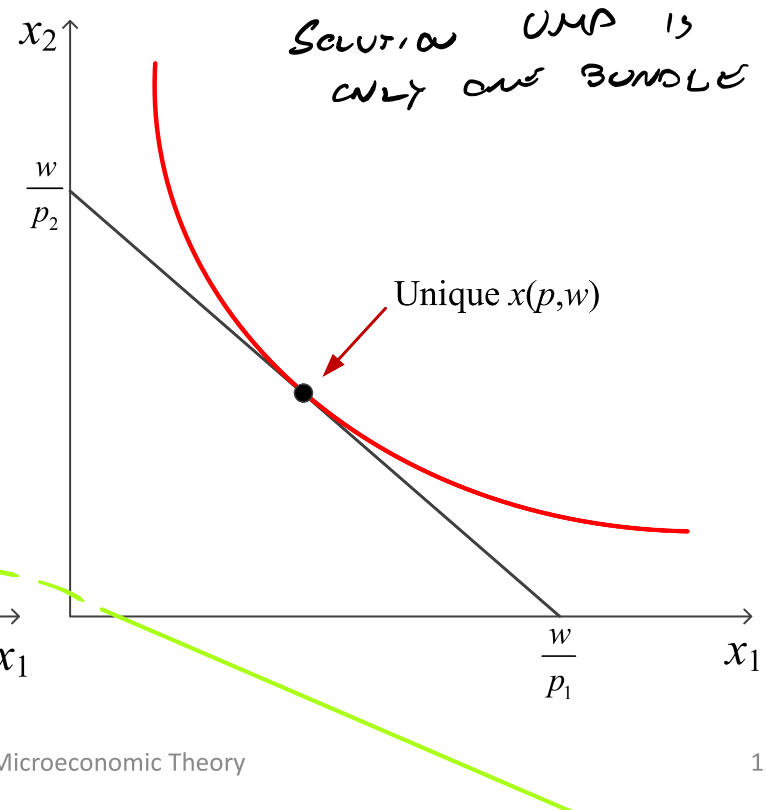
Properties of Walrasian Demand

weak

Convex preferences



Strictly convex preferences



I DON'T HAVE ONLY ONE BUNDLE BUT A SET OF BUNDLES

IN STRICTLY THIS HAVE A UNIQUE SOLUTION.

WE WILL SEE BOTH OF THE CASES.

NOW GO FOR ANALITICAL DEMONSTRATION -> TANGENT

UMP: Necessary Condition

SOLUTION NON NEGATIVE

$$\max_{x \geq 0} u(x) \quad \text{s.t. } p \cdot x \leq w$$

- We solve it using Kuhn-Tucker conditions over the Lagrangian $L = u(x) + \lambda(w - p \cdot x)$,

KKT COND

$$\frac{\partial L}{\partial x_k} = \frac{\partial u(x^*)}{\partial x_k} - \lambda p_k \leq 0 \text{ for all } k, \; = 0 \text{ if } x_k^* > 0$$

$$\frac{\partial L}{\partial \lambda} = w - p \cdot x^* = 0$$

$u(x_1, x_2) + \lambda(w - p_1 x_1 - p_2 x_2) \geq 0$

∂L

COMPLEMENTARY SLACKNESS

- That is, in a *interior* optimum, $\frac{\partial u(x^*)}{\partial x_k} = \lambda p_k$ for every good k, which implies

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k} \Leftrightarrow MRS_{l,k} = \frac{p_l}{p_k} \Leftrightarrow \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

USE LAGRANGIAN INSTEAD MAX $u(x)$ WE MAXIMISE LAGRANGIAN FUNCTION TO GET ANOTHER FUNCTION TO MAX

$$L = u(x) + \lambda(w - p \cdot x)$$

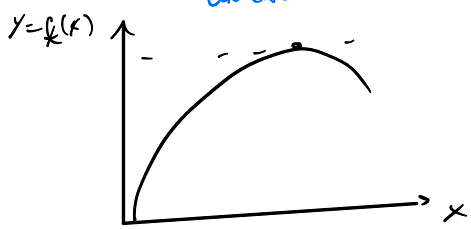
LAGRANGIAN

LAGRANGIAN MULTIPLIER

ONE VARIABLE CASE

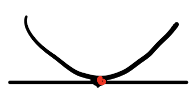
$L(x, \lambda)$

CONCAVE



1. SLOPE CURVE MUST BE ∅ (STATIONARY POINT)
2. DER. NEGATIVE IN STATIONARY POINT

CONVEX



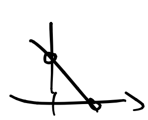
SLOPE IS MINIMUM

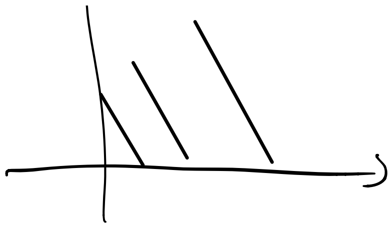
$$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$$ FOR INTERNAL SOLUTION

WALRAS LAW → OPT CONSUMER CONSUME MORE INCOME

$$\frac{\partial L}{\partial \lambda} = 0 \qquad w - p \cdot x = 0 \rightarrow w = p\,x$$

CORNER SOLUTION → INTERNAL SOLUTION OR CORNERS





IN CASE OF PERF SUB SOLUTION IN CORNERS

INSTEAD USE =0 WE USE KKT CONDITION

SEE ABOVE ↑

COMPLEMENTARY SLACKNESS

If $x_k^* > 0 \Rightarrow \frac{\partial L}{\partial x_k} = 0$ ← WE WILL CONSIDER THIS CASE

If $x_k^* = 0 \to \frac{\partial L}{\partial x_k} < 0$

1) $\frac{\partial u}{\partial x_1} - \lambda p_1 = 0$

2) $\frac{\partial u}{\partial x_2} - \lambda p_2 = 0$

MOVE TO THE RATIO! → $\frac{p_1}{p_2} = \frac{\cancel{\lambda}\frac{\partial u}{\partial x_1}}{\cancel{\lambda}\frac{\partial u}{\partial x_2}} = MRS$

LAMBDA GOES

MARGINAL UTILITIES

LHS → MRS (SLOPE I.C.)

3) $w - px = 0$

$\frac{p_1}{p_2}$ → SLOPE OF BUDGET CONSTRAINT

(IN REALITY IS THE OPPOSITE

I.C AND BUD CONSTRAIN NEGATIVELY SLOPED BUT YOU CAN MULTIPLY BY -1.

$-\frac{p_1}{p_2} \cdot -1 = \frac{p_1}{p_2}$

$\frac{\frac{\partial u}{\partial x_1}}{p_1} = \frac{\frac{\partial u}{\partial x_2}}{p_2} \Rightarrow$ INCOME AT HOW MANY ONE OF GOOD 1?

$$\frac{1€}{p_1} \cdot \frac{\partial u}{\partial x_1} = \frac{1€}{p_2} \cdot \frac{\partial u}{\partial x_2}$$ FOR GOOD 2

MU MUST BE THE SAME SPEND FOR THE SECOND GOOD

MOVING INCOME FROM 1 TO ANOTHER GOOD DOES NOT GET MORE OPTIMAL

UMP: Sufficient Condition

- When are Kuhn-Tucker (necessary) conditions, also sufficient?
  - That is, when can we guarantee that $x(p, w)$ is the $\max$ of the UMP and not the $\min$?

UMP: Sufficient Condition

- Interpretation of $\frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$

The marginal utility of the last dollar ("marginal" euro) spent in good l must produce the same utility of the last euro spent in good k. [Hint. With one dollar you buy $1/p_l$units of good l and $1/p_l$units of good k)

- When are Kuhn-Tucker (necessary) conditions, also sufficient?
  - That is, when can we guarantee that $x(p, w)$ is the max of the UMP and not the min?

UMP: Sufficient Condition

SECOND ORDER CONDITION

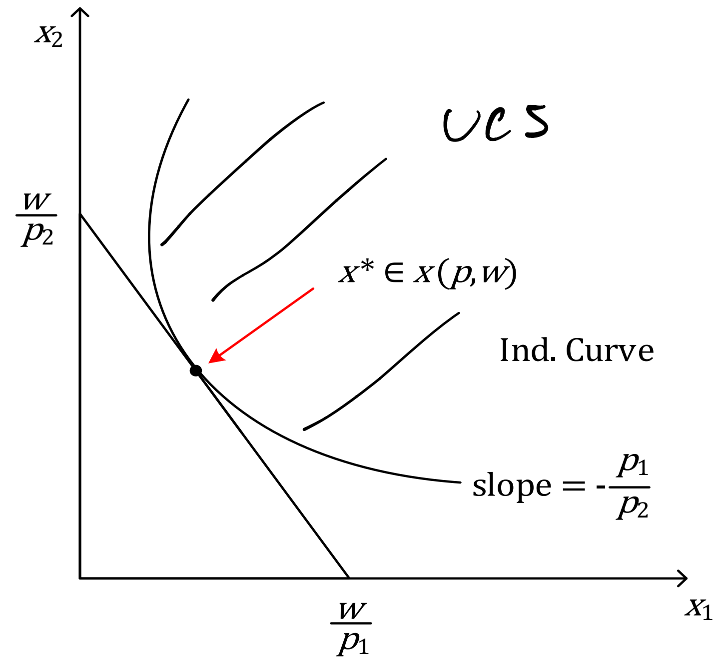
- Kuhn-Tucker conditions are sufficient for a max if:

  1) $u(x)$ is quasiconcave, i.e., convex upper contour set (UCS).

  2) $u(x)$ is monotone.

  3) $\nabla u(x) \neq 0$ for $x \in \mathbb{R}_{+}^{L}$.

  – If $\nabla u(x) = 0$ for some x, then we would be at the "top of the mountain" (i.e., blissing point), which violates both LNS and monotonicity.



UMP: Violations of Sufficient Condition

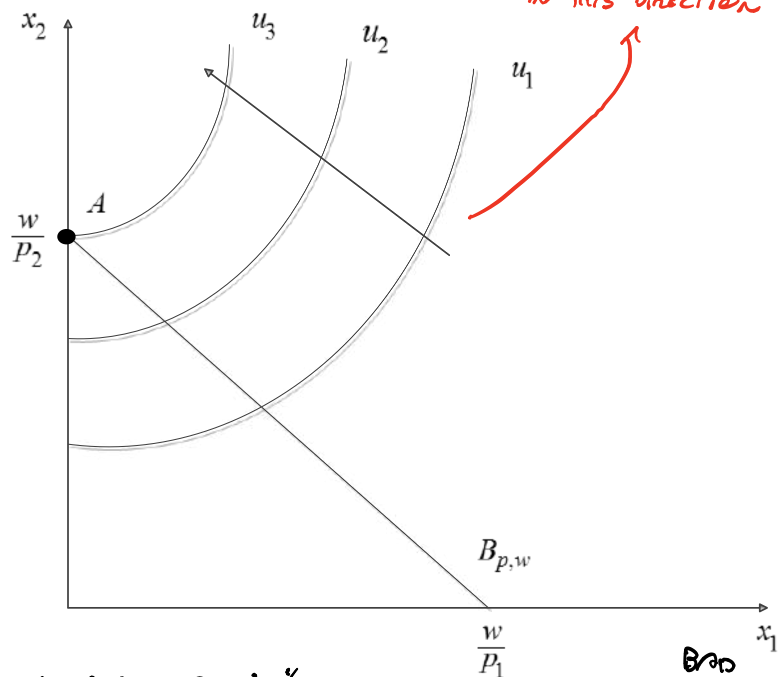
VIOLATION OF MONOTONICITY → BLISSING POINT

1) $u(\cdot)$ is non-monotone:

- The consumer chooses bundle *A* (at a corner) since it yields the highest utility level given his budget constraint.

CORNER SOLUTION

- At point *A*, however, the tangency condition $MRS_{1,2} = \frac{p_1}{p_2}$ does not hold.



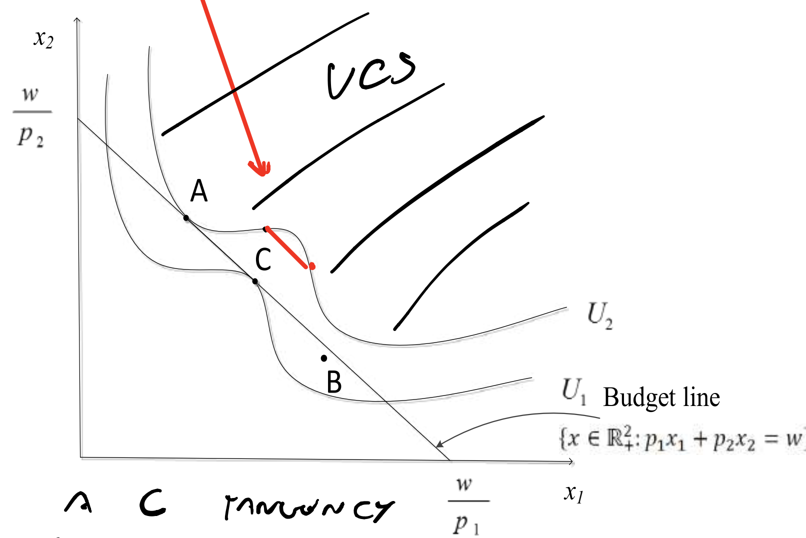
WE WILL ONLY CONSIDER CORNER SOLUTION OF PERFECT SUBSTITUTES

UMP: Violations of Sufficient Condition

2) $u(\cdot)$ is not quasiconcave:

- The upper contour sets (UCS) are not convex.
- $MRS_{1,2} = \frac{p_1}{p_2}$ is not a sufficient condition for a max.
- A point of tangency (*C*) gives a lower utility level than a point of non-tangency (*B*).
- True maximum is at point *A*.

UCS NOT CONVEX



A C TANGENCY
BUT ONE IS MAX AND THE OTHER
NO. VIOLATION OF SUFF CONDITION

UMP: Corner Solution

- Analyzing differential changes in x_l and x_l, that keep individual's utility unchanged, $du = 0$,

$$\frac{du(x)}{dx_l} dx_l + \frac{du(x)}{dx_k} dx_k = 0 \text{ (total diff.)}$$

- Rearranging,

$$\frac{dx_k}{dx_l} = -\frac{\frac{du(x)}{dx_l}}{\frac{du(x)}{dx_k}} = -MRS_{l,k}$$

- ***Corner Solution***: $MRS_{l,k} > \frac{p_l}{p_k}$, or alternatively, $\frac{\frac{du(x^*)}{dx_l}}{p_l} > \frac{\frac{du(x^*)}{dx_k}}{p_k}$, i.e., the consumer prefers to consume more of good l.

UMP: Corner Solution

- In the FOCs, this implies:
  - *a)* $\frac{\partial u(x^*)}{\partial x_k} \leq \lambda p_k$ for the goods whose consumption is zero, $x_k^* = 0$, and
  - *b)* $\frac{\partial u(x^*)}{\partial x_l} = \lambda p_l$ for the good whose consumption is positive, $x_l^* > 0$.
- *Intuition*: the marginal utility per dollar spent on good l is still larger than that on good k.

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \lambda \geq \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

UMP: Corner Solution

CASES OF CORNER SOLUTION

WE CANNOT FIND TANGENCY CONDITION

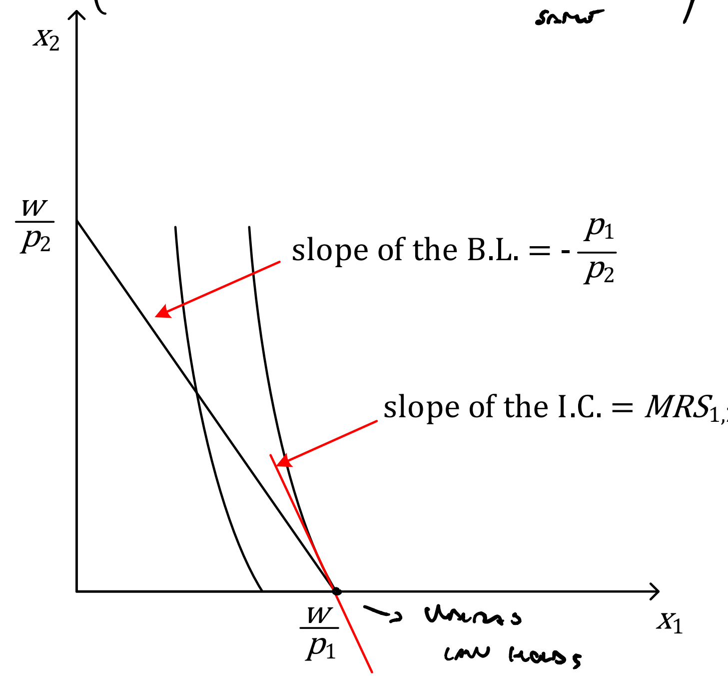
(NO TANGENCY POINT → SLOPE IS SAME)

- Consumer seeks to consume good 1 alone.

- At the corner solution, the indifference curve is steeper than the budget line, i.e.,

$$MRS_{1,2} > \frac{p_1}{p_2} \text{ or } \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$$

- Intuitively, the consumer would like to consume more of good 1, even after spending his entire wealth on good 1 alone. , BUDGET



SLOPE IC AND BC?

SLOPE IS GREATER IN IC THAN B.C.

$x_2 = 0 \quad x_1 = \frac{w}{p_1}$

$|MRS| > p_1/p_2$

$$\frac{\frac{\delta u}{\delta x_1}}{p_1} > \frac{\frac{\delta u}{\delta x_2}}{p_2}$$

YOU CONSUME ALL INCOME IN x_1 SO YOU CAN'T MOVE FROM ONE GOOD TO ANOTHER

IN CORNER SOLUTION $MRS > \frac{p_1}{p_2}$ IS NOT EQUALITY

$$\boxed{\text{ERRORS IN EXERCISES!}}$$

$$\frac{\frac{\delta u}{\delta x_1}}{\frac{\delta u}{\delta x_2}} = MRS \quad = \frac{p_1}{p_2}$$

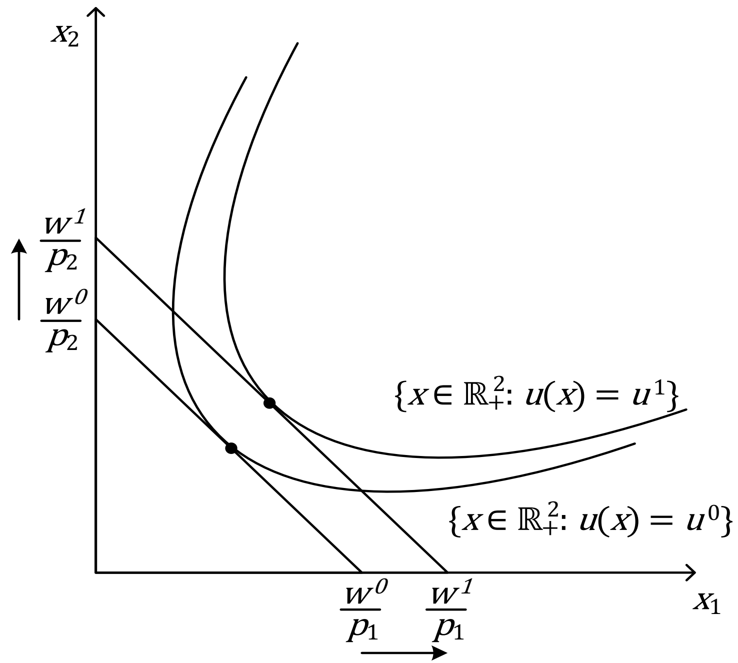
DEFINITION

THIS ONLY HOLDS FOR THE MAXIMUM!!

UMP: Lagrange Multiplier → λ

- λ is referred to as the "marginal values of relaxing the constraint" in the UMP (a.k.a. "***shadow price of wealth***").

- If we provide more wealth to the consumer, he is capable of reaching a higher indifference curve and, as a consequence, obtaining a higher utility level.
  - We want to measure the change in utility resulting from a marginal increase in wealth.



next lesson

UMP: Lagrange Multiplier

- Let us take $u(x(p,w))$, and analyze the change in utility from change in wealth. Using the chain rule yields,

$$\nabla u(x(p,w)) \cdot D_w x(p,w)$$

- Substituting $\nabla u(x(p,w)) = \lambda p$ (in interior solutions),

$$\lambda p \cdot D_w x(p,w)$$

NB. ∇ means differential with respect to a vector, $x = (x_1, x_2, \dots . x_n)$ the result is a vector next lesson

UMP: Lagrange Multiplier

- From Walras' Law, $p \cdot x(p, w) = w$, the change in expenditure from an increase in wealth is given by
$$p \cdot D_w x(p, w) = D_w[p \cdot x(p, w)] = D_w(w) = 1$$

- Hence,
$$\nabla u(x(p, w)) \cdot D_w x(p, w) = \lambda \underbrace{p \cdot D_w x(p, w)}_{1} = \lambda$$

- *Intuition*: If $\lambda = 5$, then a \$1 increase in wealth implies an increase in 5 units of utility. At the maximum this must be the same for all goods, otherwise we are not at the maximum

Walrasian Demand: Wealth Effects

PARTIAL DER. IN RESPECT OF WEALTH IF > NORM
IF < INF

- Normal vs. Inferior goods

$$\frac{\partial x(p,w)}{\partial w} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{normal} \\ \text{inferior} \end{Bmatrix}$$

INCREASING OF INCOME
DOESN'T MAKE YOU TAKE
MORE CRAPPY WINE
BUT YOU GET
BETTER WINE

- Examples of inferior goods:
  - Two-buck chuck (a really cheap wine)
  - Walmart during the economic crisis
  - POTATOES
  - A CRAP BY (ANY CHEAP) WINE

Walrasian Demand: Wealth Effects

PRICES CONSTANT → SLOPE SAME

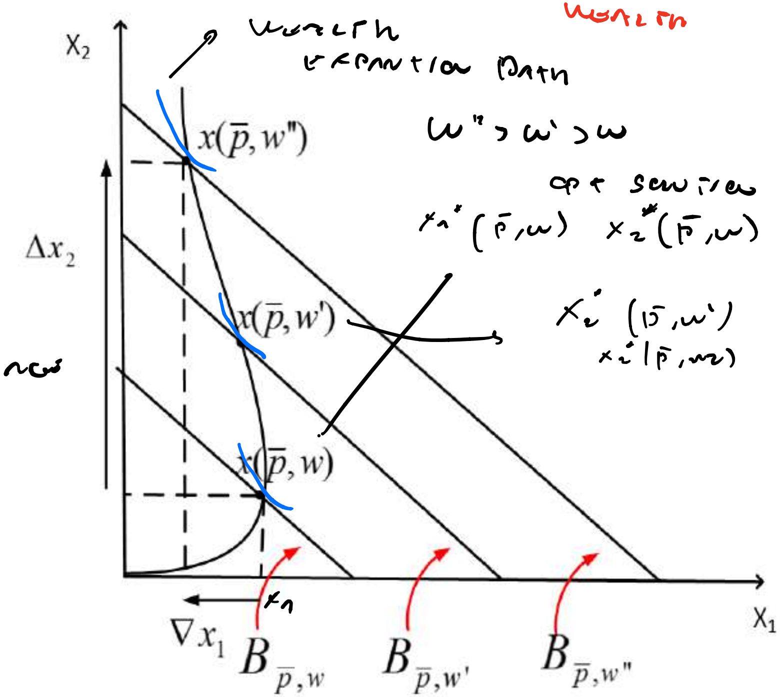
INTERCEPT CHANGES → IF INCR. WEALTH

- An increase in the wealth level produces an outward shift in the budget line.

- x_2 is normal as $\frac{\partial x_2(p,w)}{\partial w} > 0$, while x_1 is inferior as $\frac{\partial x(p,w)}{\partial w} < 0$.

HOW DEMAND CHANGES IF WEALTH CHANGES

- ***Wealth expansion path:***
  - connects the optimal consumption bundle for different levels of wealth
  - indicates how the consumption of a good changes as a consequence of changes in the wealth level



WEALTH EXPANSION PATH

$w'' > w' > w$

OPT SOLUTION

$x_1^*(\bar{p}, w)$ $x_2^*(\bar{p}, w)$

$x_2'(\bar{p}, w')$

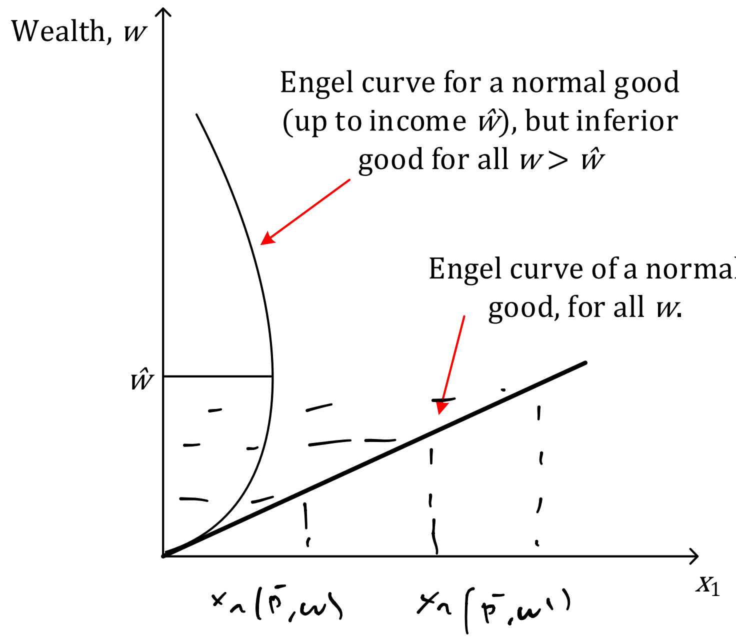
$x_1'(\bar{p}, w')$

x_1

INC INCREASES, YOU CONSUME MORE x_2 (NORMAL) AND LESS x_1 (INFERIOR)

Walrasian Demand: Wealth Effects

- ***Engel curve*** depicts the consumption of a particular good in the horizontal axis and wealth on the vertical axis.
- The slope of the Engel curve is:
  - positive if the good is normal
  - negative if the good is inferior
- Engel curve can be positively slopped for low wealth levels and become negatively slopped afterwards.



DRAW POINT AND I OBTAIN THE ENGEL CURVE

If price for sugar increase, then you demand less coffe. If prime derivative is positive

Walrasian Demand: Price Effects

- Own price effect:

$$\frac{\partial x_k(p,w)}{\partial p_k} \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Usual} \\ \text{Giffen} \end{Bmatrix}$$

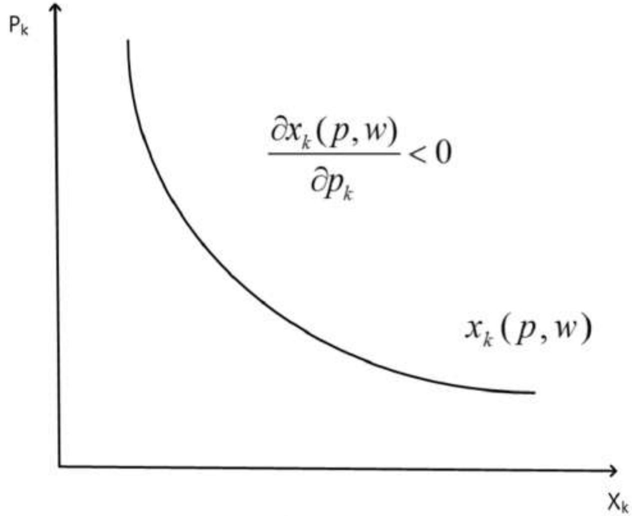
- Cross-price effect:

$$\frac{\partial x_k(p,w)}{\partial p_l} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Substitutes} \\ \text{Complements} \end{Bmatrix}$$

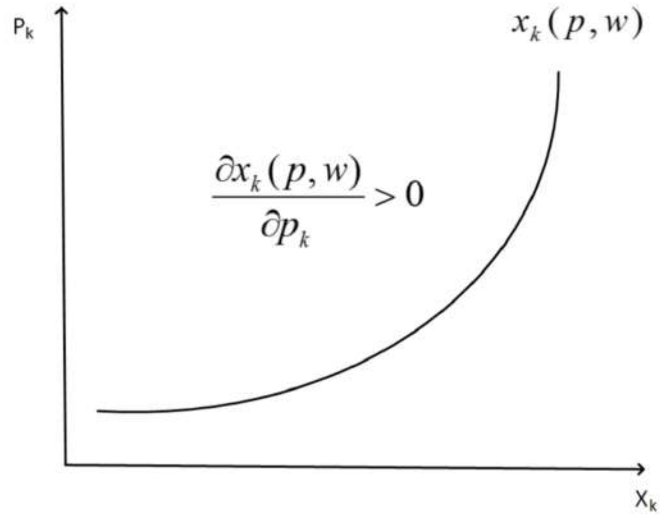
  – *Examples of Substitutes*: two brands of mineral water, such as Sant'Anna vs. Acqua Panna (Disclaimer: I did not receive money from any of the two....)
  – *Examples of Complements*: coffee and sugar.

Walrasian Demand: Price Effects

- Own price effect (inverse demand is graphed, i.e. P in vertical axis and the good in horizontal axis)



Usual good
(law of price holds, if P increases quantity decreases)



Giffen good
(law of price does not hold)

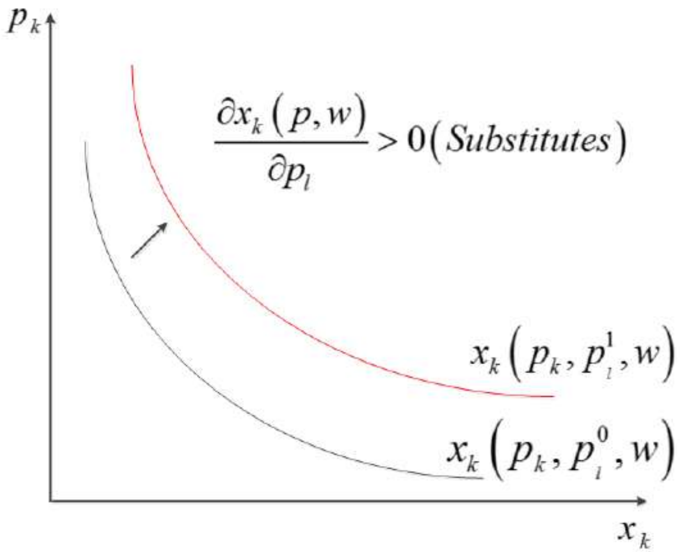
Vertical axis is the demand. We have said that if price increases the quantity increase and the walras' demand is positively sloped.

What if we want to see graphically if demand for one good is the same as the second good?
We want to see how demand depends. On another wealth. We can't use this curve because represent the realtion between quiantity of k and price of k.

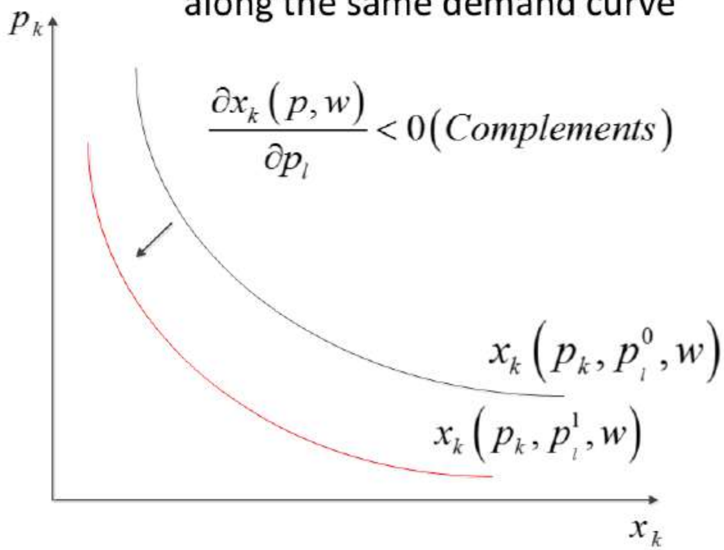
Walrasian Demand: Price Effects

- Cross-price effect

- Two-dimensions graph: change in p_1 means moving to another demand curve, while changes in p_k means moving along the same demand curve



Substitutes



Complements

Warla's demand. Level 0 of price pk. Walras demand. If other two variable, like price in the other good change, the curves could change up or down. If good increase and the goods are substitute the curve moves up.
For a given pk do you demand more or less pk. So curve goes up right.
Complements good is the opposite. If the price of the other good increase the second one will decrease.

Different goods can be classified using walras' demand.

Indirect Utility Function

- The Walrasian demand function, $x(p,w)$, is the solution to the UMP (i.e., argmax, i.e. value of the argument that maximizes utility).

- What would be the utility function evaluated at the solution of the UMP, i.e., $x(p,w)$?
  - This is the ***indirect utility function*** (i.e., the highest utility level), $v(p,w) \in \mathbb{R}$, associated with the UMP.
  - It is the "value function" of this optimization problem.

(I.e the function evaluated at the maximum)

If good normal or inferior we expect demand of the good will increase or decreases.
After solving the UMP getting the argmax yesterday, the solution of this problem is called walras demand. We have found this solution called x(p,w). Now we can compute the utilty function of this argument. If we compute utility function at the optimal level.

$u(x(p,w)) =$ IND UTILITY FUNCTION

$$\max_{x \geq 0} u(x) \quad \text{such that} \quad x \cdot p \leq w$$

Degree of homogeneity of the indirect utilty function?
What happened to the value function if the prices and the wealth increase by the same proportion? [value alpha]. And we want to see what happen to the maximum likehood. What we have found? Walra's demeaned is homogeneous of degree 0 since the budget constraint the solution will be the same.
What happen to the utility function if p and w change for the small propotion of alpha. The value of utility doesn't change so I directed utility function is homogeneous of degree 0.

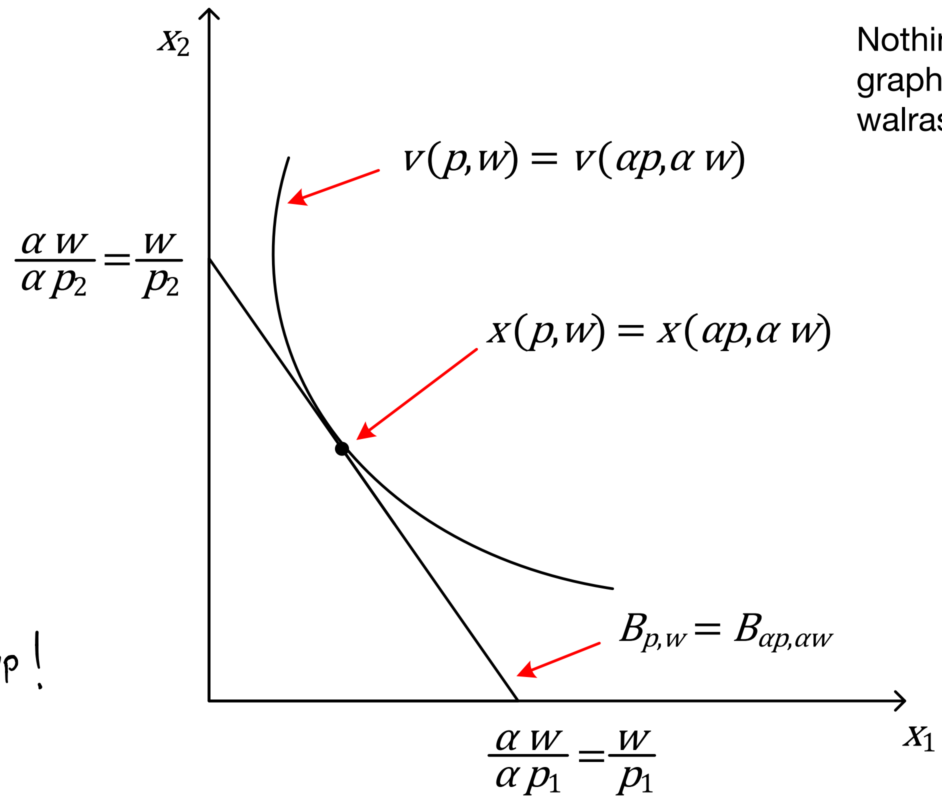
The indirect utilty function is homogeneous of degree 0.

Properties of Indirect Utility Function

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}_{+}^{L}$, then the indirect utility function $v(p, w)$ satisfies:

1) ***Homogenous of degree zero:*** Increasing p and w by a common factor $\alpha > 0$ does not modify the consumer's optimal consumption bundle, $x(p, w)$, nor his maximal utility level, measured by $v(p, w)$.

Properties of Indirect Utility Function



Nothing change graphically: same IC, walras demand ecc..

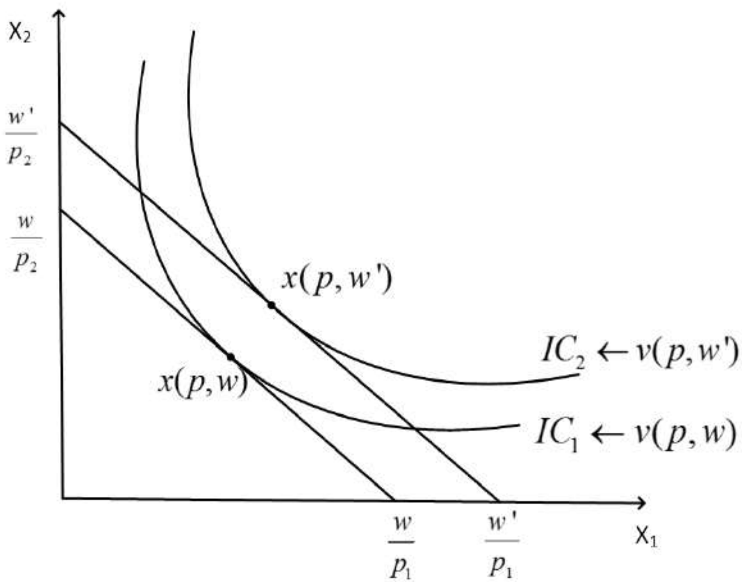
If w ↑ /
IUF ? shift up!

Properties of Indirect Utility Function

If wealth increase, i will get more

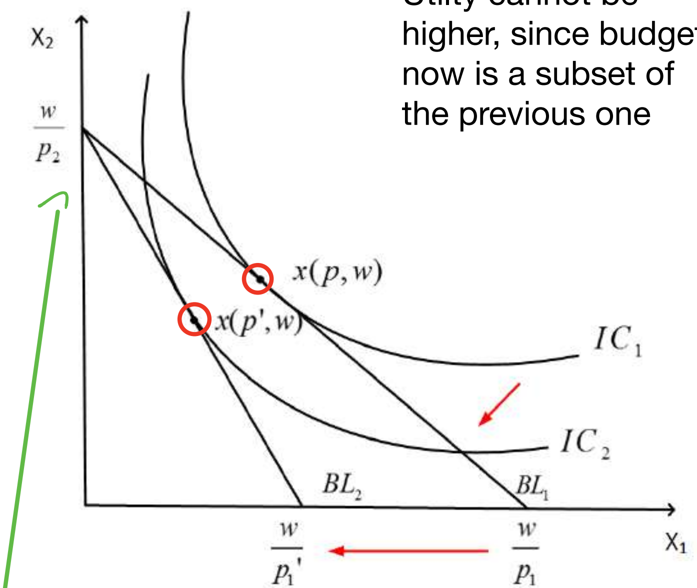
2) Strictly increasing in w:

$v(p, w') > v(p, w)$ for $w' > w$.



3) non-increasing (i.e., weakly decreasing) in p_k

Utilty cannot be higher, since budget now is a subset of the previous one



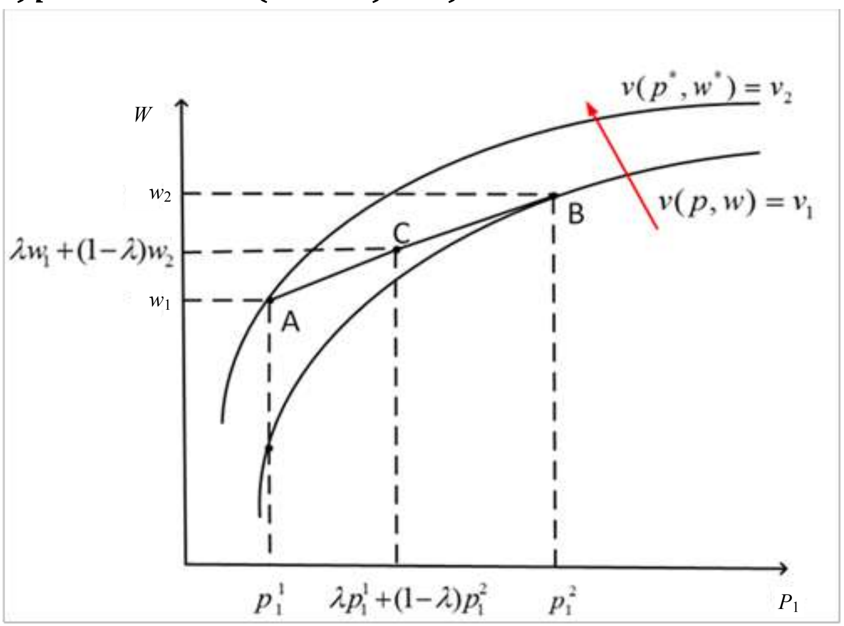
Imagine corner solution, the demand remain the same (x2), the supply will decrease. So is not increasing in pk\

NOT SO IMPORTANT FOR EX!!

Properties of Indirect Utility Function

4) Quasiconvex: The set $\{(p,w): v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

- ***Interpretation I***: If $(p^1,w^1) \succsim^* (p^2,w^2)$, then $(p^1,w^1) \succsim^* (\lambda p^1 + (1-\lambda)p^2,\ \lambda w^1 + (1-\lambda)w^2)$; i.e., if $A \succsim^* B$, then $A \succsim^* C$.

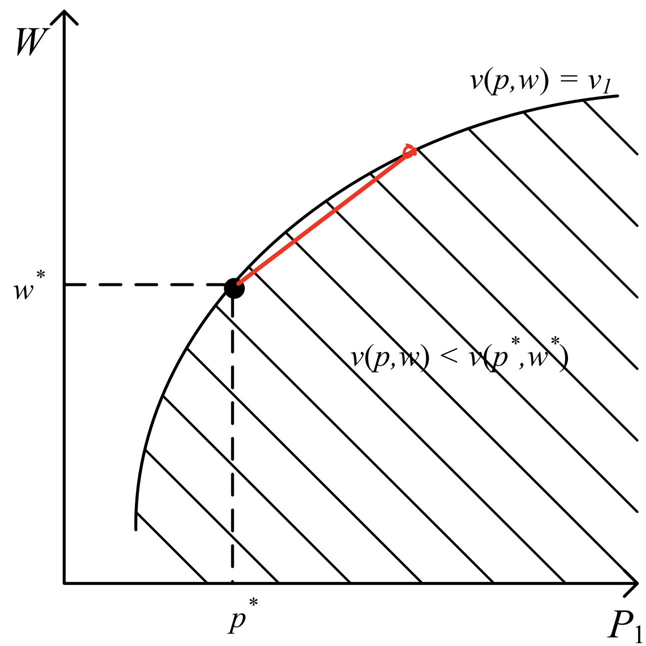


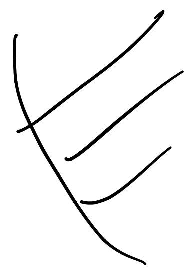
Properties of Indirect Utility Function

- ***Interpretation II***: $v(p, w)$ is quasiconvex if the set of (p, w) pairs for which $v(p, w) < v(p^*, w^*)$ is convex.

UCS -> CONVEX
LWS -> CONCAVE

IUF IS CONCAVE

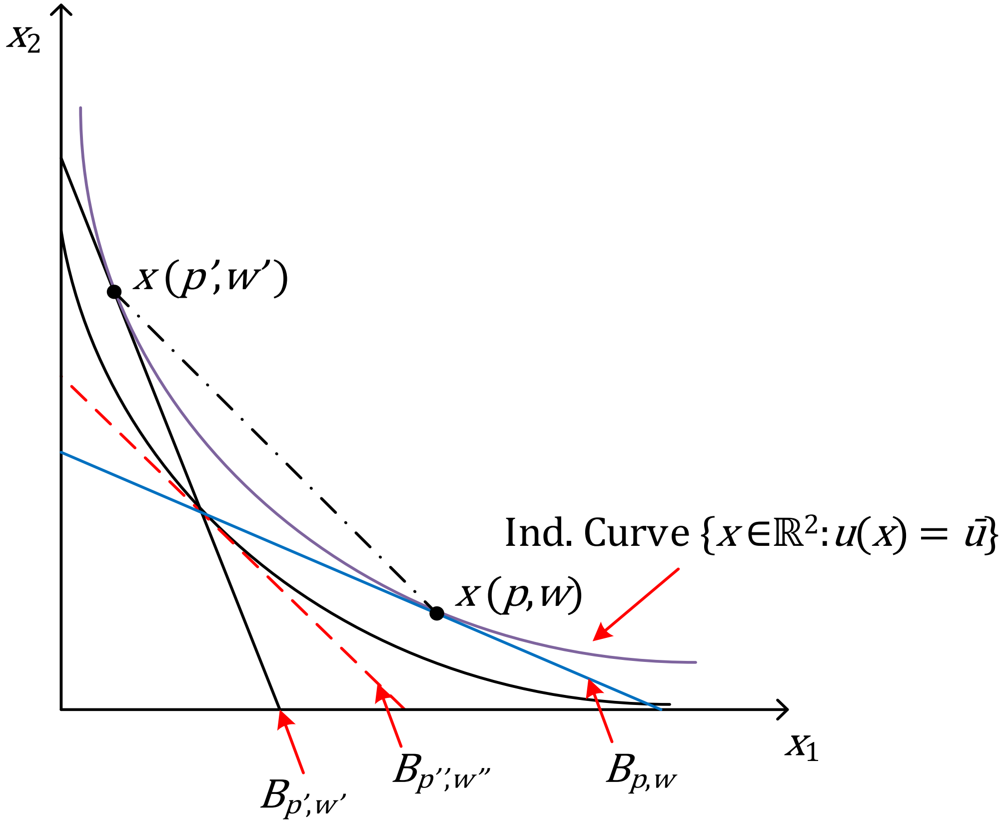




Properties of Indirect Utility Function

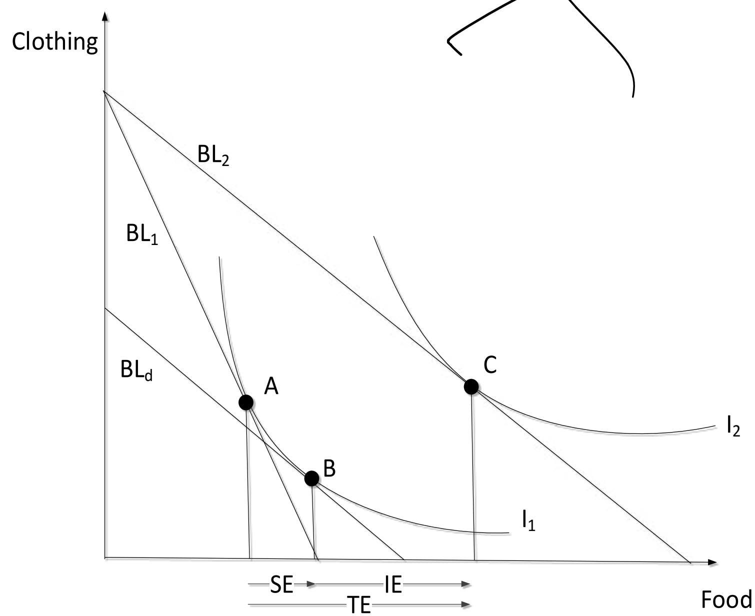
- ***Interpretation III***: Using x_1 and x_2 in the axis, perform following steps:
  1) When $B_{p,w}$, then $x(p,w)$
  2) When $B_{p',w'}$, then $x(p',w')$
  3) Both $x(p,w)$ and $x(p',w')$ induce an indirect utility of $v(p,w) = v(p',w') = \bar{u}$
  4) Construct a linear combination of prices and wealth:
  $$\left.\begin{aligned} p'' &= \alpha p + (1-\alpha)p' \\ w'' &= \alpha w + (1-\alpha)w' \end{aligned}\right\} \; B_{p'',w''}$$
  5) Any solution to the UMP given $B_{p'',w''}$ must lie on a lower indifference curve (i.e., lower utility)
  $$v(p'',w'') \leq \bar{u}$$

Properties of Indirect Utility Function



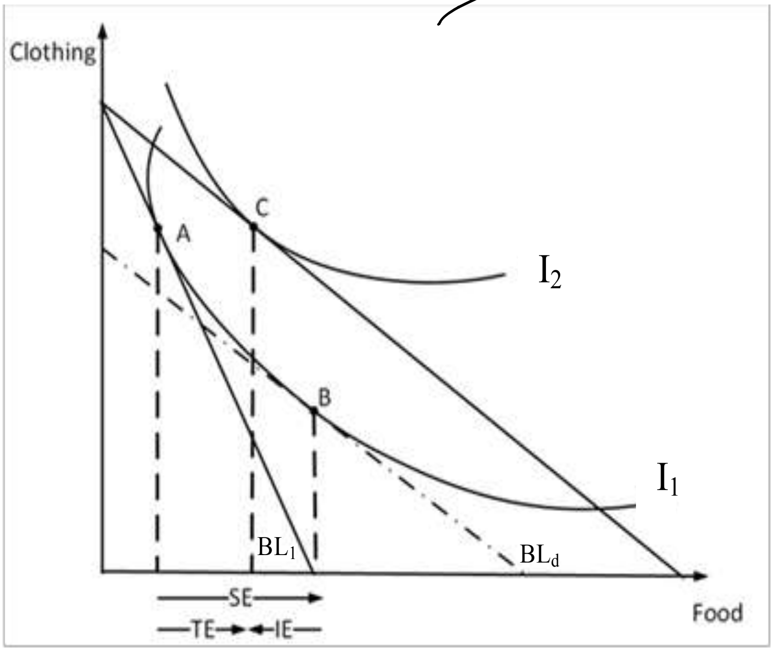
Substitution and Income Effects: Normal Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The substitution effect (SE) moves in the opposite direction as the price change.
  - A reduction in the price of food implies a positive substitution effect.
- The income effect (IE) is positive (thus it reinforces the SE).
  - The good is normal.



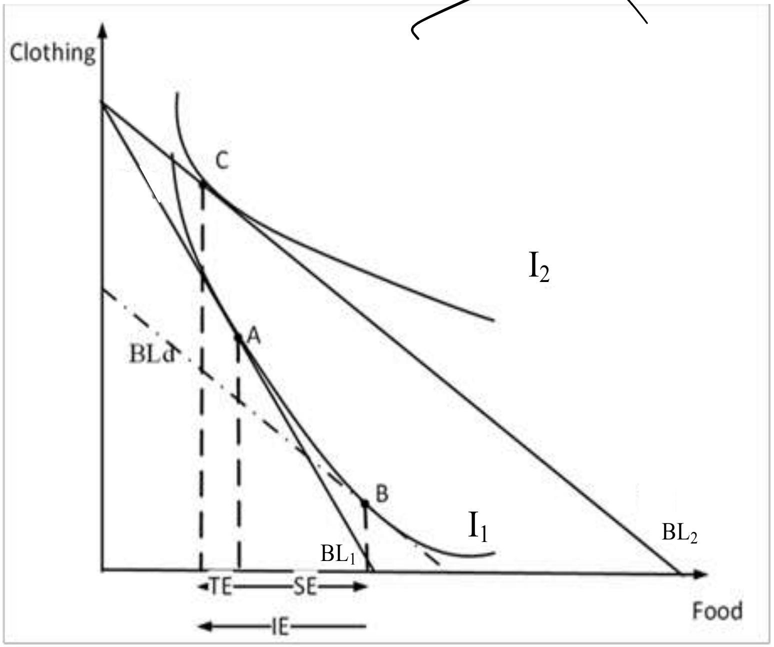
Substitution and Income Effects: Inferior Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is now negative (which partially offsets the increase in the quantity demanded associated with the SE).
  - The good is inferior.
- *Note*: the SE is larger than the IE.



Substitution and Income Effects: Giffen Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is still negative but now completely offsets the increase in the quantity demanded associated with the SE.
  - The good is Giffen good.
- *Note*: the SE is less than the IE.



Substitution and Income Effects

	SE	IE	TE
Normal Good	+	+	+
Inferior Good	+	-	+
Giffen Good	+	-	-

- Not Giffen: Demand curve is negatively sloped (as usual)
- Giffen: Demand curve is positively sloped

ESERCIZI

EX 2 CH.2

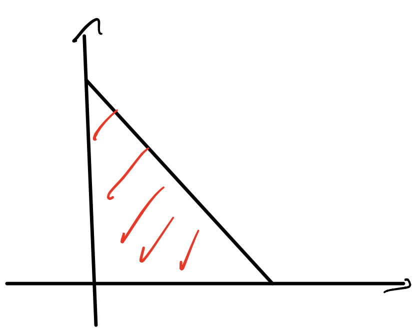
$$u(x_1, x_2) = x_1^{a} x_2^{\frac{1}{2}-a}$$

1. WALRAS DEMAND x_1, x_2
2. RESTRICTION ON a SUCH THAT $x_1, x_2 > 0$

$p_1 x_1 + p_2 x_2 \leq w$ CONSTRAINT

1) $$L = x_1^{a} x_2^{\frac{1}{2}-a} + \lambda (w - p_1 x_1 - p_2 x_2)$$

$$\text{FOCS} = \begin{cases} \dfrac{\delta L}{\delta x_1} = a x_1^{a-1} x_2^{\frac{1}{2}-a} - \lambda p_1 = 0 \\ \dfrac{\delta L}{\delta x_2} = \left(\dfrac{1}{2} - a\right) x_2^{-\frac{1}{2}-a} x_1^{a} - \lambda p_2 = 0 \\ \dfrac{\delta L}{\delta \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \leftarrow \text{WALRAS LAW} \end{cases}$$



CORNER SOLUTION CAN'T BE THE SOLUTION BECAUSE THE $u(\cdot)$ IS A COUB-DOUGLAS BECAUSE IN CORNER UTILITY CAN BE $\emptyset$

NOW WE HAVE 3 EQUATION AND 3 VARIABLES SO:

$$\frac{a x_1^{a-1} x_2^{\frac{1}{2}-a}}{\left(\frac{1}{2}-a\right) x_2^{-\frac{1}{2}-a} x_1^{a}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2} \Rightarrow \frac{a x_1^{a-1} x_2^{\frac{1}{2}-a}}{\left(\frac{1}{2}-a\right) x_2^{-\frac{1}{2}-a} x_1^{a}} = \frac{p_1}{p_2}$$

$$\frac{\alpha}{\frac{1}{2}-\alpha}\cdot\frac{x_2}{x_1}=\frac{p_1}{p_2}$$

$$\frac{\cancel{x_1}^{\alpha-1}}{x_1^{\alpha}} \qquad \frac{x_2^{\frac{1}{2}-\alpha}}{x_2^{-\frac{1}{2}-\alpha}}$$

$$x_2=\frac{p_1}{p_2}\cdot\frac{\frac{1}{2}-\alpha}{\alpha}\,x_1 \qquad \boxed{x_2=\frac{1-2\alpha}{2\alpha}\cdot\frac{p_1}{p_2}\,x_1}$$

THE SUBSTITUTE x_2 IN THE CONSTRAINT

$$w - p_1x_1 - p_2x_2 = 0 \to w - p_1x_1 - \cancel{p_2}\left(\frac{1-2\alpha}{2\alpha}\,\frac{p_1}{\cancel{p_2}}\,x_1\right)=0$$

$$w = p_1x_1 + \frac{1-2\alpha}{2\alpha}\,p_1\,x_1 \qquad w=\left(1+\left(\frac{1-2\alpha}{2\alpha}\right)\right)x_1p_1$$

$$w = x_1\left(\frac{2\alpha+1-2\alpha}{2\alpha}\right) \to x_1=\frac{2\alpha\, w}{p_1}$$

now find x_2:

$$x_2=\frac{1-2\alpha}{2\alpha}\cdot\frac{p_1}{p_2}\,x_1=\frac{1-2\alpha}{p_2}\cdot w$$

$$\Rightarrow x^{*}=\left(\underbrace{\frac{2\alpha\, w}{p_1}}_{x_1}\quad\underbrace{\left(\frac{1-2\alpha}{p_2}\right)w}_{x_2}\right)$$

LOOKING AT THIS, CONSIDERING x_1 WHAT ABOUT THE DERIVATIVE? IS Ø SO THE TWO GOODS ARE INDEPENDENT

THE SHARE OF INCOME SPENT IN x_1 = $\dfrac{\cancel{p_1}\cdot\dfrac{2\alpha\cdot\cancel{w}}{\cancel{p_1}}}{\cancel{w}}=2\alpha$

THE SHARE OF INCOME SPENT IN x_2 = $\dfrac{\cancel{p_2}\quad\dfrac{1-2\alpha}{\cancel{p_2}}\cdot w}{\cancel{w}}=1-2\alpha$

2) Condition on α? α should be positive

$$x^{\alpha} = \frac{2\alpha w}{p_1} \;;\; \frac{1-2\alpha}{p_2} w$$

$\alpha > 0$

$\rightarrow 1 - 2\alpha > 0 \rightarrow -2\alpha > -1$

$\rightarrow 2\alpha < 1 \quad \alpha < \frac{1}{2}$

$\alpha \in \left(0, \frac{1}{2}\right)$ for interior condition, both x_1 and x_2 are positive

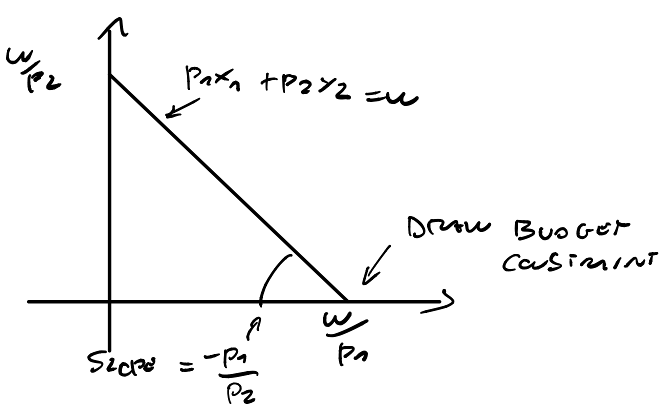
So also $\alpha \in \left[0, \frac{1}{2}\right]$

$\alpha x_1 + \beta x_2$ if x_1 or x_2 is $\emptyset$ the value of utility can still not be $\emptyset$

$\boxed{\text{Ex. 1 Ch. 2}}$

Linear function

$u(x) = 3x_1 + 4x_2$ s.t. $p_1 x_1 + p_2 x_2 \leq w$



$$L = 3x_1 + 4x_2 - \lambda(w - p_1 x_1 - p_2 x_2)$$

$$\begin{cases} \frac{\delta L}{x_1} = 3 - \lambda p_1 = 0 \\ \frac{\delta L}{x_2} = 4 - \lambda p_2 = 0 \\ \frac{\delta L}{\lambda} = w + p x_1 + p_2 x_2 \end{cases} \Rightarrow \boxed{\frac{3}{4} = \frac{p_1}{p_2}}$$

|MRS| SLOPE OF B.C.

THIS CAN BE TRUE ONLY IF $\frac{p_1}{p_2}$ IS $\frac{3}{4}$

(INTERNAL?)

THERE IS NO INTERIOR SOLUTION

$\Rightarrow$ WALRAS DEMAND IS NOT A FUNCTION BUT CORRESPONDE THE OPT SOLUTIONS ARE ALL POSITIVE ON THE BUDGET CONSTRAINT

NO TANGENCY POINT BUT THEY OVERLAP

OTHER WAY

UTILITY LEVEL K

$$3x_1 + 4x_2 = K$$

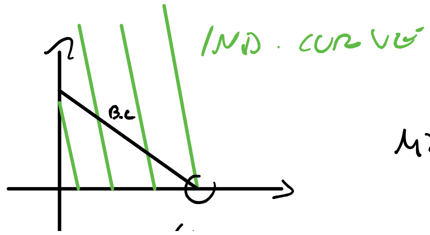
$$x_2 = -\frac{3x_1}{4} + \frac{K}{4}$$

THEN WHAT IS THE SLOPE? IN ABSOLUTE VALUE

IS THE COEFFICIENTS OF $x_1 = -\frac{3}{4}$ SO $\left|-\frac{3}{4}\right| = \frac{3}{4}$

NOW IMAGINE $MRS > \frac{p_1}{p_2}$ SO $\frac{3}{4} > \frac{p_1}{p_2}$

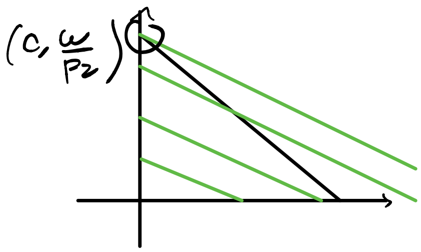
IND. CURVES ARE ... THAN THE B.C



$$MRS > \frac{p_1}{p_2}$$

1 $\left(\frac{w}{p_1}, 0\right)$

corner solution with $x_2 = 0$ $x_1 = \frac{w}{p_1}$



MRS $\left(\frac{3}{4}\right)$ $< \frac{p_1}{p_2}$ $\rightarrow$ corner solution with $x_1 = 0$

$x_2 = \frac{w}{p_2}$

$x^* = \left(0, \frac{w}{p_2}\right)$

Expenditure Minimization Problem

and connection between functions

Expenditure Minimization Problem

- Expenditure minimization problem (EMP):

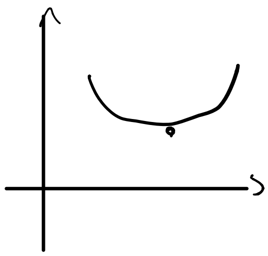
$$\min_{x \geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$
$$(\text{i.e. } u(x) - u \geq 0)$$

- Alternative to utility maximization problem
- NB. $\min_{x \geq 0} p \cdot x = \max_{x \geq 0} -(p \cdot x)$ I can set up this as a maximization problem, and use what we already know.

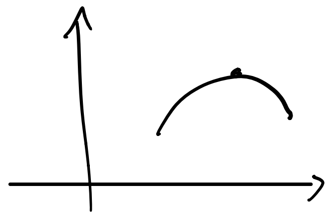
In the previous problem we have the budget constraint and we have ...

If you want to translate this problem in a optimization problem we can maximise the opposite of the max.

IF I WANT MIN



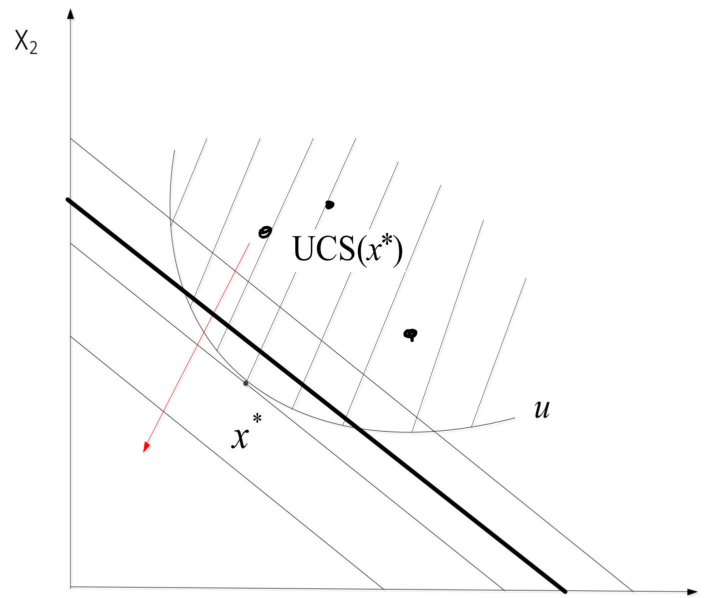
INVERT AND TAKE MAX



Expenditure Minimization Problem

WE WANT TO FIND BUDGET CONSTRAINT

- Consumer seeks a utility level associated with a particular indifference curve, while spending as little as possible.
- Bundles strictly above x^* cannot be a solution to the EMP:
  - They reach the utility level u
  - But, they do not minimize total expenditure
- Bundles on the budget line strictly below x^* cannot be the solution to the EMP problem:
  - They are cheaper than x^*
  - But, they do not reach the utility level u



LOWER POINT IS THE TANGENCY POINT BETWEEN CONSTRAINT AND LOWEST POSSIBLE BUDGET CONSTRAINT

Expenditure Minimization Problem

VECTORS AND PRODUCT $p_1x_1+p_2x_2$

TAKE OPPOSITE

$-p_1x_1-p_2x_2+w[u(x_1,x_2)]$

DERIVATIVE

$-p_1+w\dfrac{\partial u}{\partial x_1}\le 0$

ALSO HERE COMPLEMENTARY SLACKNESS COND

- Lagrangian

$$L=-p\cdot x+\mu[\underbrace{u(x)-u}_{\geq 0}]$$

- FOCs (necessary conditions)

$$\frac{\partial L}{\partial x_k}=-p_k+\mu\frac{\partial u(x^*)}{\partial x_k}\le 0$$

[= 0 for interior solutions]

$=0\;;\;x_k>0$

$<0\;;\;x_k=0$

$k=1,2$

$$\frac{\partial L}{\partial \mu}=u(x^*)-u\ge 0$$

Take the opposite of the maximal function.

The second is the constraint and i add the la grangian multiplier (mu) which multiply the budget constrain.

INTERIOR SOLUTION

$x_k > 0$

BY GRAPHICAL SOLUTION

POINT IN UCS ARE NOT OPTIMAL

BUT MUST BE EXACTLY IN THE IC SO = !!

AND WE CAN FOCUS ON

= CASE

⇓

$u(x^*) - \bar{u} = 0$

↓

THEN MOVE

ON INTERIOR

AND (NEXT SLIDE)

Expenditure Minimization Problem

- For interior solutions,

$K = 1,2$

$$p_k = \mu \frac{\partial u(x^*)}{\partial x_k} \quad \text{or} \quad \frac{1}{\mu} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

SINCE BOTH EQUAL TO $\frac{1}{\mu}$

for any good k. This implies,

SLOPE OF BUDGET CON = MRS

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} = \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} \quad \text{or} \quad \frac{p_k}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{\frac{\partial u(x^*)}{\partial x_l}}$$

$$\frac{1}{\mu} = \frac{\frac{\partial u}{\partial x_1}}{p_1}$$

$$\frac{1}{\mu} = \frac{\frac{\partial u}{\partial x_2}}{p_2}$$

SO THEY ARE THE SAME

SAME COND. OF 1€ IN ONE GOOD THEN I HAVE TO SPEND 1€ IN OTHER GOODS

- The consumer allocates his consumption across goods until the point in which the marginal utility per dollar spent on each good is equal across all goods (i.e., same “bang for the buck”).

- That is, the slope of indifference curve is equal to the slope of the budget line. **(i.e. the “usual tangency condition”)**

EMP: Hicksian Demand

- The bundle $x^* \in \text{argmin}\, p \cdot x$ (the argument that solves the EMP) is the ***Hicksian demand***, which depends on p and u (while **Walrasian demand depends on *p* and *w***),

$$x^* \in h(p, u)$$

- Recall that if such bundle x^* is unique, we denote it as $x^* = h(p, u)$ (i.e. it is a function not a correspondance).

Walras demand is the solution of maximisation problem. Similar we get the same with minimum problem and is called the Hicksian demand.

Walras demand depends on the price and the wealth that are the parameter in the budget constraint. While x is the choice variable.

$x(p, w)$ x -> choice

price wealth

Parameters appearing? Price parameter, is u parameter? Yes.
Hicksian depend on price and utility! So it’s different.

Solution is unique.... set of bundle???
[24]

If both price and u increase by alpha then ratio between price doesn’t change. Bundle does’t change but expenditure does change! P X* —> alpha P X*.
To reach that utilty level you spend more!

Properties of Hicksian Demand

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $h(p, u)$ satisfies: is just increasing p not u

  1) ***Homog(0)*** in p, i.e., $h(p, u) = h(\alpha p, u)$ for any p, u, and $\alpha > \cancel{0}$ 1

     - If $x^* \in h(p, u)$ is a solution to the problem
$$\min_{x \geq 0} p \cdot x$$
     then it is also a solution to the problem
$$\min_{x \geq 0} \alpha p \cdot x$$
     - *Intuition*: a common change in all prices does not alter the slope of the consumer's budget line.

Properties of Hicksian Demand

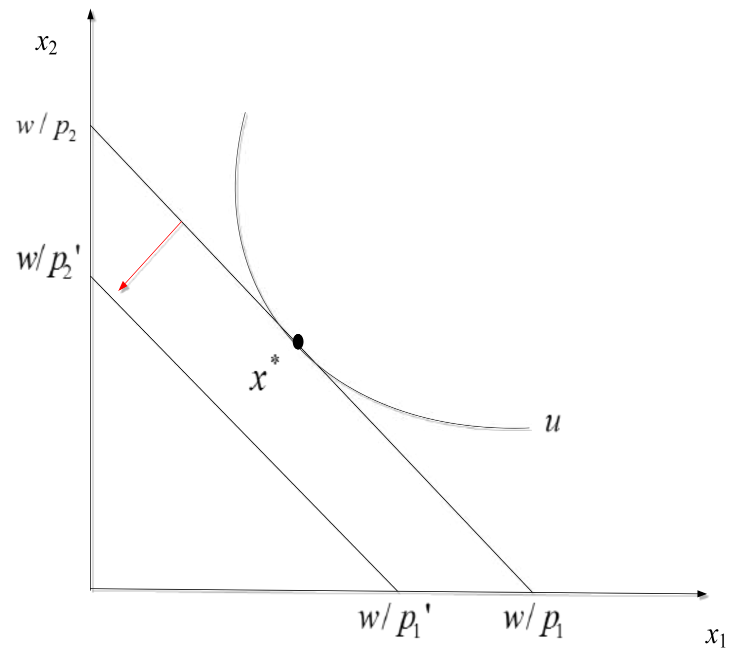
- x^* is a solution to the EMP when the price vector is $p = (p_1, p_2)$.
- Increase all prices by factor α

$$p' = (p_1', p_2') = (\alpha p_1, \alpha p_2)$$

- Downward (parallel) shift in the budget line, i.e., the slope of the budget line is unchanged.
- But I have to reach utility level u to satisfy the constraint of the EMP!
- Spend more to buy bundle $x^*(x_1^*, x_2^*)$, i.e.,

$$p_1' x_1^* + p_2' x_2^* > p_1 x_1^* + p_2 x_2^*$$

- Hence, $h(p, u) = h(\alpha p, u)$

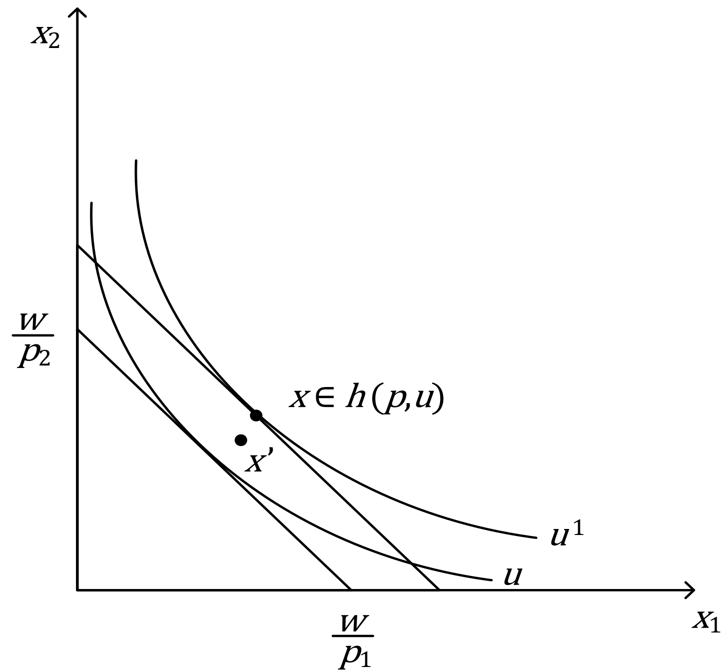


Properties of Hicksian Demand

2) *No excess utility:*

for any optimal consumption bundle $x \in h(p, u)$, utility level satisfies $u(x) = \bar{u}$.

(That is the level of utility fixed in the constraint)



NB. Equivalent of Walras' Law in UMP
(constraint holds with equality)

Properties of Hicksian Demand

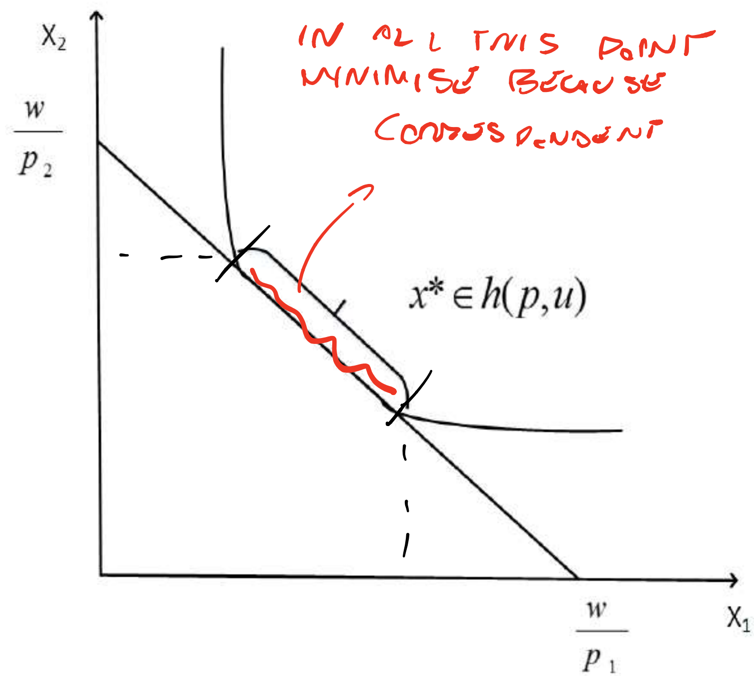
- *Intuition*: Suppose there exists a bundle $x \in h(p, u)$ for which the consumer obtains a utility level $u(x) = u^1 > u$, which is higher than the utility level u he must reach when solving EMP.
- But we can then find another bundle $x' = x\alpha$, where $\alpha \in (0,1)$, very close to x ($\alpha \rightarrow 1$), for which $u(x') > u$.
- Bundle x':
  - is cheaper than x since it contains fewer units of all goods; and
  - exceeds the minimal utility level u that the consumer must reach in his EMP.
- We can repeat that argument until reaching bundle x.
- In summary, for a given utility level u that you seek to reach in the EMP, bundle $h(p, u)$ does not exceed u. Otherwise you can find a cheaper bundle that exactly reaches u.

Properties of Hicksian Demand

CORRESPONDENTS

*3) **Convexity:***

If the preference relation is convex, then $h(p,u)$ is a convex set.

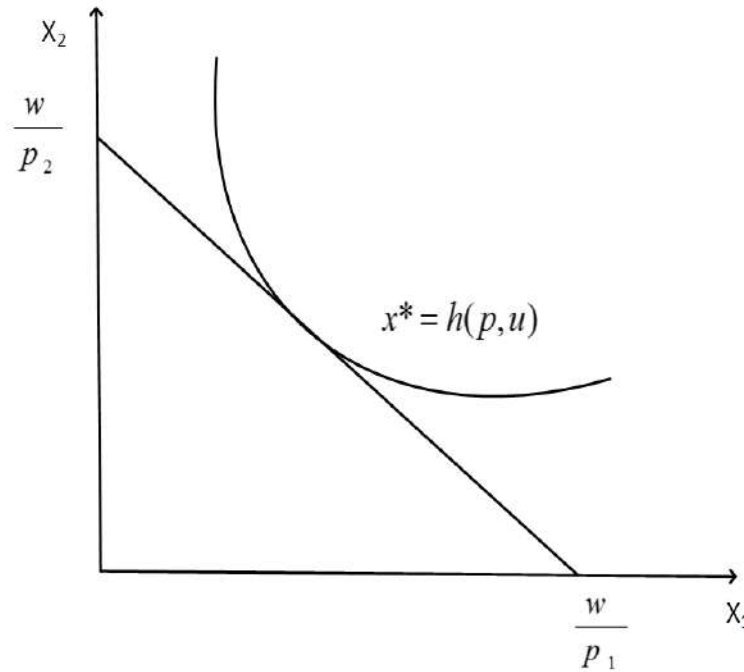


Properties of Hicksian Demand

IF CONVEX, SOLUTION IS UNIQUE

4) Uniqueness:

If the preference relation is strictly convex, then $h(p,u)$ contains a single element.



COMPENSATED DEMAND

PRODUCT MUST BE SO THIS HOLDS FOR
EVERY GOOD IN BUNDLE

$p' > p$ OR $p' < p$

$$\underbrace{(p'-p)}_{\overset{>}{<0}} \cdot \underbrace{[h(p',u) - h(p,u)]}_{\overset{<0}{>0}} \leq 0$$

IF P INCR HICK DEMAND INCR.
SAMOUNT IN OPPOSITE DIRECTION

Properties of Hicksian Demand

- ***Compensated Law of Demand:*** for any change in prices p and p',

$$(p'-p) \cdot [h(p',u) - h(p,u)] \leq 0$$

  - *Implication*: for every good k,

$$(p'_k - p_k) \cdot [h_k(p',u) - h_k(p,u)] \leq 0$$

  - This is true for Hicksian (also named "compensated") demand, but not necessarily true for Walrasian demand (which is uncompensated). This means that movements in prices and movements in quantities must go in **opposite direction**.
    - The following will be clear later, when we introduce income and substitution effects:
    - Recall the figures on Giffen goods, where a decrease in p_k in fact decreases $x_k(p,u)$ when wealth was left uncompensated.
    - Meaning: changes in prices and changes in compensate demand always go in opposite directions (if price increases demand falls, if price falls demand increases)

$p \cdot x \rightarrow$ EXPENDITURE

$\uparrow$ SUBSTITUTE

$h(p,w)$

The Expenditure Function

- Plugging the result from the EMP, $h(p,u)$, into the objective function, $p \cdot x$, we obtain the value function of this optimization problem,

$$p \cdot h(p,u) = e(p,u)$$

THIS IS CALLED EXPENDITURE FUNCTION

where $e(p,u)$ represents the ***minimal expenditure*** that the consumer needs to incur in order to reach utility level u when prices are p.

This is called expenditure function.

Properties of Expenditure Function

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $e(p, u)$ satisfies:

$e(\alpha p, u) = (\alpha p)\, h(\alpha p, u) = \alpha\, p \cdot h(p, u) =$

$= \alpha \cdot e(p, u)$ → This is homog of deg 1.

1) *Homog(1)* in p,

$\alpha > 0$

$$e(\alpha p, u) = (\alpha p) h(\alpha p, u) = \alpha[p \cdot h(p, u)] = \alpha \cdot e(p, u)$$

for any p, u, and $\alpha > 0$.

  - We know that the optimal bundle is not changed when all prices change, since the optimal consumption bundle in $h(p, u)$ satisfies homogeneity of degree zero.
  - Such a price change just makes it more or less expensive to buy the same bundle.

Properties of Expenditure Function

→ INCR UTILITY WE NEED TO HIGH I.C. SO BUDGET WILL BE HIGHER

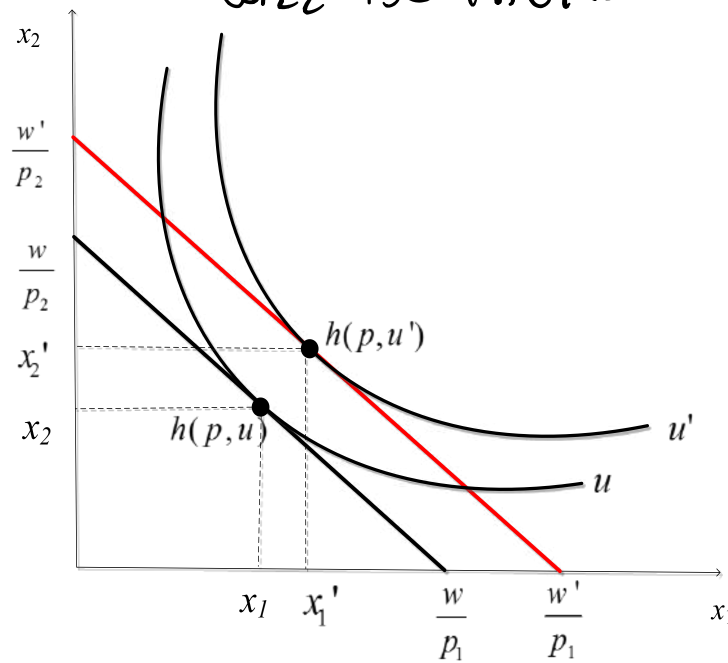
2) *Strictly increasing in u:*
For a given price vector, reaching a higher utility requires higher expenditure:

$$p_1 x_1' + p_2 x_2' > p_1 x_1 + p_2 x_2$$

where $(x_1, x_2) = h(p, u)$ and $(x_1', x_2') = h(p, u')$.

Then,

$$e(p, u') > e(p, u)$$



IF P ↑, MINIMIZE EXP ↑

Properties of Expenditure Function

3) *Non-decreasing in p_k for every good k:*

Higher prices mean higher expenditure to reach a given utility level.

- Let $p' = (p_1, p_2, \dots, p'_k, \dots, p_L)$ and $p = (p_1, p_2, \dots, p_k, \dots, p_L)$, where $p'_k > p_k$.
- Let $x' = h(p', u)$ and $x = h(p, u)$ from EMP under prices p' and p, respectively.
- Then, $p' \cdot x' = e(p', u)$ and $p \cdot x = e(p, u)$. Mc

SIMC

$$e(p', u) = p' \cdot x' \ \geq \ p \cdot x' \ \geq \ p \cdot x = e(p, u)$$

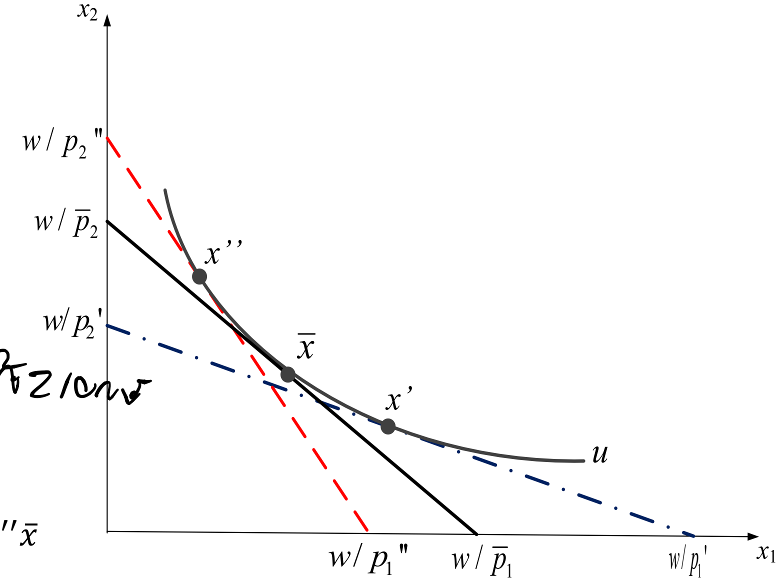
- 1st inequality due to $p' \geq p$
- 2nd inequality: at prices p, bundle x minimizes EMP.

Properties of Expenditure Function

4) Concave in p:

Let $x' \in h(p', u) \Rightarrow p'x' \leq p'x$
$\forall x \neq x'$, e.g., $p'x' \leq p'\bar{x}$
and
$x'' \in h(p'', u) \Rightarrow p''x'' \leq p''x$
$\forall x \neq x''$, e.g., $p''x'' \leq p''\bar{x}$
where $\bar{x} = \alpha x' + (1-\alpha)x''$

NO DIMOSTRAZIONE



This implies

$$\alpha p'x' + (1-\alpha)p''x'' \leq \alpha p'\bar{x} + (1-\alpha)p''\bar{x}$$

$$\alpha \overbrace{p'x'}^{e(p',u)} + (1-\alpha)\overbrace{p''x''}^{e(p'',u)} \leq [\underbrace{\alpha p' + (1-\alpha)p''}_{\bar{p}}]\bar{x}$$

$$\alpha e(p',u) + (1-\alpha)e(p'',u) \leq e(\bar{p},u)$$

as required by concavity

Connections

Relationship between the Expenditure and Hicksian Demand

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}^L_+$. For all p and u,

WE CAN DERIVE HICKSIAN FROM EXPENDITURE

$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) \text{ for every good } k$$

This identity is "***Shepard's lemma***": if we want to find $h_k(p,u)$ and we know $e(p,u)$, we just have to differentiate $e(p,u)$ with respect to prices.

- *Proof*: three different approaches
    1) the support function
    2) first-order conditions
    3) the envelope theorem

(See Appendix 2.2)

IN EXERCISES THEY COULD GIVE US EXPENDITURE IN MINIMUM → WE CAN COMPUTE HICKSIAN USING DER IN RESPECT TO p_k

Proof of Shephard’s lemma (using “Envelope theorem”)

$$e(p,u) = \min_{x \geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

To see how e(.) changes when a parameter p_k changes we can use the Langrangian

$L = -(p \cdot x)+\mu(u(x) - u)$ (remember we set it as a max problem)

In particular

$$\frac{\partial e(p,u)}{\partial p_k} = -\left[\frac{\partial L}{\partial p_k}|_{x=x^*(p)}\right] = -\frac{\partial[-p \cdot x(p))+\mu(u\big(x(p)\big) - u]}{\partial p_k}|_{x=x^*(p)}$$

$$= -[-x_k(p) - p\frac{\partial x}{\partial p_k} + \mu\frac{\partial u}{\partial x}\frac{\partial x}{\partial p_k}]|_{x=x^*(p)}$$

But $-p + \mu\frac{\partial u}{\partial x} = 0$ from FOCs then $\frac{\partial e(p,u)}{\partial p_k}$= $x_k(p)|_{x=x^*(p)} = h_k(p,u)$

(NB. p, $x(p)$, $\frac{\partial u}{\partial x} = \nabla u\big(x(p)\big)$, $\frac{\partial x}{\partial p_k} = D_{p_k}x(p)$ are vectors, while μ a scalar)

- Take opposite
- Write lagrangian

The opt will be x* so computing minimum deriving la grandina in respect of pk. The values of the problem computed in the opt should be the same. I take der of Exp in respect to pk that will be der of L with respect to pk.

Next i take a minus since I translated the min problem in the max problem.
X is a function of p since if we change p then will change the opt solution that is x*. We write x as a function of p. Also x is a function in p computing the derivative.

With a composite function we have first to derive in respect to the second function moltiply by the derive the second function in respect with the parameter par.

$$\frac{\partial u}{\partial x(p)}(x\ p) \cdot \frac{\partial x(p)}{\partial p}$$

$$\frac{\partial x}{\partial p_k}\left(-p + \lambda \frac{\partial u}{\partial x}\right)$$

look at EMP in which $p = \lambda \frac{\partial u}{\partial x}$

$$\frac{\partial x}{\partial p_k}\left(\overset{0}{-p + p}\right)$$

so $h_k(p, u)$

this is env theorem

If we have opt problem you can forget all the der involving the constraint, you can just derive in the Expednciture function the part that is related to the objective function.

Relationship between Hicksian and Walrasian Demand

- We can formally relate the Hicksian and Walrasian demand as follows:
  - Consider $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}^L_+$.
  - Consider a consumer facing $(\bar{p}, \bar{w})$ and attaining utility level $\bar{u}$ (i.e. solution of UMP) [SOLUTION OF UMP]
  - Note that $\bar{w} = e(\bar{p}, \bar{u})$, i.e. the min expenditure that the consumer bear to reach utility $\bar{u}$ is $\bar{w}$. In addition, we know that for any (p, u), $h_l(p, u) = x_l(p, \underbrace{e(p, u)}_{w})$. Differentiating this expression with respect to p_k, and evaluating it at $(\bar{p}, \bar{u})$, we get:

COMPOSITE FUNCTION

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \boxed{\frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial e(\bar{p}, \bar{u}))} \boxed{\frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}}}$$

$\hat{u}$ max UMP $x(\bar{p}, \bar{w})$ $\bar{w}$

Utilty maximisation problem: How much consumer spend in opt solution in this UMP?
W (barrato). So u(bar) is the max utility in UMP.
To reach u maximising u and the level of wealth then it must be the case is the w(bar)

SOLVE SEPARATELY EQUAL THEN HICKSIAN DEMAND

I have p bar and w bar. Reach level of utilty b bar and w bar. What is the expenditure of this walras demand? Is the w bar. Now I'm saying, what is the min exp to reach

$$h_\ell(\bar p, \bar w) = x_\ell(\bar p, \bar e(\bar p, \bar w))$$

WALRAS DEPEND ON P AND W. REPLACE WITH IDENTITY BETWEEN AND THIS IS $\bar w$

WE HAVE EXP OPT PROBLEM. IMAGINE TO SOLVE THESE GIVING MAXIMUM, SO THIS IS UTILITY

$\ell = k$

$\ell \neq k$

Relationship between Hicksian and Walrasian Demand

- Using the fact that $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$ (Shepard's lemma),

(SHEPPARD LEMMA)

I CAN REPLACE $\bar{w}$ WITH EXPENDITURE

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial e(\bar{p},\bar{u}))} h_k(\bar{p},\bar{u})$$

- Finally, since $\bar{w} = e(\bar{p},\bar{u})$ and $h_k(\bar{p},\bar{u}) = x_k\big(\bar{p},e(\bar{p},\bar{u})\big) = x_k(\bar{p},\bar{w})$, then

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k} + \frac{\partial x_l(\bar{p},\bar{w})}{\partial \bar{w}} x_k(\bar{p},\bar{w})$$

HOW I CAN LINK HICKSIAN DEM WITH WALRAS DEMAND

I CAN REPLACE e WITH w

Slutsky equation correspond to total effect and income effect. So

$$\underbrace{\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k}}_{SE} = \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k}}_{TE} + \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial w} x_k(\bar{p},\bar{w})}_{IE}$$

$$\underbrace{\frac{\partial x_l}{\partial p_k}}_{TE} = \frac{\partial h_l}{\partial p_k} - \frac{\partial x_l}{\partial w} \cdot x_k \quad IE$$

Tot diff of two two

Relationship between Hicksian and Walrasian Demand

- This is the so called **Slutsky equation:** The total effect of a price change on Walrasian demand can be decomposed into a substitution effect and an income effect:

$$\underbrace{\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k}}_{SE} = \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k}}_{TE} + \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial \bar{w}} x_k(\bar{p},\bar{w})}_{IE}$$

Or, more compactly, $SE = TE + IE$ or TE = SE - IE

Where SE = **substitution effect**

TE = **total effect**

IE = **income effect**

TE, IE, SE

- ***Total Effect:*** measures how the quantity demanded is affected by a change in the price of good l, when we leave the wealth uncompensated (Walras demand is also called **uncompensated demand**).
- ***Substitution Effect***: measures how the quantity demanded is affected by a change in the price of good l, after the wealth adjustment which allows the consumer to reach the same utility as before the price change. Is given by Hicksian demand that is also called **compensated demand**.
  - That is, the substitution effect only captures the change in demand due to variation in the price ratio, but abstracts from the larger (smaller) purchasing power that the consumer experiences after a decrease (increase, respectively) in prices.
- ***Income Effect:*** measures the change in the quantity demanded as a result of the wealth adjustment.

IF P ONE GOOD MORE EXPENS
THEN I W/CC BUY LESS SECOND GOOD

Rent increase I’ll go to the second house, i consume a little bit houses. This means that we are left with less income to buy less good. So increasing price of one good will reduce the consumption of others good even if you don’t change the consumption of one good.

Inflation is an exemple. If i consume the same bundle ...[1.31]
So this is the income effect.

Substitution effect relate to the fact of compensate the Hicksian demand. When computing Hicksian demand we gave a utilty level .. to the price before. How the bundle changes when we keep the consumer in the same IC as the prices changes. So neutralising the effect on well. Slutsky ...

This is the slutsky equation. In the left wee have SE that is the change in Hicksian demand. The change in the Hicksian demand depend in the price change = TE + IE.

We can compute this effect for each good: der first good with respect to price of first good or der in the quantity and changing price. With 2 good we have 4 derivatives. This 4 can be put in a matrix called Slutsky matrix.

A generic term slk(p,w)

Slutsky matrix

- All these derivatives can be collected into a matrix, the so called ***Slutsky (or substitution) matrix***

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

Cross price effect out of the main diagonal

where each element in the matrix is

[illegible] EFFECT

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

$\frac{\partial h_l}{\partial p_k}$

Implications of WARP: Slutsky Matrix

Just know this about WARP

- ***Proposition:*** If preferences satisfy LNS and strict convexity, and they are represented with a continuous utility function, then the Walrasian demand $x(p, w)$ generates a Slutsky matrix, $S(p, w)$, which is symmetric.

- The above assumptions are really common.
  – Hence, the Slutsky matrix will then be symmetric.

- However, the above assumptions are not satisfied in the case of preferences over perfect substitutes (i.e., preferences are convex, but not strictly convex).

Implications of WARP: Slutsky Equation

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect}} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\text{Total effect}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\text{Income effect}}$$

- ***Total Effect:*** measures how the quantity demanded is affected by a change in the price of good l, when we leave the wealth uncompensated.
- ***Income Effect:*** measures the change in the quantity demanded as a result of the wealth adjustment.
- ***Substitution Effect***: measures how the quantity demanded is affected by a change in the price of good l, after the wealth adjustment.
  - That is, the substitution effect only captures the change in demand due to variation in the price ratio, but abstracts from the larger (smaller) purchasing power that the consumer experiences after a decrease (increase, respectively) in prices.

Why is useful to decompose total effect changing? We see how quantity changes depending on the characteristics of the goods.

Implications of WARP: Slutsky Matrix

If weak (WARP) .. holds then substitution effect is negative —> Hicksian demand decrease

- Let us focus now on the signs of the IE and SE (implied by WARP, i.e. of the utilities that we will use) in case of $P_l \uparrow$
- Non-positive substitution effect, $s_{ll} \leq 0$:

WALRASIAN ↑

SE always non positive

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect }(-)} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\substack{\text{Total effect:} \\ (-)\text{ usual good} \\ (+)\text{ Giffen good}}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\substack{\text{Income effect:} \\ (+)\text{ normal good} \\ (-)\text{ inferior good}}}$$

Giffen: if price increase, demand increases so this derivative increase

- Substitution Effect = Total Effect + Income Effect

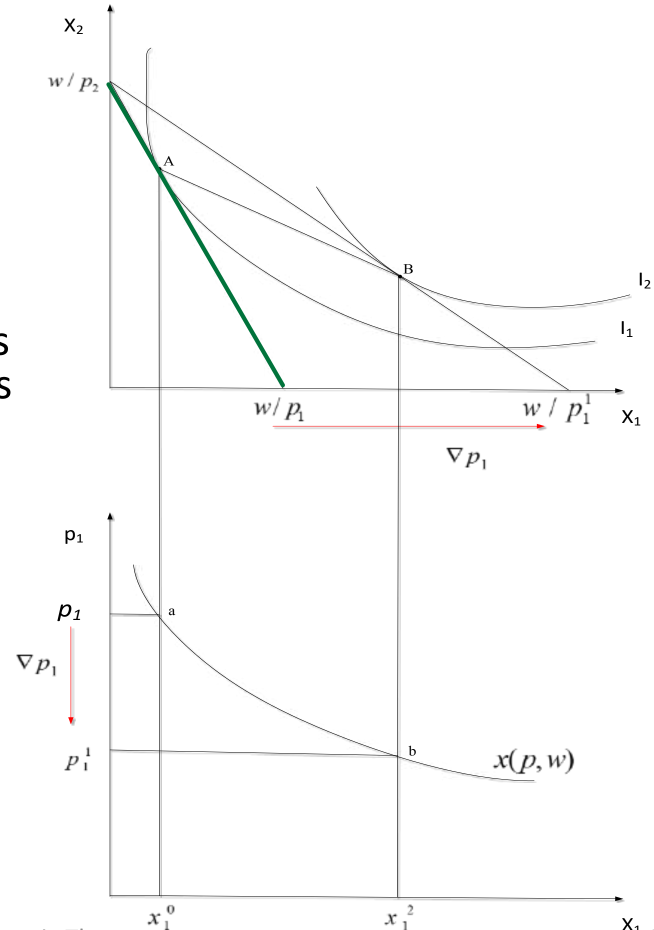
  $\Rightarrow$ Total Effect = Substitution Effect - Income Effect

If SE decrease and TE positive mean that IE should be negative and greater than TE. So x(p,w) should be > 0 so derivative is negative. Giffen good can only be inferior good by definition. But not only inferior good are Giffen. If income increase i call it normal.

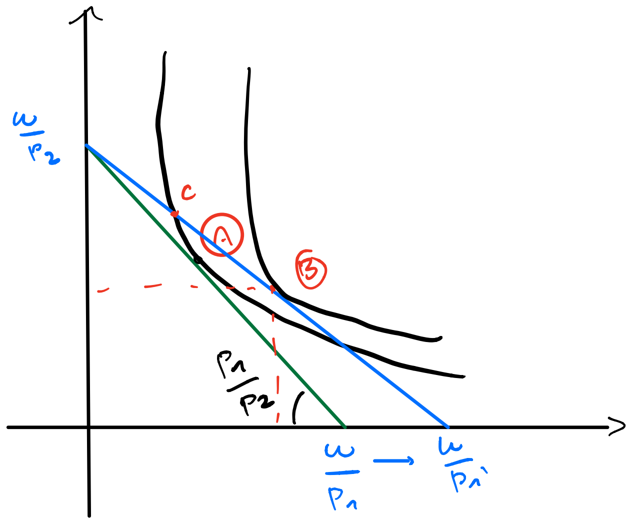
Decompose the two effect graphically.

Graphical representation: Slutsky Equation

- Reduction in the price of x_1.
  - It enlarges consumer's set of feasible bundles.
  - He can reach an indifference curve further away from the origin.
- The Walrasian demand curve indicates that a decrease in the price of x_1 leads to an increase in the quantity demanded.
  - This induces a negatively sloped Walrasian demand curve (so the good is "normal").
- The increase in the quantity demanded of x_1 as a result of a decrease in its price represents the ***total effect (TE)***.



We start from a given budget constraint with price p1. The solution of consumer problem is the tangency point between the IC and the budget constraint. We call this point A.



- IF $p_1 \downarrow$: p_1'

→ x_1 USUAL GOOD SINCE QUANTITY INCR.

IF C NEW TANG POINT, THEN WE HAVE GIFFEN GOOD (P↑, DEMAND↑)

GIFFEN GOOD

$p \uparrow \quad q \uparrow$

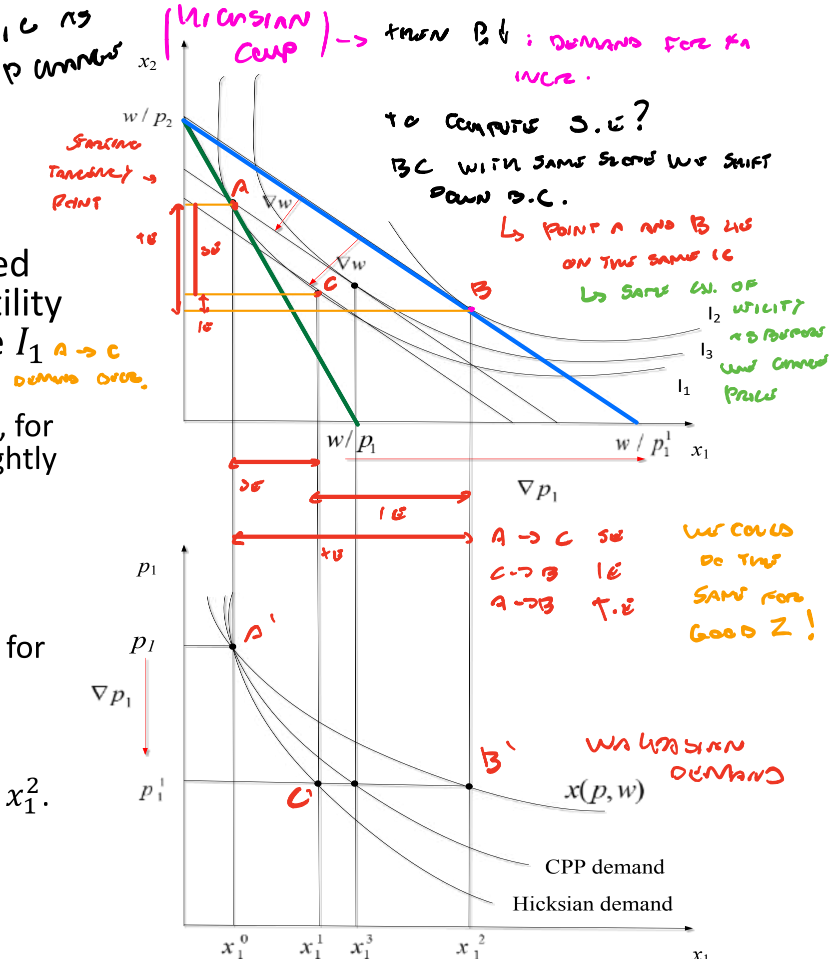
$p \downarrow \quad q \downarrow$

→ P AND Q GO IN THE SAME DIRECTION

Graphical representation: Slutsky Equation

- Reduction in the price of x_1.
  - ***Hicksian wealth compensation*** (i.e., "constant utility" demand curve)?

same IC as before p changes



- The consumer's wealth level is adjusted so that she can still reach her initial utility level (i.e., the same indifference curve I_1 as before the price change).
  - The Hicksian demand curve reflects that, for a given decrease in p_1, the consumer slightly increases her consumption of good one.

- In summary, a given decrease in p_1 produces:
  - A small increase in the Hicksian demand for the good, i.e., from x_1^0 to x_1^1.
  - (neglect CCP)
  - A substantial increase in the Walrasian demand for the product, i.e., from x_1^0 to x_1^2.

Focus on good x_1 and compare A and C.

A to B for Walrasian demand

A to C is Hicksian demand

For Walrasian demand the changing demand is larger

What does it imply?

Point C is lower than A and B

So Walrasian demand is higher

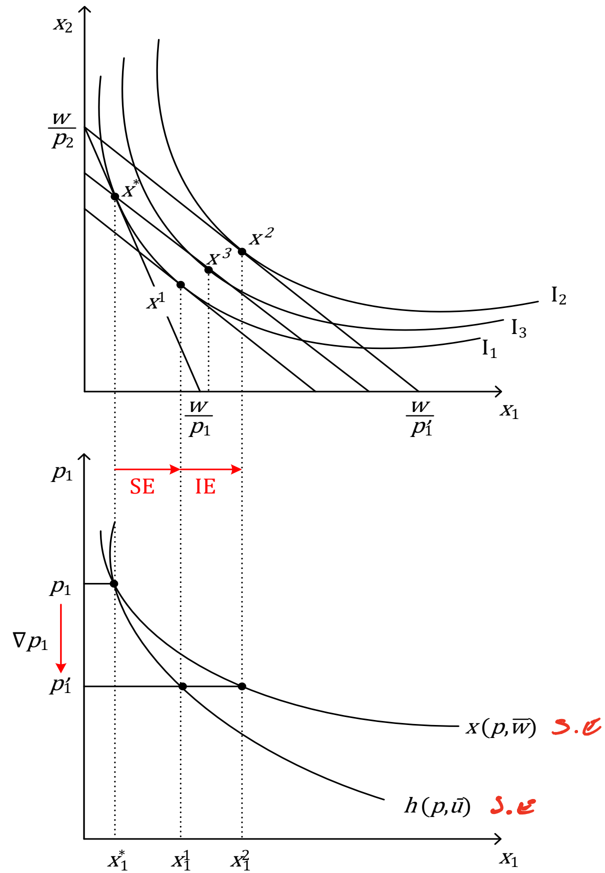
Implications of WARP: Slutsky Equation

- A decrease in price of x_1 leads the consumer to increase his consumption of this good, Δx_1, but:
  - The Δx_1 which is solely due to the price effect (measured by the Hicksian demand curve) is smaller than the Δx_1 measured by the Walrasian demand, $x(p, w)$, which also captures wealth effects.

Relationship between Hicksian and Walrasian Demand

$P_1 \downarrow$: $h \uparrow x \uparrow$

- When income effects are positive (*normal goods*), then the Walrasian demand $x(p, w)$ is **above** the Hicksian demand $h(p, u)$.
  - The Hicksian demand is *steeper* than the Walrasian demand.



$P_1 \downarrow$: I.I. ↑ DE

↓

YOU ARE RICHER
SINCE PURCHASING POWER
INCREASES (INFLATION)

S.E ALWAYS DRIVE CONSUMPTION UP
IF GOOD NORMAL

IF GOOD INFERIOR P↓ D↓
IE DRIVE CONSUMPTION DOWN

IF GOODS INCREASE DIR OF SE
WILL BE HICKSIAN DEMAND ABOVE
OR BELOW WALRASIAN ?
ABOVE

Relationship between Hicksian and Walrasian Demand

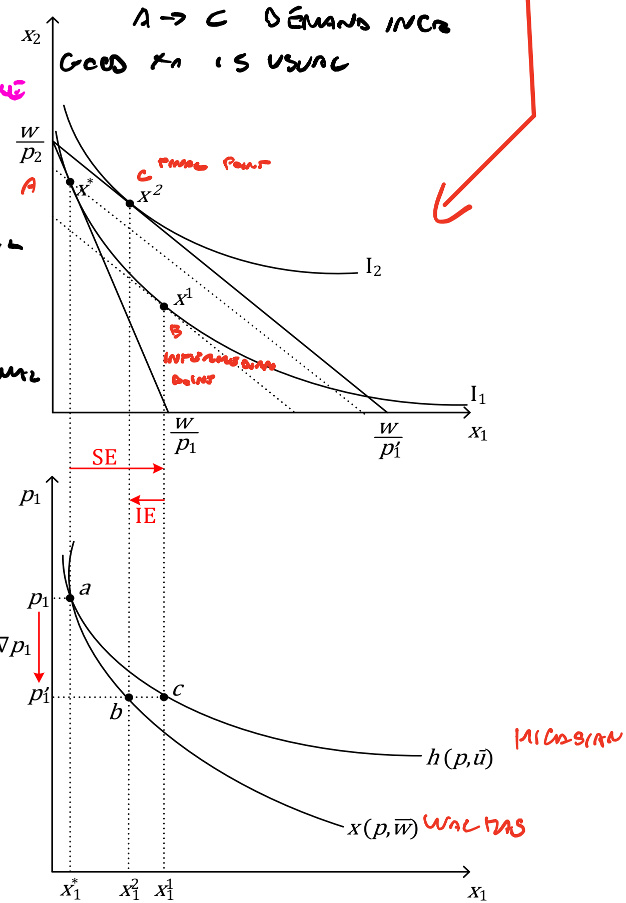
- When income effects are negative (*inferior goods*), then the Walrasian demand $x(p,w)$ is **below** the Hicksian demand $h(p,u)$.
  - The Hicksian demand is *flatter* than the Walrasian demand.

USUAL PRICE
NORMAL INCOME

$\frac{\delta x_1}{\delta p_1} < 0$ USUAL

$\frac{\delta x_1}{\partial w} > 0$ NORMAL

hence $p \downarrow$ $D \uparrow$ so



A → B SUBSTITUTION EFFECT (CONSUMPT INC)

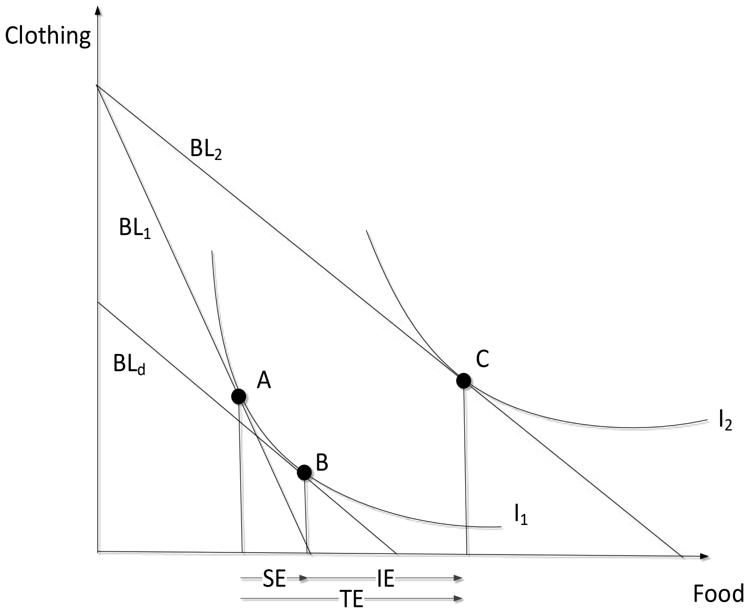
B → C INCOME EFFECT (CONSUMP DECRE)

SE > IE ↓ GOOD IS INFERIOR → SE GO TO IE CONTRARY

↳ HICKSIAN > WALRAS

Substitution and Income Effects: Normal Goods

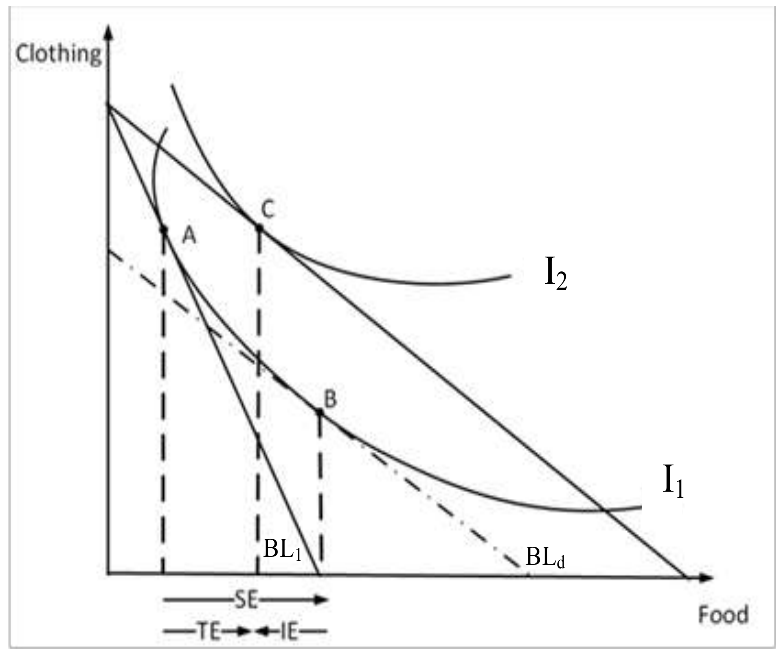
- Decrease in the price of the good in the horizontal axis (i.e., food).
- The substitution effect (SE) moves in the opposite direction as the price change.
  - A reduction in the price of food implies a positive substitution effect.
- The income effect (IE) is positive (thus it reinforces the SE).
  - The good is normal.



SE, IE SAME DIRECTION

Substitution and Income Effects: Inferior Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is now negative (which partially offsets the increase in the quantity demanded associated with the SE).
  - The good is inferior.
- *Note*: the SE is larger than the IE (Law of price still holds)

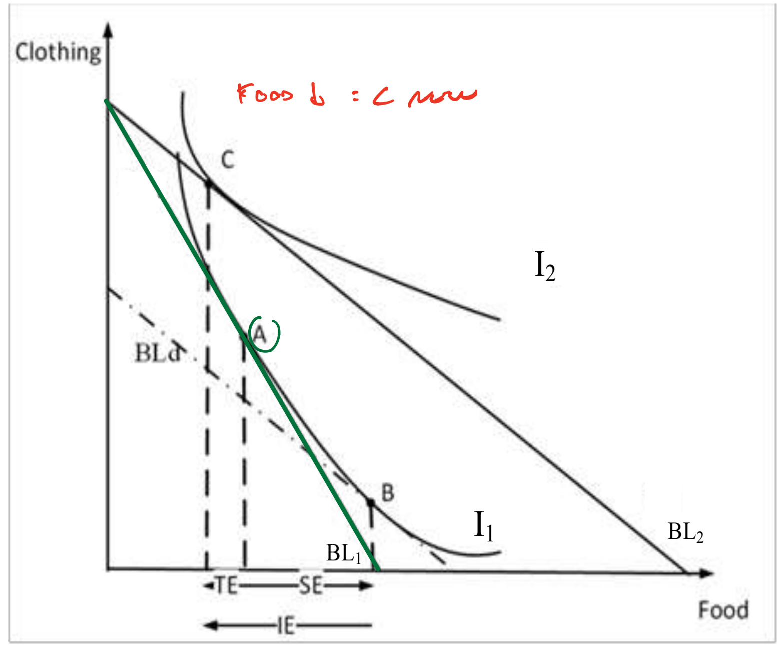


TE < SE

Substitution and Income Effects: Giffen Goods

→ When you are poor you consume it a lot!

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is still negative but now completely offsets the increase in the quantity demanded associated with the SE.
  - The good is Giffen good.
- *Note*: the SE is less in absolute value than the IE (Law of price does not hold)



Food is Giffen good.

In point B compensation demand

TE negative.

When you get richer you consume is very low

Substitution and Income Effects (e.g. effects P_l on x_l)

IE $P\downarrow$

	SE	IE	TE
Normal Good	$\uparrow$	$\uparrow$	$\uparrow$
Inferior Good	$\uparrow$	$\downarrow$	$\uparrow$
Giffen Good	$\uparrow$	$\downarrow$	$\downarrow$

- Not Giffen: Demand curve is negatively sloped (as usual)
- Giffen: Demand curve is positively sloped

Substitution and Income Effects

- ***Summary:***

1) SE is negative (since $\downarrow p_1 \Rightarrow \uparrow x_1$, they move in opposite directions)
   - SE < 0 does not imply $\downarrow x_1$ just implies that **the two move in opposite directions**
2) If good is inferior, IE < 0. Then,

$$\text{TE} = \underbrace{\underbrace{\text{SE}}_{-} - \underbrace{\text{IE}}_{-}}_{+} \quad \Rightarrow \quad \text{if } |\text{IE}| \begin{Bmatrix} > \\ < \end{Bmatrix} |\text{SE}|, \text{ then } \begin{Bmatrix} \text{TE}(+) \\ \text{TE}(-) \end{Bmatrix}$$

For a price decrease $\downarrow p_1$, this implies

$$\begin{Bmatrix} \text{TE}(+) \\ \text{TE}(-) \end{Bmatrix} \Rightarrow \begin{Bmatrix} \downarrow x_1 \\ \uparrow x_1 \end{Bmatrix} \quad \begin{matrix} \text{Giffen good} \\ \text{Non–Giffen good} \end{matrix}$$

3) Hence,
   a) A good can be inferior, but not necessarily be Giffen
   b) But all Giffen goods must be inferior.

NB. The signs of SE and IE are opposite if we consider $\downarrow p_1$ or $\uparrow p_1$

Relationship between the Expenditure and Hicksian Demand

- The relationship between the Hicksian demand and the expenditure function $\frac{\partial e(p,u)}{\partial p_k} = h_k(p, u)$ can be further developed by computing the second derivative. That is,

$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_k} = \frac{\partial h_k(p,u)}{\partial p_k}$$

or

$$D_p^2 e(p,u) = D_p h(p,u)$$

- Since $D_p h(p,u)$ provides the Slutsky matrix, $S(p,w)$, then

$$S(p,w) = D_p^2 e(p,u)$$

Thus the **Slutsky matrix can be obtained from the observable Walrasian demand (rather than from the unobservable Hicksian or compensated demand)**.

Relationship between Walrasian Demand and Indirect Utility Function

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_{+}^{L}$. Suppose also that $v(p, w)$ is differentiable at any $(p, w) \gg 0$. Then,

$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p, w) \text{ for every good } k$$

RESULT IS WALRASIAN DEMAND

- This is ***Roy's identity*** (I don't do this proof, is ex. 28 Ch. 2)
- Powerful result, since in many cases it is easier to compute the derivatives of $v(p, w)$ than solving the UMP with the system of FOCs. Hint. Having the indirect utility function allows you to derive the Walrasian demand functions.

IUtilty is walrasian demand on maximum.
Roy identity to derive walrasian demand just computation ratio of the two derivative.
[1.02

Taking stock: Summary of Relationships

- The Walrasian demand, $x(p,w)$, is the solution of the UMP.
  – Its value function is the indirect utility function, $v(p,w)$.
- The Hicksian demand, $h(p,u)$, is the solution of the EMP.
  – Its value function is the expenditure function, $e(p,u)$.

Summary of Relationships

The UMP **The EMP**

$$\begin{tikzcd} x(p,w) \arrow[d, "(1)"'] \\ v(p,w) \end{tikzcd} \qquad\qquad \begin{tikzcd} h(p,u) \arrow[d, "(1)"] \\ e(p,u) \end{tikzcd}$$

Summary of Relationships

- Relationship between the value functions of the UMP and the EMP (lower part of figure):
  - EMP
    $e(p, v(p, w)) = w$, i.e., the minimal expenditure needed in order to reach a utility level equal to the maximal utility that the individual reaches at her UMP, $u = v(p, w)$, must be w.
  - UMP
    $v(p, e(p, u)) = u$, i.e., the indirect utility that can be reached when the consumer is endowed with a wealth level w equal to the minimal expenditure she optimally bear in the EMP, i.e., $w = e(p, u)$, is exactly u.

In the expenditure prices and utilty in constraint. Since EMP the expenditure function will be function of price and utilty.

IUF depends on wealth and price.
What maximise price p and wealth w. When we give max utilty level in price p and wealth w and by definition is w.
We can do the same with Indirect utilty function.

Summary of Relationships

The UMP **The EMP**

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{x(p,w)} \arrow[d, "(1)"'] & & \boxed{h(p,u)} \arrow[d, "(1)"] \\
\boxed{v(p,w)} \arrow[rr, leftrightarrow, "(2)"] & & \boxed{e(p,u)}
\end{tikzcd}
$$

$$e(p,\ v(p,w))=w$$

$$v(p,\ e(p,w))=u$$

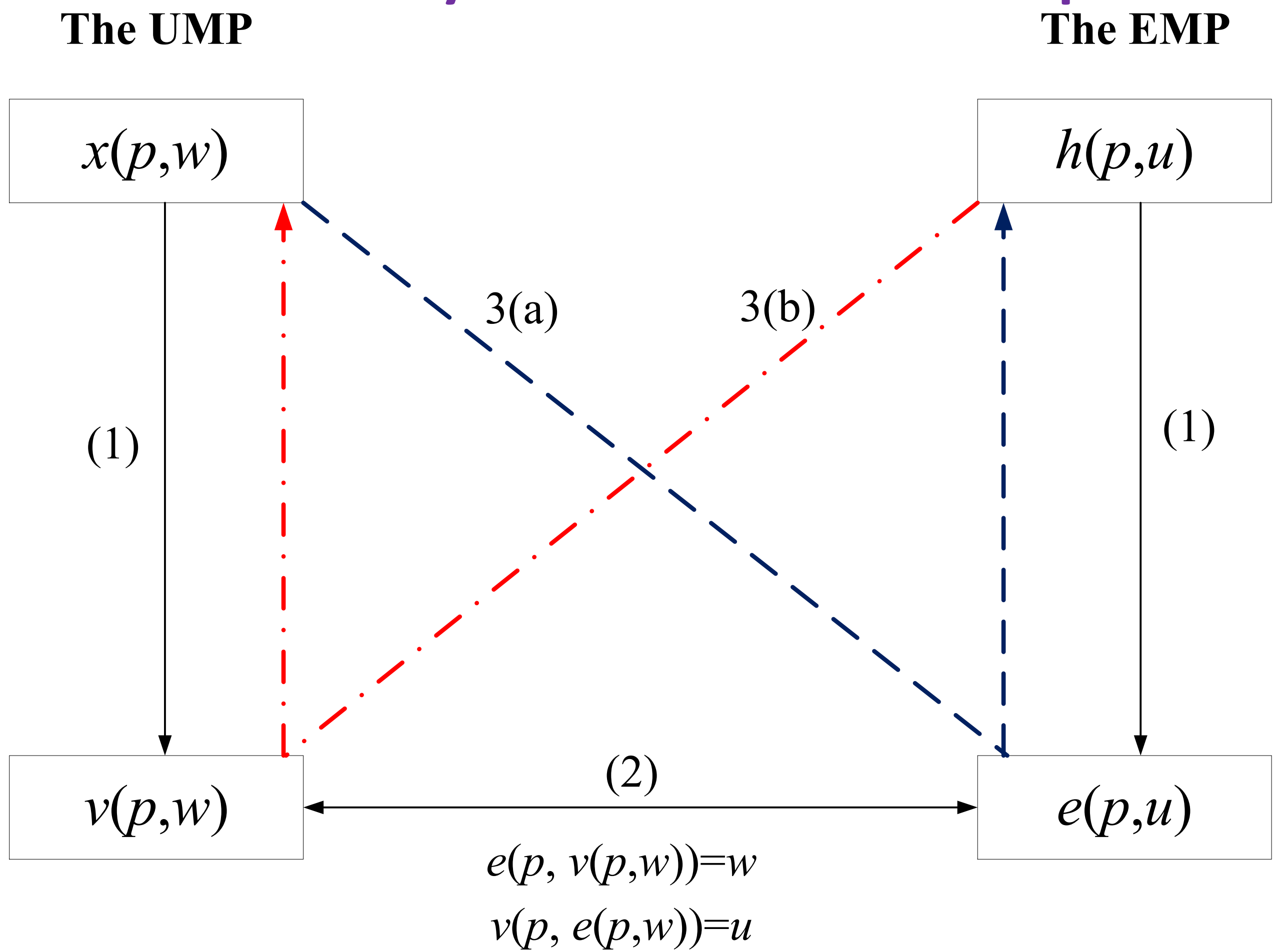
Summary of Relationships

- Relationship between the argmax of the UMP (the Walrasian demand) and the argmin of the EMP (the Hicksian demand):
  - $x(p, e(p,u)) = h(p,u)$, i.e., the (uncompensated) Walrasian demand of a consumer endowed with an adjusted wealth level w (equal to the expenditure she optimally bear in the EMP), $w = e(p,u)$, coincides with his Hicksian demand, $h(p,u)$.
  - $h(p, v(p,w)) = x(p,w)$, i.e., the (compensated) Hicksian demand of a consumer reaching the maximum utility of the UMP, $u = v\ (p,w)$, coincides with his Walrasian demand, $x(p,w)$.

H = W if UMP

W = H if EMP

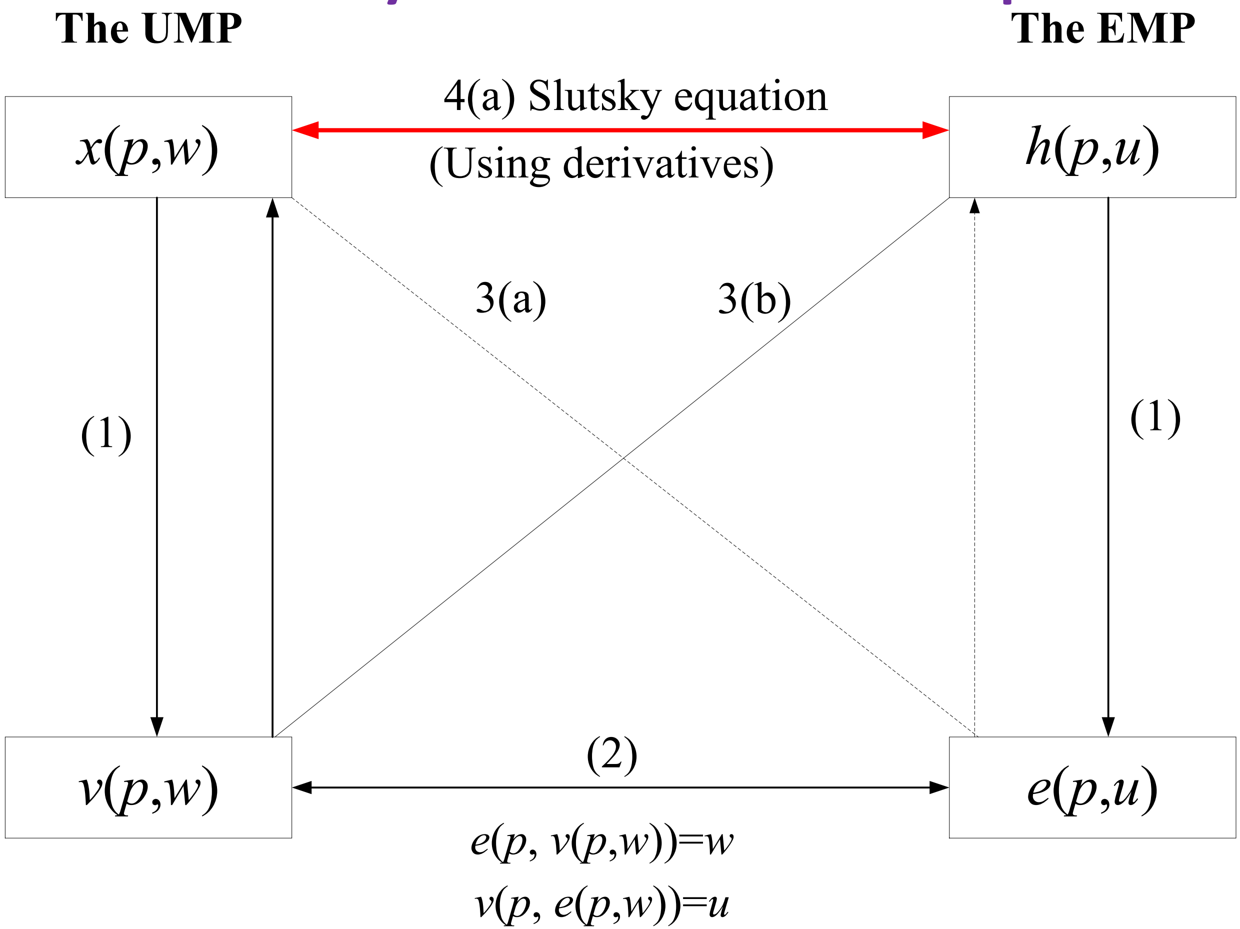
Summary of Relationships



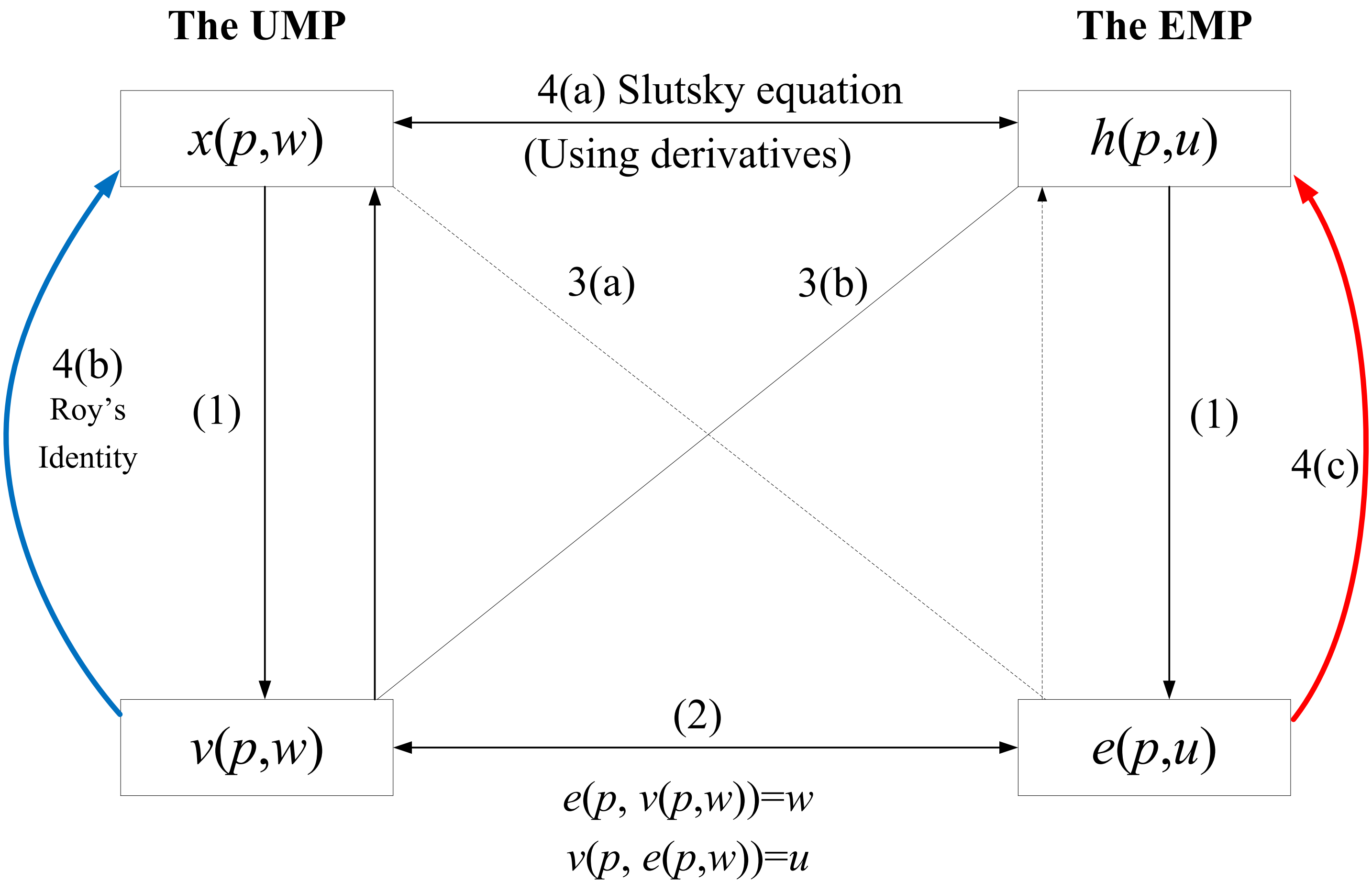
Summary of Relationships

- Finally, we can also use:
  - The ***Slutsky equation***:
    $$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$
    to relate the derivatives of the Hicksian and the Walrasian demand.
  - ***Shepard's lemma***:
    $$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$$
    to obtain the Hicksian demand from the expenditure function.
  - ***Roy's identity***:
    $$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p,w)$$
    to obtain the Walrasian demand from the indirect utility function.

Summary of Relationships



Summary of Relationships



Take away

- It is time to study hard guys!!!

- To defuse Micro:

A physicist, a chemist and an economist are stranded on an island, with nothing to eat. A can of soup washes ashore.
The physicist says, "Lets smash the can open with a rock."
The chemist says, "Lets build a fire and heat the can first."
The economist says, "Lets assume that we have a can-opener..."

ESERCIZI

EX. 1) FINDING WALRASIAN DEMAND

UTILITY $= \ln x_1 + x_2$ $\quad p_1x_1 + p_2x_2 \leq w$ BUDGET CONSTRAINT

WALRASIAN $\rightarrow$ MAX $\ln x_1 + x_2$

SUCH THAT $p_1x_1 + p_2x_2 \leq w$
AS BUDGET CONSTRAINT

$$L = \ln x_1 + x_2 + \lambda(w - p_1x_1 - p_2x_2)$$

$$\frac{\delta L}{\delta x_1} = \frac{1}{x_1} + \lambda p_1 \leq 0 \qquad x_1 \geq 0 \text{ WITH CS}$$

$$\frac{\delta L}{\delta x_2} = 1 - p_2\lambda \leq 0 \qquad x_2 \geq 0 \text{ WITH CS}$$

$$\frac{\delta L}{\delta \lambda} = w - p_1x_1 - p_2x_2 = 0$$

INTERIOR SOLUTION $\rightarrow$ FOCUS ON 1° DERIVATIVE x_1, x_2 NEEDS

$$\begin{cases} \frac{1}{x_1} = \lambda p_1 \\ 1 = p_2\lambda \end{cases} \qquad x_1^* = \frac{p_2}{p_1}$$

OPTIMAL SINCE ONLY x_1 IS IN THIS EQUATION

WALRASIAN DEMAND FOR x_2 $\rightarrow$ REPLACE x_1 IN BUDGET CONSTRAINT

$$w - p_1\frac{p_2}{p_1} - p_2x_2 = 0 \qquad w - p_2 = p_2x_2$$

$$x_2^* = \frac{w}{p_2} - 1$$

WE FIND WALRASIAN DEMAND $\rightarrow$ WE SHOULD CHECK FOR CORNER SOLUTION

(IF $x_1 \cdot x_2$ $\rightarrow$ NO CORNER SINCE $x_1 = 0$ THEN $x_2 = 0$ $\rightarrow$ NO CORNER)

(IF $x_1 = 0$ FEASIBLE CORNER SOLUTION)

$x_1 = \frac{w}{p_2}$ AND $x_2 = 0$ BUT THIS CANNOT BE OPT SOLUTION.

IN GENERAL IT'S OK TO FOCUS ON INTERIOR SOLUTION AT THE EXAM

2.) NOW FIND IND. UTILITY FUNCTION → UTILITY ON WALRASIAN DEMAND

SAME PARAMETER OF UMP

$$v(p_1, p_2, w) = \ln x_1^* + x_2^* = \ln \frac{p_2}{p_1} + \frac{w}{p_2} - 1 =$$

$$= \ln p_2 - \ln p_1 + \frac{w}{p_2} - 1$$

3.) EXP. FUNCTION → MIN EXPENDITURE TO REACH A GIVEN UTILITY LEVEL

⇓

SOLUTION OF EMP

SO WE MINIMISE

<u>ALSO</u>

EXP FROM IUF

MIN EXPENDITURE TO REACH u (UTILITY LV.) IS THE WEALTH

$$u = \ln p_2 - \ln p_1 + \frac{w}{p_2} - 1 \qquad e(p_1, p_2, u)$$

$$\left[u - \ln p_2 + \ln p_1 + 1\right] p_2 = e(p_1, p_2, u)$$

IN CASE EASY FROM IUF TO ISOLATE w SO WE CAN GET EXPENDITURE FUNCTION

IN CASE $(w^{1/2} + w)$ NOT EASY

4) FIND HICKSIAN DEMAND OF BOTH GOODS

SOLUTION OF EMP BUT IN THIS CASE WE CAN USE

SHEPPARD $\rightarrow \dfrac{\delta \cdots}{\delta p}$

CONTINUE OF 1

EXPENDITURE FUNCTION

$$e = p_2 \cdot u - p_2 \ln\left(\frac{p_2}{p_1}\right) + p_1 =$$

$$p_2 \cdot u \underbrace{- p_2 \left[\ln p_2 - \ln p_1\right]}_{p_2 \ln p_2 - p_2 \ln p_1} + p_2$$

$$h_1(p_1, p_2, u), \; h_2(p_1, p_2, u)$$

$$\frac{\partial e}{p_1} = h_1 \qquad \frac{\partial e}{p_2} = h_2$$

- $\dfrac{\partial e}{p_1} = \dfrac{p_2}{p_1} \rightsquigarrow -p_2 \cdot \left(-\dfrac{1}{p_1}\right) = \dfrac{p_2}{p_1}$

- $\dfrac{\partial e}{p_2} = u + 1 - \ln p_2 - \cancel{p_2} \cdot \dfrac{1}{\cancel{p_2}} + \ln p_1 =$

  $= u + 1 + \ln p_1 - \ln p_2 - 1 = u + \ln \dfrac{p_1}{p_2}$

- $\dfrac{\partial h_2}{\partial p_2} = -\dfrac{p_1}{p_2^2} < 0$ • $\dfrac{\partial h_1}{\partial p_1} = \dfrac{-p_2}{p_1^2}$

CROSS PRICE EFFECT	CROSS (WALRASIAN DEMAND)	NET (HICKSIAN DEMAND
COMPLEMENTS	$\frac{\partial x_1}{\partial p_2} < 0$ $p_2 \uparrow$ $x_1 \downarrow$ (OPPOSITE DIRECTION)	$\frac{\partial h_1}{\partial p_2} < 0$
SUBSTITUTES	$\frac{\partial x_1}{\partial p_2} > 0$ $p_2 \uparrow$ $x_1 \uparrow$ (SAME DIRECTION)	$\frac{\partial h_1}{\partial p_2} > 0$

→ THIS DEFINITION CAN BE ASYMMETRIC

$$\frac{\partial h_1}{\partial p_2} = \frac{1}{p_1} > 0$$ (PRICE CAN'T BE NEGATIVE) => NET SUBSTITUTES

SO THEY ARE

$$h_1 = \frac{p_2}{p_1}$$

$$\frac{\partial h_2}{\partial p_1} = \frac{1}{p_1} > 0$$ ⇝ NET SUBSTITUTES

$$h_2 = u + \ln\left(\frac{p_1}{p_2}\right) = u + \ln p_1 - \ln p_2$$

SYMMETRIC

LOOK AT THE WALRASIAN DEMAND

$\boxed{\text{Ex}}$ 1.b

$$u = \sqrt{x_1} + \sqrt{x_2} = x_1^{1/2} + x_2^{1/2}$$

$$\max_{x_1, x_2 > 0} \quad u = x_1^{1/2} + x_2^{1/2}$$

$$\text{s.t.} \quad x_1 p_1 + x_2 p_2 \leq w$$

$$L \cdot x_1^{1/2} + x_2^{1/2} + \lambda(w - p_1 x_1 + p_2 x_2)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = \dfrac{1}{2} x_1^{-1/2} - \lambda p_1 \leq 0 & x_1 \geq 0 \\[2ex] \dfrac{\partial L}{\partial x_2} = \dfrac{1}{2} x_2^{-1/2} - \lambda p_2 \leq 0 & x_2 \geq 0 \\[2ex] \dfrac{\partial L}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \end{cases}$$

$$\boxed{\dfrac{\frac{1}{2} x_1^{-1/2}}{\frac{1}{2} x_2^{-1/2}}} = \dfrac{\lambda p_1}{\lambda p_2} \quad \text{slope of I.C.}$$

MRS

$$\Downarrow$$

$$\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{p_1}{p_2} \qquad x_2 = \left(\frac{p_1}{p_2}\right)^2 \cdot x_1$$

$$w - p_1 x_1 - p_2 \left(\frac{p_1}{p_2} \right)^2 x_1 = 0$$

$$w - p_1 x_1 - \frac{p_1^2}{p_2} x_1 = 0$$

$$x_1 \left(p_2 + \frac{p_1^2}{p_2} \right) = w$$

$$x_1^* = \frac{w}{p_1 + \frac{p_1^2}{p_2}} = \frac{w}{p_1\left(1 + \frac{p_1}{p_2}\right)}$$

$$x_1^* = \left(\frac{p_1^2}{p_2} \right) x_1 = \frac{p_1^2}{p_2} \cdot \frac{w}{p_1\left(1+\frac{p_1}{p_2}\right)} = \frac{p_1}{p_2^2} \cdot \frac{w}{\left(1+\frac{p_1}{p_2}\right)}$$

INDIRECT UTILITY FUNCTION

$$u(x_1^*, x_2^*) = v(p_1, p_2, w) =$$

$$\left(\frac{w}{p_1\left(1+\frac{p_1}{p_2}\right)} \right)^{1/2} + \left(\frac{w}{\frac{p_1}{p_2}\left(1+\frac{p_1}{p_2}\right)} \right)^{1/2}$$

$h_1, h_2 \longrightarrow$ SOLUTION OF EMP

$$\min_{x_1, x_2 \geq 0} \quad p_1 x_1 + p_2 x_2 \quad \text{S.T} \quad x_1^{1/2} + x_2^{1/2} \geq u$$

$$\downarrow$$

$$\max_{x_1, x_2 \geq 0} \quad -(p_1 x_1 + p_2 x_2) \quad \text{S.T.} \quad \dots \quad \geq 0$$

$$L = -p_1 x_1 - p_2 x_2 + \lambda \left(x_1^{1/2} + x_2^{1/2} - u \right)$$

INTERIOR SOLUTION

FOC

$$\frac{\partial L}{\partial x_1} = -p_1 + \lambda \frac{1}{2} x_1^{-1/2} = 0$$

$$\frac{\partial L}{\partial x_2} = -p_2 + \lambda \frac{1}{2} x_2^{-1/2} = 0$$

$$\longrightarrow$$

$$\frac{\partial L}{\partial \lambda} = x_1^{1/2} + x_2^{1/2} = u$$

$$\longrightarrow \frac{\lambda \left(\frac{1}{2} x_1^{-1/2} \right)}{\lambda \left(\frac{1}{2} x_2^{-1/2} \right)} = \frac{p_1}{p_2}$$

$$\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{p_1}{p_2}$$

$$x_2 = \left(\frac{p_1}{p_2}\right)^2 \cdot x_1$$

$$x_2^{1/2} = \frac{p_1}{p_2} \cdot x_1^{1/2}$$

$$x_1^{1/2} + \left(\frac{p_1}{p_2}\right) x_1^{1/2} = u$$

$$\left(1 + \frac{p_1}{p_2}\right) x_1^{1/2} = u$$

$$\Rightarrow x_1 = \left(\frac{u}{1+\frac{p_1}{p_2}}\right)^2 = h_1$$

$$x_2 = \left(\frac{p_1}{p_2}\right)^2 \cdot \left(\frac{u}{1+p_1/p_2}\right)^2$$

EXPENDITURE FUNCTION

$$e(p_1, p_2, u) = p_1 h_1 + p_2 h_2 =$$

$$= p_1 \cdot \left(\frac{u}{1 \cdot \frac{p_1}{p_2}}\right)^2 + p_2 \left(\frac{p_1}{p_2}\right)^2 \left(\frac{u}{1+\frac{p_1}{p_2}}\right)^2$$

EX #

I.C. A S.C WITH PERFECT SUBSTITUTES

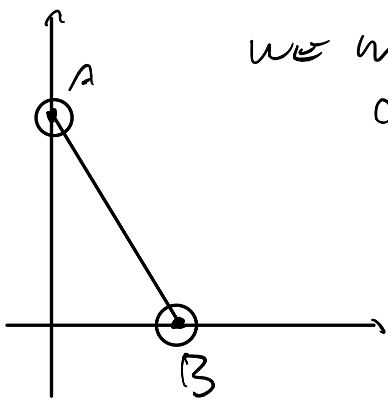
WALRAS DEMAND $u(x_1, x_2) = \underbrace{x_1 + 2x_2}_{\text{LINEAR}}$

S.T $p_1 x_1 + p_2 x_2 \le w$

LINEAR

$\downarrow$

CORNER SOLUTION



WE HAVE TO COMPARE THE SLOPE OF B.C AND I.C.

$$|MRS| > \frac{p_1}{p_2} \rightarrow B \qquad x_2 = 0, \quad x_1 = \frac{w}{p_1}$$

$$|MRS| < \frac{p_1}{p_2} \rightarrow A \qquad x_1 = 0, \quad x_2 = \frac{w}{p_2}$$

EXAMPLE $p_1 = 1, p_2 = 3$ $\rightarrow$ WE ARE IN CASE B

$$\frac{1}{2} > \frac{1}{3} \Rightarrow x_1^* = w, \quad x_2^* = 0$$

WALRASIAN DEMAND OF GOOD 2

WHAT HAPPEN IF $p_2 = 4$?

$\frac{1}{2} > \frac{1}{4} \rightsquigarrow |MRS| > \frac{p_1}{p_2} \Rightarrow$ CORNER SOLUTION $x_1^* = w \quad x_2^* = 0$

$\boxed{\text{TOTAL EFFECT OF PRICE CHANGE}}$ ON GOOD 1 AND GOOD 2 ?

$TE \; x_1 = w - w = 0$

END POINT (AFTER THE PRICE CHANGE)

$TE \; x_2 = 0 - 0 = 0$

$\boxed{\text{SUBSTITUTION EFFECT}}$

CASE 2 $\quad p_2 \downarrow \quad p_2 = 3 \rightsquigarrow p_2 \; \frac{1}{2}$

OPTIMAL SOLUTION WITH $p_1 = 1 \quad p_2 = \frac{1}{2}$

$|MRS| < \frac{p_1}{p_2}$

$\frac{1}{2} < 2 \rightarrow$ OPTIMAL SOLUTION IMS $x_1^* = 0$

$x_2^* = \frac{w}{1/2} = 2w$

NEW SOLUTION $(0, 2w)$

$\boxed{\text{TOTAL EFFECT}}$

$TE: x_1 = 0 - w = -w$

$TE: x_2 = 2w - 0 = 2w$
(END POINT - START POINT)

? I prove this

$(x_1, x_2) \succsim (y_1, y_2) \longrightarrow x_1 \geq y_1 - 1$

compare 1° components of bundles

MONOTONICITY

- (x_1, x_2) vs $(a x_1, x_2)$
- (x_1, x_2) vs $(x_1, a x_2)$
- (x_1, x_2) vs $(a x_1, a x_2)$

- $a x_1$ $\quad x_1 - 1$ $\quad$ $\underline{a > 1}$ $\quad$ then $\quad a x_1 > x_1 - 1$

  so $(a x_1, x_2) \succsim (x_1, x_2)$

- x_1 $\quad x_1 - 1$ $\rightarrow x_1 > x_1 - 1$ so $(x_1, a x_2) \succsim (x_1, x_2)$

- is like case 1!

STRONG MONOTONE MAINLY BECAUSE

$a x_1 > x_1 - 1$

YESTERDAY T.E $x_1 = -w$ $A(w, 0)$

T.E $x_2 = 2w$ $C(0, 2w)$

WE COMPUTE T.E. BY TAKING DIFF. BETWEEN A AND C

NOW COMPUTE SUBSTITUTION EFFECT (HICKSIAN COMPENSATED)

WE HAVE TO GIVE SO MANY AS BEFORE UTILITY LEVEL BEFORE PRICE CHANGE

$u(x_1, x_2) = x_1 + 2x_2$

UTILITY LEVEL? ON A $u(x_1, x_2) = w$

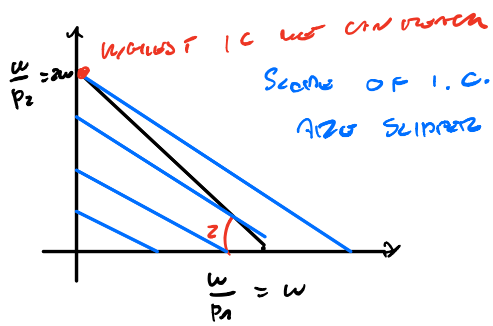
$p_1 = 1$ $p_2 = \frac{1}{2}$ AFTER PRICE CHANGE

HICKSIAN COMPENSATION

$$\begin{cases} \underbrace{x_1 + 2x_2}_{\text{UTILITY LEVEL}} = w \\ \frac{1}{2} < 2 \end{cases}$$

NO TANGENCY CONDITION

DER x_1 / DER x_2 RATIO PRICE



SO CORNER SOLUTION

$x_2 = 2w = h_1$

$x_1 = 0 = h_2$

B(0, 2w) INTERMEDIATE POINT.

TO SUMO FROM A TO B (B - A)

SE x_1 = 0 - w = -w

SE x_2 = 2w - 0 = 2w

S.E. AND TE ARE EQUAL

INCOME EFFECT B TO C (C - B)

IE x_1 = 0 - 0 = 0

IE x_2 = 2w - 2w = 0

IN PERFECT SUBSTITUTES I.E. IS Ø

TE = SE + IE

BY SUMMING UP THIS WE OBTAIN TOTAL EFFECT

TE_{x_1} = -w + 0 = -w

TE_{x_2} = 0 + 2w = 2w

WE GET STARTING T.E. SO IS THE CASE

$\boxed{\text{EXAMPLE}}$

INCOME & SUBSTITUTION EFFECT OF FUNCTION WILL BEHAVE (COBB-DOUGLAS)

$$\text{MAX } U(x_1, x_2) = x_1^{1/2} \, x_2^{1/3} \qquad P_1 = 3 \quad P_2 = 2$$

$$x_1, x_2 \geq 0 \qquad \text{s.t. } 3x_1 + 2x_2 = 50 \qquad w = 50$$

$$L = x_1^{1/2} \, x_2^{1/3} + \lambda(50 - 3x_1 - 2x_2)$$

NOT CORNER SINCE WITH U ∅ WE GET $u \to [10, \infty]$

⇓ WE EXCLUDE CORNER SOLUTION

$$\text{FOC} = \frac{\partial L}{\partial x_1} = \frac{1}{2} x_1^{-1/2} \, x_2^{1/3} - 3\lambda = 0 \qquad x_1 > 0$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{3} x_2^{-1/3} \, x_1^{1/2} - 2\lambda = 0 \qquad x_2 > 0$$

$$\frac{\partial L}{\partial \lambda} = 50 - 3x_1 - 2x_2$$

$$\frac{\frac{1}{2}}{\frac{1}{3}} \left(\frac{1}{2} \; x_1^{-1/2} \cdot x_1^{-1/2} \; x_2^{1/3} \; x_2^{2/3} \right) = \frac{3}{2}$$

$$\boxed{\cancel{\frac{3}{2}} \, \frac{x_2}{x_1} = \cancel{\frac{3}{2}}} \rightarrow x_2 = x_1$$

NOW WE REPLACE THIS IN BUDGET CONSTRAINT

$$50 - 3x_1 - 2x_1 = 0 \qquad x_1^* = \frac{50}{5} = 10 \qquad x_2^* = 10$$

$$A(10, 10)$$

WHAT MEANS IF $\boxed{P_2 = 4}$

$\downarrow$

$MRS = \frac{P_1}{P_2}$

SO $\begin{cases} |MRS| = \frac{P_1}{P_2} \\ \text{BUDGET CONSTRAINT} \end{cases}$ $\begin{cases} \frac{\cancel{3}}{2} \frac{x_2}{x_1} = \frac{\cancel{3}}{4} \\ 50 - 3x_1 - 4x_2 = 0 \end{cases}$

UTILITY REMAIN THE SAME

$\downarrow$

PRICE APPEAR ONLY ON BUDGET CONSTRAINT

$\begin{cases} x_2 = \frac{1}{2} x_1 \\ 50 - 3x_1 - 2x_1 = 0 \end{cases}$ $\begin{cases} x_1 = 10 \\ x_2 = 5 \end{cases}$

NEW POINT AFTER PRICE CHANGE

NOW COMPUTE T.E x_1 AND x_2

$C(10, 5)$

$(C-A)$ $\begin{cases} TE \quad x_1 = 10 - 10 = 0 \\ TE \quad x_2 = 5 - 10 = -5 \end{cases}$

FOR x_1 T.E. NOT CHANGE.

FOR x_2 CHANGE OF -5

$x_1^* = 10 \quad x_2^* = 10 \quad u(10,10) = 10^{\frac{1}{2}+\frac{1}{3}} = 10^{\frac{5}{6}}$

$\nearrow$ A(10,10)

to COMPUTE S. E. → HICKSIAN DEMAND

$x_1^{1/2} \, x_2^{1/3} = 10^{5/6}$

$|MRS| = \frac{p_1}{p_2}$ → AFTER PRICE CHANGE

$\frac{3}{2} \frac{x_2}{x_1} = \frac{3}{4} \rightarrow x_2 = \frac{1}{2} x_1$

SAME TANGENCY POSITION

$x_1^{1/2} \cdot \left(\frac{1}{2} x_1\right)^{1/3} = 10^{5/6}$

$x_2 = \frac{1}{2} x_1$

$x_1^{\frac{1}{2}+\frac{1}{3}} = 10^{\frac{5}{6}} \left(\frac{1}{2}\right)^{-\frac{1}{3}}$

$x_2 = \frac{1}{2} x_1$

SQUARE ROOT!

$x^2 = 25 \quad x^{2 \cdot \frac{1}{2}} = 25^{\frac{1}{2}}$

$x = 5$

$x_1^{5/6 \cdot 6/5} = 10^{\frac{5}{6} \cdot \frac{6}{5}} \left(\frac{1}{2}\right)^{-\frac{1}{3} \cdot \frac{6}{5}}$

↓

so $x_1^* = 10 \left(\frac{1}{2}\right)^{-2/5}$

$x_2^* = 5 \left(\frac{1}{2}\right)^{-\frac{3}{5}}$

$B\left(10\left(\frac{1}{2}\right)^{-2/5}, 5\left(\frac{1}{2}\right)^{-3/5}\right)$

(B - A)

S.E x_1 = ..

S.E x_2 = ..

(C - B)

I.E x_1 = ...

I.E x_2 = ...

CHECK THAT SUMMING S.E + I.E = T.E. !

SOLUTIONS ARE OK AGAIN

ENGEL CURVE → LINK QUANTITY WITH WEALTH (OR INCOME)

$$U(x_1, x_2) = x_1^{1/2} x_2^{2/3} \qquad p_1 = 3 \quad p_2 = 2 \quad w = 50$$

$$\text{s.t.} \quad 3x_1 + 2x_2 = w$$

VARIABLE

RELATION BETWEEN x AND w

SO SHOULD BE LIKE

$$x_1 = f(w)$$

$$x_2 = f(w)$$

w IS CONSTANT → TAKE w AS VARIABLE IN B.C.

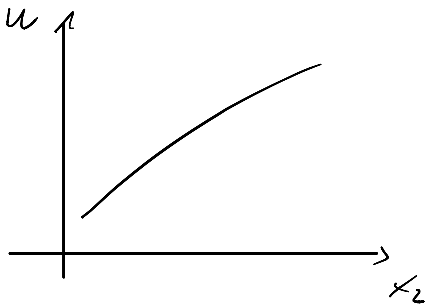
$$\begin{cases} |MRS| = \dfrac{p_1}{p_2} \\ \text{B.C} \end{cases} \qquad \begin{cases} \cancel{\dfrac{3}{2}} \dfrac{x_2}{x_1} = \cancel{\dfrac{3}{2}} \rightarrow x_2 = x_1 \\ \end{cases}$$

$$\begin{cases} x_2 = x_1 \\ 3x_1 + 2x_1 = w \end{cases} \qquad \begin{cases} x_2 = x_1 \\ 5x_1 = w \end{cases} \begin{matrix} \rightarrow x_2 = \dfrac{w}{5} \\ \rightarrow x_1 = \dfrac{w}{5} \end{matrix}$$

ENGEL C. FOR x_2

ALSO ISOLATING w

$w = 5x_1$ $\quad$ $w = 5x_2$



ENGEL CURVE FOR GOOD 1

INC ↑ D ↑ → NORMAL GOOD

$$\frac{\text{DER DEMAND}}{\text{DER } w} \rightarrow \frac{\partial x_2}{\partial w} = \boxed{\frac{1}{5} > 0}$$

NORMAL

IF DER $\frac{\delta x_k}{\delta w} < 0$

INFERIOR GOOD

EX SET II

d) $X(p_x, p_y, w) = 200 - 4p_x - 1.5\,p_y + 0.008w$

↓ DEMAND FOR X

GOODS ARE GROSS SUBSTITUTE OR GROSS COMPLEMENTS WITH RESPECT TO Y?

- IF I GOT X GROSS SUB THEN I GOT GROSS FOR Y
- IF NOT WELL BEHAVE Y IS IND. FROM X!

EX

$Y = 6p_y$ → Y IND FROM X SINCE DOES NOT APPEAR p_x

CROSS PRICE DERIVATIVE

$$\frac{\partial x}{\partial p_y} = -1.5 < 0$$ → SO 2 GOODS ARE GROSS COMPLEMENTS

ARE THE NET COMPLEMENTS OR NET SUBSTITUTES?

→ HICKSIAN DEMAND

IS X NORMAL OR INFERIOR?

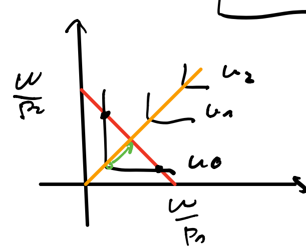
$$\frac{\partial x}{\partial w} = 0.008 > 0$$ SO GOOD IS NORMAL

INCOME & SUBSTITUTION EFFECT WITH PERFECT COMPLEMENTS

↓ KINK

$$u(x_1, x_2) = \min\{x_1, 2x_2\}$$

$x_1, x_2 \geq 0$ s.t. $p_1x_1 + p_2x_2 \leq w$

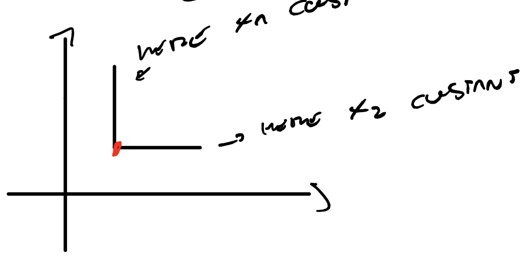


I PUT FIRST COMPONENT EQUALS TO THE SECOND TO FIND KINK

I CAN MOVE IN THIS DIR IN WHICH IT TOUCH B.C WITH ONLY ONE POINTS

(KINK SOLUTION)

$x_1 = 2x_2$



$x_1 = 2x_2 \rightarrow x_2 = \frac{1}{2} x_1$ SLOPE $= \frac{\delta x_2}{\delta x_1} = \frac{1}{2}$

FIND WALRASIAN DEMAND SINCE SOLUTION IS IN THE KINK

$$\begin{cases} \text{KINK} \rightarrow x_2 = \frac{1}{2} x_1 \\ \text{B.C.} \rightarrow p_1 x_1 + p_2 x_2 = w \end{cases} \quad \Big\} \; p_1 x_1 + \frac{p_2}{2} x_1 = w$$

WE SOLVE EVEN IF WE DON'T HAVE PRICES

$$\Big\{ \left(p_1 + \frac{p_2}{2}\right) x_1 = w \qquad \begin{cases} x_2^* = \dfrac{w/2}{p_1 + \frac{p_2}{2}} \\ x_1^* = \dfrac{w}{p_1 + \frac{p_2}{2}} \end{cases}$$

$$x \left(\frac{w}{p_1 + \frac{p_2}{2}} , \frac{w/2}{p_1 + \frac{p_2}{2}} \right)$$

WHAT TO DO TO FIND UTILITY IN WALRASIAN DEMAND?

CHECK LAST LECTURE

U(WALRASIAN) ⇒ WITHOUT INDIRECT UTILITY. $v\left(\frac{w}{2} \dots\right)$

WHAT HAPPEN IF PRICE DECREASES?

$P_2 \uparrow$ $\quad \overset{\text{NEW}}{p'_2 = 4} \quad \overset{\text{OLD}}{p_2 = 2} \quad p_1 = 1$

$$\begin{cases} x_1 = \frac{1}{2} x_2 \ \text{(KINK)} \longrightarrow x_1 + 2x_1 = w \\ 1 \cdot x_1 + 4x_2 = w \end{cases}$$

$x_1^* = \frac{w}{3}$ $\qquad \overset{\text{OLD}}{A\left(\frac{w}{2}, \frac{w}{4}\right)}$

$x_2^* = \frac{w}{6}$ $\qquad C\left(\frac{w}{3}, \frac{w}{6}\right)$

T.E. FROM A TO C

T.E $x_1 = \frac{w}{3} - \frac{w}{2} = -\frac{w}{6}$

T.E $x_2 = \frac{w}{6} - \frac{w}{4} = -\frac{w}{12}$

S.E.?

↓

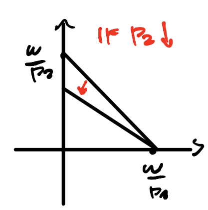
HICKSIAN COMPENSATION

1. SAME UTILITY
2. B.C. MUST BE TANGENT TO OLD UTILITY LV OF I.C.

NEW B. LINE WILL BE TANGENT TO OLD UTILITY LEVEL

B.C $= x_1 + 4x_2 = w$

$$\begin{cases} x_2 = \frac{1}{2} x_1 \\ \min\{x_1, 2x_2\} = \frac{w}{2} \end{cases}$$



$x_1 = 2x_2 = \frac{w}{2}$ $\qquad x_1^* = \frac{w}{2} \qquad x_2^* = \frac{w}{4}$

$B\left(\frac{w}{2}, \frac{w}{4}\right)$

A to B

S.E $x_1 = 0$ NE S.E SINCE THE GOODS

S.E. $x_2 = 0$ ARE CONSUMED IN THE SAME PROPORTION

B to C $B\left(\frac{w}{2}, \frac{w}{4}\right)$ $C\left(\frac{w}{3}, \frac{w}{6}\right)$

I.E.

$IE \quad x_1 = \frac{w}{3} - \frac{w}{2} = -\frac{w}{6}$

$IE \quad x_2 = \frac{w}{6} - \frac{w}{4} = -\frac{w}{12}$

T.E = IE BECAUSE SE = 0 FOR BOTH GOODS

WE CANNOT SUBSTITUTE ONE WITH THE EACH OTHER

Advanced Microeconomic Theory

Chapter 3: Welfare evaluation

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

Measuring the Welfare Effects of a Price Change

Measuring the Welfare Effects of a Price Change

- How can we measure the welfare effects of:
  - a price decrease/increase
  - the introduction of a tax/subsidy

    $P\uparrow$ $P\downarrow$

- Why not use the difference in the individual's utility level, i.e., from u^0 to u^1?
  - Two problems:
    1) *Within a subject criticism*: Only ranking matters (ordinality), not the difference;
    2) *Between a subject criticism*: Utility measures would not be comparable among different individuals.

- Instead, we will pursue **monetary evaluations** of such price/tax changes.

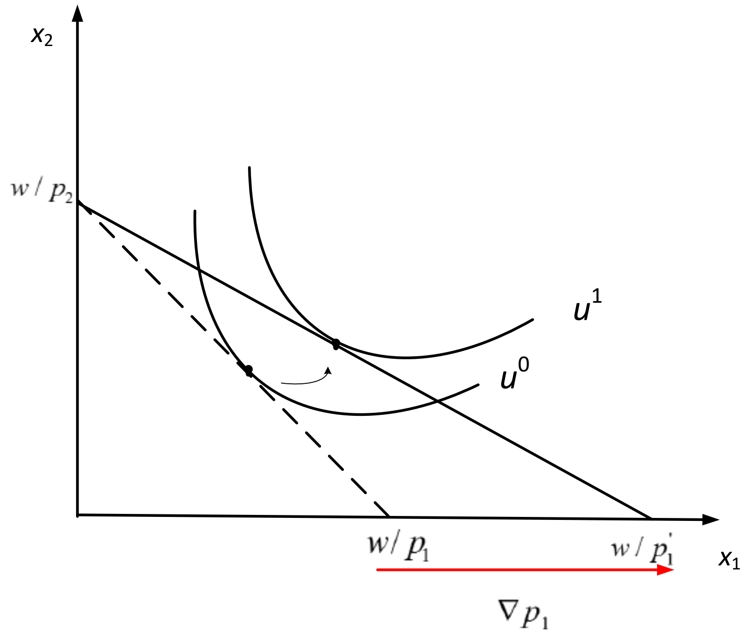
How to evaluate the welfare with different level of utilty? In reality different guys have different utilty function.

2) utility may be different between individuals.

We use money to evaluate welfare

Measuring the Welfare Effects of a Price Change

- Consider a price decrease from p_1^0 to p_1^1.
- We cannot compare u^0 to u^1.
- Instead, we will find a **money-metric** measure of the consumer's welfare change due to the price change.



Measuring the Welfare Effects of a Price Change

How much money i will have to transfer to the consumer after a price change(decrease or increase) to be as well off as before the price change

- ***Compensating Variation (CV)***:
  – How much money a consumer would **be willing to give up** *after* a reduction in prices to be just as well off as *before* the price decrease (After-Before, AB)

    same utility level

- ***Equivalent Variation (EV)***:
  – How much money a consumer **would need** *before* a reduction in prices to be just as well off as *after* the price decrease (Before-After, BA)

We could use Hicksian demand or expenditure function

Hoping with Lower price is better than with higher prices. This means that after price decrease we have higher utility level. After price change utility level was lower.
To let the guy reach the same utility level before the price decrease the guy should have more or less income? We have to reduce the income.

If we consider a increase in price is the opposite. Willing to give up is only fro reduction in price.
Transfer can be positive or negative. Positive mean increasing income, negative decreasing income.

Measuring the Welfare Effects of a Price Change

- Two approaches:
  1) Using expenditure function
  2) Using the Hicksian demand

CV using Expenditure Function

$p \hat{x} = w$ $\quad$ $e(p^0, u^0) = w$

- $CV(p^0, p^1, w)$ using $e(p, u)$: Utility level solving the UMP

$$CV(p^0, p^1, w) = \boxed{e(p^1, \overset{=w}{u^1})} - e(p^1, u^0)$$

- The amount of money the consumer is willing to give up *after* the price decrease (after price level is p^1 and her utility level has improved to u^1) to be just as well off as *before* the price decrease (reaching utility level u^0).

CV is new price and new unity level - new price and old utilty level.
The vector of prices goes from p0 to p1, so wealth remain the same. So BC remain the same.

→ HICKSIAN COMPENSATION

CV using Expenditure Function

HOW MUCH MONEY YOU WILL TRANSFER TO CONSUMER AFTER PRICE CHANGE

$p_1 \downarrow$ $p_2 \leftrightarrow$

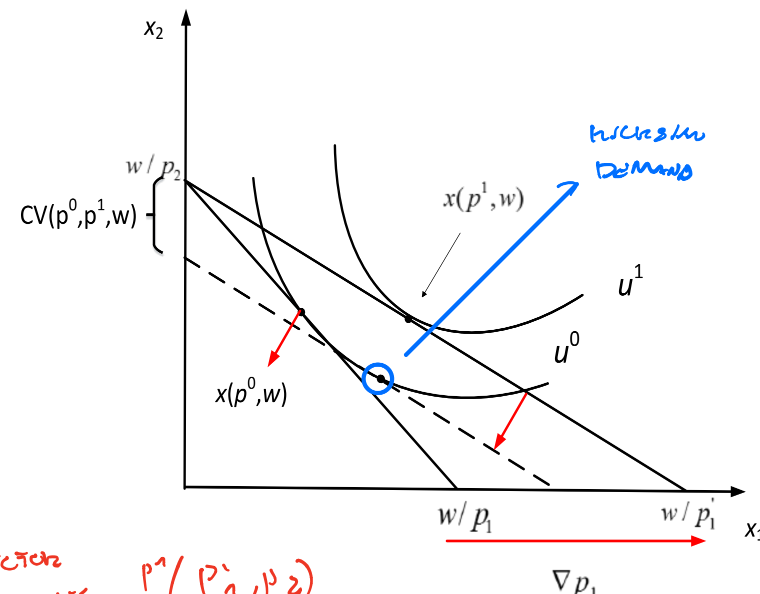
NEW PRICE

1) When $B_{p^0,w}$, $x(p^0,w)$
2) ∇p_1 and $x(p^1,w)$ under $B_{p^1,w}$
3) Adjust final wealth (*after* the price change) to make the consumer as well off as *before* the price change
4) Difference in expenditure:

IN $x(p^1,w)$ $w =$

$$CV(p^0,p^1,w) = \underbrace{e(p^1,u^1)}_{\text{at } B_{p^1,w}} - \underbrace{e(p^1,u^0)}_{\text{dashed line}}$$

→ VECTOR OF PRICES



$p^1(p_1', p_2)$ → $p_1 h_1 + p_2 h_2$

$p^0(p_1, p_2)$

This is **Hicksian wealth compensation**!

HOW MUCH MONEY TRANSFER BEFORE PRICE CHANGE
TO BE WELL OFF AFTER PRICE CHANGE

EV using Expenditure Function

- $EV(p^0, p^1, w)$ using $e(p, u)$:

MINIMUM EXPENDITURE
WITH OLD PRICES.
ALSO
UMP
BEFORE
PRICE CHANGES

$$EV(p^0, p^1, w) = \boxed{e(p^0, u^1)} - e(p^0, u^0)$$

SO WE WANT TO FIND THIS ↗.

$= w$

- The amount of money the consumer needs to receive *before* the price decrease (at the initial price level p^0 when her utility level is still u^0) to be just as well off as *after* the price decrease (reaching utility level u^1).

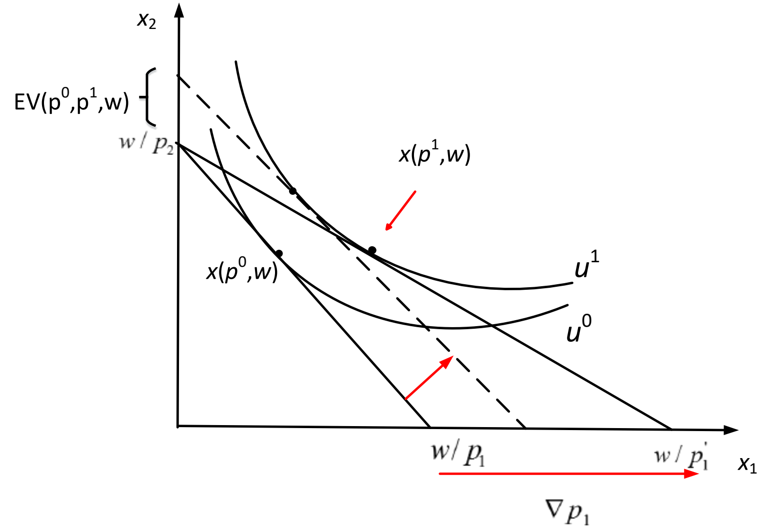
HOW WE FIND $e(p^0, u^1)$? SEE IT GRAPHICALLY

EV using Expenditure Function

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust initial wealth (*before* the price change) to make the consumer as well off as *after* the price change
4) Difference in expenditure:

$$EV(p^0, p^1, w) = \underbrace{e(p^0, u^1)}_{\text{dashed line}} - \underbrace{e(p^0, u^0)}_{\text{at } B_{p^0,w}} = w$$

$p_1 \downarrow$



NOW WE ARE ABLE TO DO ALL

EXAM 1 EXERCISES

CV using Hicksian Demand

new old

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $\underline{p_k^1 = p_k^0}$ for all $k \neq 1$, then

Good 1!

For other good deman the same

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0)$$

$$= w - e(p^1, u^0)$$

$e(p^0, u^0) =$

Utility found solving UMP with p0

(since $e(p^1, u^1)$= $e(p^0, u^0)$=w)

$$= e(p^0, u^0) - e(p^1, u^0) \; (*)$$

Primitive of derivative of exp. function

Hicksian

$$= \int_{p_1^1}^{p_1^0} \boxed{\frac{\partial e(p_1, \bar{p}_{-1}, u^0)}{\partial p_1}} dp_1 \; (**)$$

(since (**) is the solution of (*))

$$= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0) \, dp_1$$

Derivative then is Hicksian compensated demand

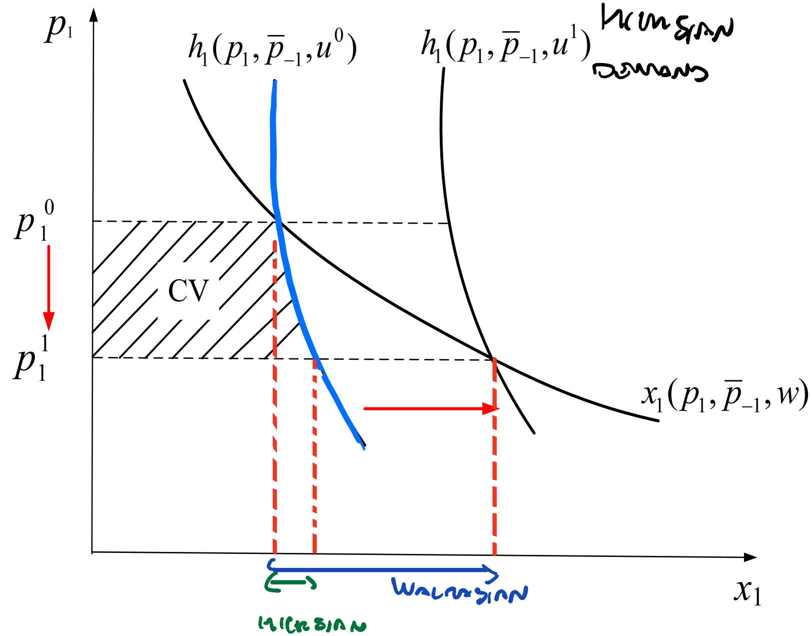
NO

Graphically is area under Hicksian comp demand

CV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, CV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^0, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.

NORMAL GOOD! WALRASIAN DEM IS FLATTER THAN HICKSIAN DEMAND



SO CHANGE IN WALRASIAN IS LARGER THAN HICKSIAN

EV using Hicksian Demand

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

OLD NEW UTI OLD PRICES

$$EV(p^0, p^1, w) = e(p^0, u^1) - \boxed{e(p^0, u^0)}$$

w

$$= e(p^0, u^1) - w$$

$w = e(p^0, u^0)$

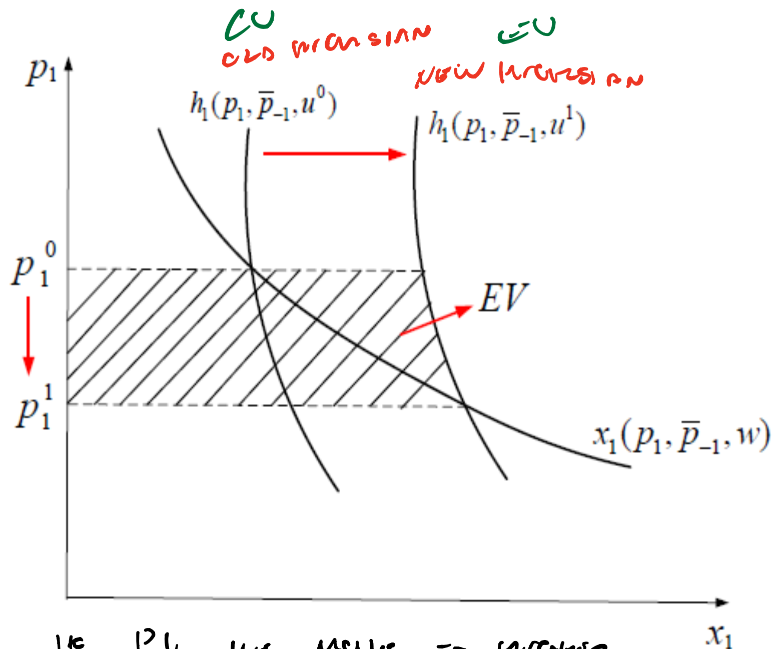
$$= e(p^0, u^1) - e(p^1, u^1)$$

$$= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \bar{p}_{-1}, u^1)}{\partial p_1} dp_1$$

$$= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$$

EV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, EV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^1, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.



If $P_1 \downarrow$ we move to higher utility level $u_1 > u_0$ so we considering Hicksian demand shifted to the right

What about a price increase?

- The Hicksian demand associated with initial utility level u^0 (before the price increase, or before the introduction of a tax) experiences an inward shift when the price increases, or when the tax is introduced, since the consumer's utility level is now u^1, where $u^0 > u^1$. Hence,

$$h_1(p_1, \bar{p}_{-1}, u^0) > h_1(p_1, \bar{p}_{-1}, u^1)$$

What about a price increase?

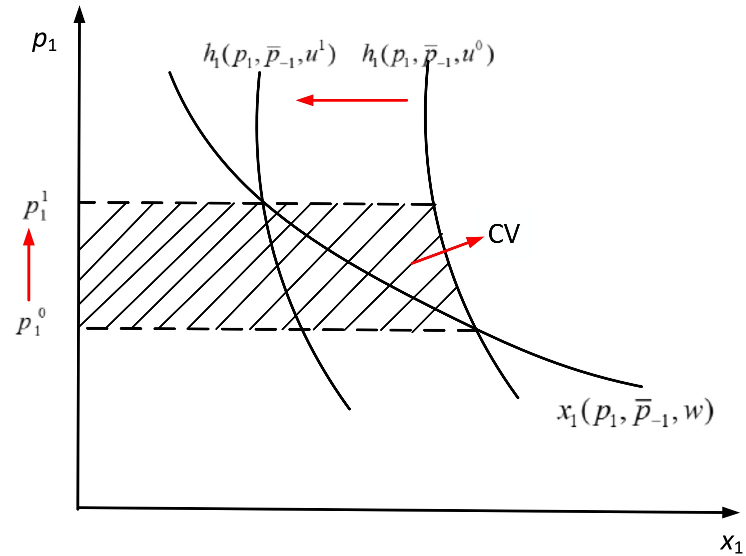
- The definitions of CV and EV would now be:
  - CV: the amount of money that a consumer would need *after* a price increase to be as well off as *before* the price increase.
  - EV: the amount of money that a consumer would be willing to give up *before* a price increase to be as well off as *after* the price increase.
- Graphically, it looks like the CV and EV areas have been reversed:
  - CV is associated to the area below $h_1(p_1, \bar{p}_{-1}, u^0)$ as usual
  - EV is associated with the area below $h_1(p_1, \bar{p}_{-1}, u^1)$.

What about a price increase?

VALUE
AFTER PRICE
INCREASE

- CV is always associated with $h_1(p_1, \bar{p}_{-1}, u^0)$

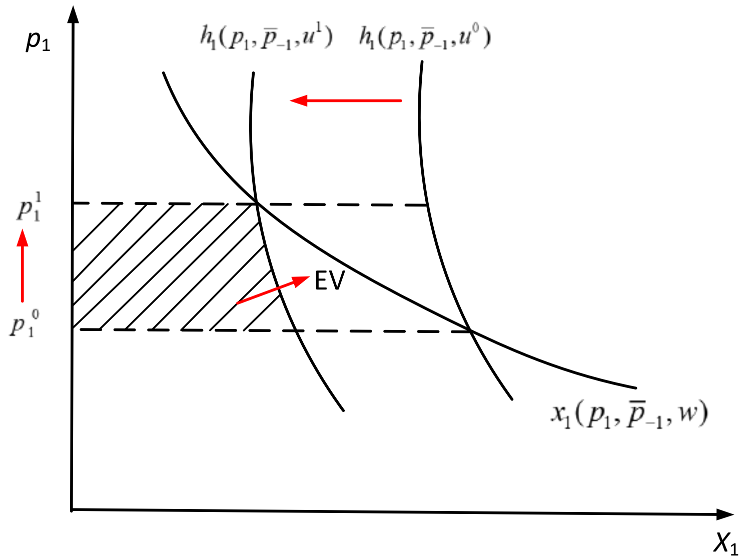
- $CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$



What about a price increase?

- EV is always associated with $h_1(p_1, \bar{p}_{-1}, u^1)$

- $EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$



this are useful for welfare analysis introducing tax

Introduction of a Tax

- The introduction of a **tax** can be analyzed as a **price increase**.
- *The main difference*: we are interested in the area of CV and EV that is *not* related to tax revenue.

how much state receive with taxes

- Tax revenue is:

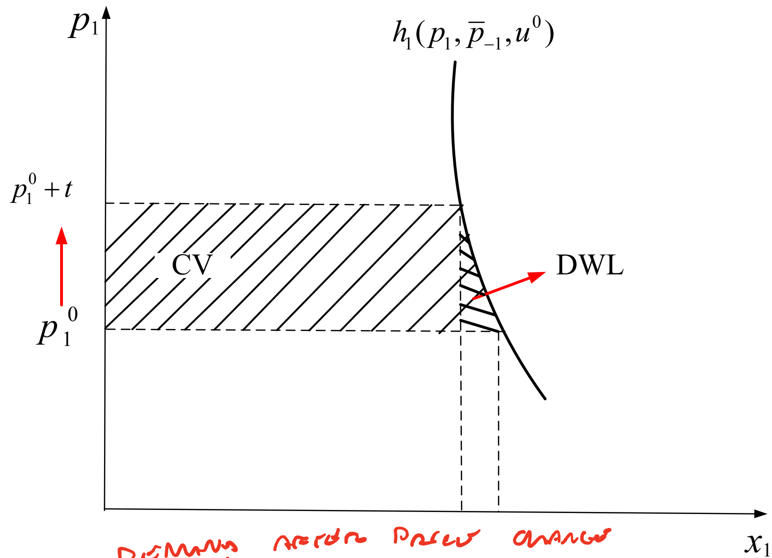
per unit tax

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^0) \text{ (using CV)}$$

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^1) \text{ (using EV)}$$

Introduction of a Tax

- CV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^0)$:
$CV(p^0, p^1, w)$
$$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$$



DEMAND AFTER PRICE CHANGE

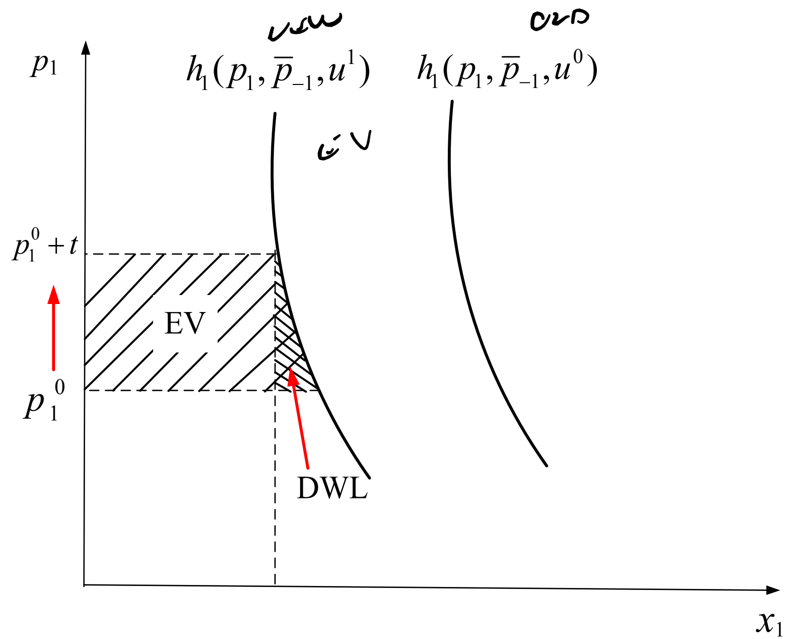
- Welfare loss (DWL) is the area of the CV not transferred to the government via tax revenue:

$DWL = CV - T$

TAX REVENUE = TAX · HICKSIAN DEMAND

Introduction of a Tax

- EV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^1)$:
  $EV(p^0, p^1, w)$
  $$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$$
- Welfare loss (DWL) is the area of the EV not transferred to the government via tax revenue:
  $$DWL = EV - T$$

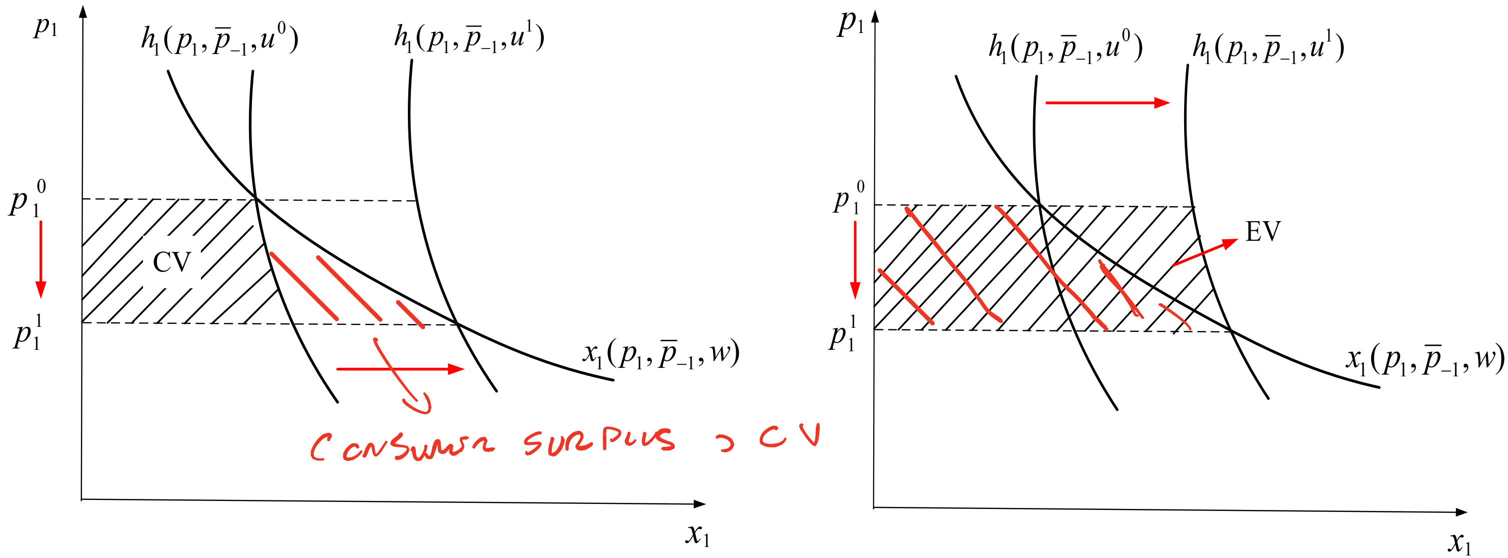


Why not use the Walrasian demand?

- Walrasian demand is easier to observe, so we could use the variation in consumer's surplus as an approximation of welfare changes.
- This is only valid when income effects are *zero*:
  - Recall that the Walrasian demand measures both income and substitution effects resulting from a price change, while
  - **The Hicksian demand measures only substitution effects from such a price change**.
- Hence, there will be a difference between CV and Consumer Surplus (CS), and between EV and CS (area under the Walrasian demand, between prices).

Why not use the Walrasian demand?

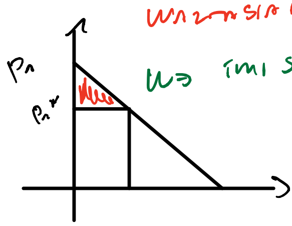
- Normal goods (i.e. W-demand flatter than H-demand)



$CV < CS$ $\quad\quad$ $CS < EV$

WALRASIAN

WE THIS AREA IS THE CONSUMER SURPLUS

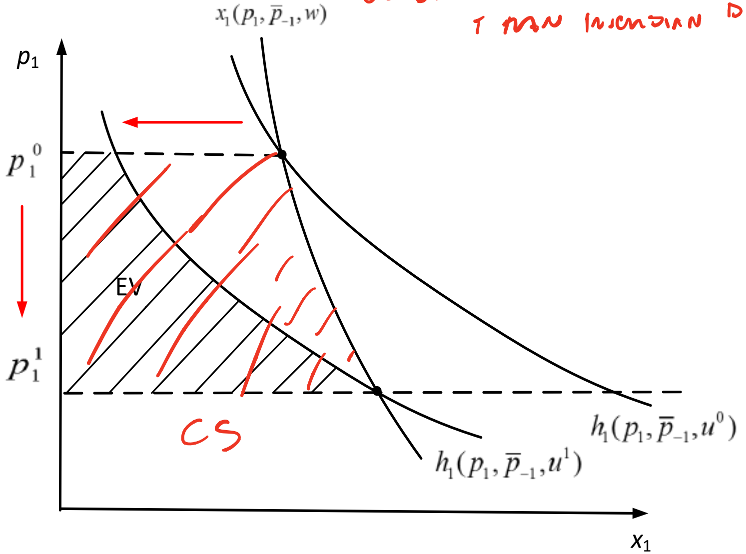


REDUCTION IN PRICE

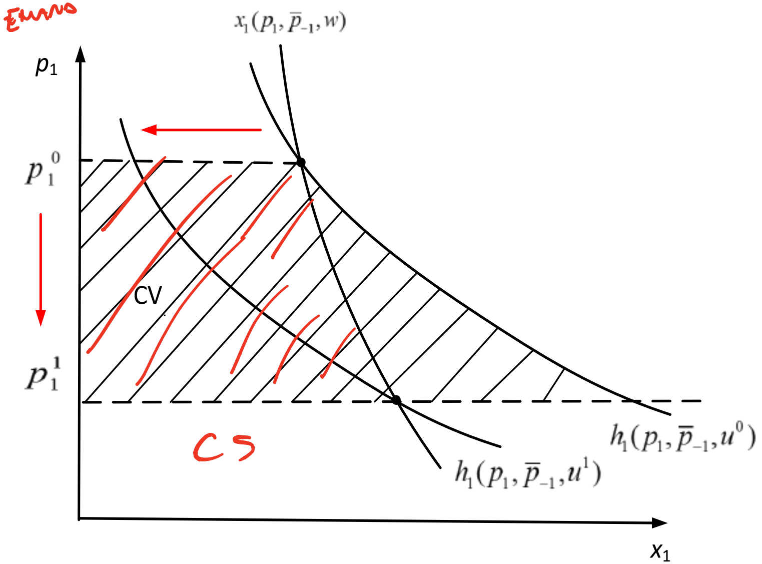
Why not use the Walrasian demand?

- Inferior goods: (i.e. H-demand flatter than W-demand)

WALRASIAN DEMAND IS STEEPER THAN HICKSIAN DEMAND



$EV < CS$



$CS < CV$

SUMMARISE

Why not use the Walrasian demand?

- For normal goods:
  - Price decrease: $CV < CS < EV$
  - Price increase: $CV > CS > EV$

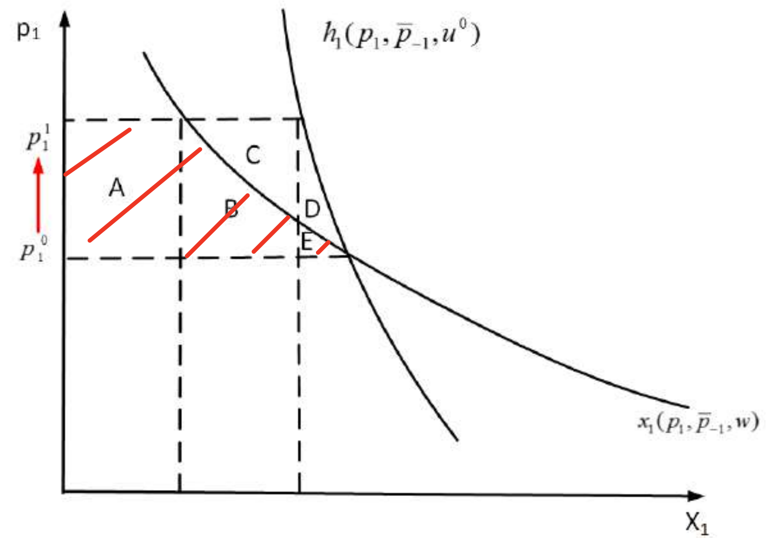
- For inferior goods we find the opposite ranking:
  - Price decrease: $CV > CS > EV$
  - Price increase: $CV < CS < EV$

- NOTE: consumer surplus is also referred to as the ***area variation*** (AV).

When can we use the Walrasian demand?

- When the price change is small (using AV):
  - $CV = A + B + C + D + E$
  - $CS = A + B + E$
  - Measurement error from using CS (or AV) is $C + D$

Area under demand



When can we use the Walrasian demand?

- The measurement difference between CV (and EV) and CS, $C + D$, is relatively small:
  1) When **income effects are small**:
     - Graphically, $x(p, w)$ and $h(p, u)$ almost coincide.
     - The welfare change using the CV and EV coincide too.
  2) When the **price change is very small**:
     - The error involved in using *AV*, i.e., areas $C + D$, as a fraction of the true welfare change, becomes small. That is,

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{CV} = 0$$

When can we use the Walrasian demand?

- However, if we measure the approximation error by $\frac{C+D}{DW}$, where $DW = D + E$, then

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{DW}$$

does not necessarily converge to zero.

Application of IE and SE

TE = SE + IE → SE = TE − IE

- From the Slutsky equation, we know

$$\frac{\partial h_1(p,u)}{\partial p_1} = \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)$$

BUDGET ↑ PRICE CHANGE

- Multiplying both terms by $\frac{p_1}{x_1}$,

$$\frac{\partial h_1(p,u)}{\partial p_1} \boxed{\frac{p_1}{x_1}} = \frac{\partial x_1(p,w)}{\partial p_1} \boxed{\frac{p_1}{x_1}} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \frac{p_1}{x_1}$$

And multiplying all terms by $\frac{w}{w} = 1$,

$$\underbrace{\frac{\partial h_1(p,u)}{\partial p_1} \frac{p_1}{x_1}}_{\substack{\text{Substitution Price} \\ \text{elasticity of demand} \\ \tilde{\varepsilon}_{p,Q}}} = \underbrace{\frac{\partial x_1(p,w)}{\partial p_1} \frac{p_1}{x_1}}_{\substack{\text{Price elasticity} \\ \text{of demand} \\ \varepsilon_{p,Q}}} + \boxed{\underbrace{\frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \frac{p_1}{x_1} \frac{w}{w}}_{?}}$$

Elasticity of walrasian demand with respect to price

$$\left[\frac{\partial x_1}{\partial w} \frac{w}{\delta x_1}\right] \frac{x_1 p_1}{w}$$

SHARE ON THE BUDGET [illegible] ON GOOD 1

Elasticity is the percentage change of a variable divided by the percentage in a second variable.

$$\varepsilon_x = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} \longrightarrow \frac{\Delta x}{\Delta p} \cdot \frac{p}{x} \rightarrow \frac{\partial x}{\partial p} \cdot \frac{p}{x}$$

To get elasticity we moltiply both side by the same ratio (p1/x1)
Also then multiply by w/w for the last term (w/w which is 1) but convenient.

For elasticity then, we can write this in this way

Application of IE and SE

- Rearranging the last term, we have

$$\frac{\partial x_1(p,w)}{\partial w} x_1(p,w) \frac{p_1}{x_1} \frac{w}{w}$$

$$= \underbrace{\frac{\partial x_1(p,w)}{\partial w} \frac{w}{x_1}}_{\substack{\text{Income elasticity} \\ \text{of demand} \\ \varepsilon_{w,Q}}} \cdot \underbrace{\frac{p_1 x_1(p,w)}{w}}_{\substack{\text{Share of budget} \\ \text{spent on good 1, } \theta}}$$

- We can then rewrite the Slutsky equation in terms of elasticities as follows

$$\underbrace{\tilde{\varepsilon}_{p,Q}}_{SE} = \underbrace{\varepsilon_{p,Q}}_{TE} + \underbrace{\varepsilon_{w,Q} \cdot \theta}_{IE}$$

If income very close to 0 then SE = TE. So if eps is 0 no income effect or if income effect is very small. So this one case we can use walrasian demand instead of Hicksian demand to do welfare analysis. Also if share of budget spent on good 1 is closer to 0

Application of IE and SE

- ***Example***: consider a good like housing, with $\theta = 0.4$, $\varepsilon_{w,Q} = 1.38$, and $\varepsilon_{p,Q} = -0.6$.
- Therefore,

Much smaller than walrasian demand! (1.38)

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta = -0.6 + 1.38 \cdot 0.4 = \boxed{-0.05}$$

- If price of housing rises by **10%**, and consumers do not receive a wealth compensation to maintain their welfare unchanged, consumers reduce their consumption of housing by 6%.
- However, if consumers receive a wealth compensation, the housing consumption will only fall by 0.5%.
  - Intuition: Housing is such an important share of my monthly expenses, that higher prices lead me to significantly reduce my consumption (if not compensated), but to just slightly do so (if compensated).

Share on the budget is not small in housing. In this example testa is 0.4 so 40% of IE. So this term is not close to 0. We can use walrasian demand instead of Hicksian demand to have some infos about elasticity of housing with respect to income.

What does elasticity of 1.38 means?
You cannot by a piece of house so we can measure it with square feet. So 1.38 if your income increase by 1% the demand for housing increase 1.38% so demand increase more than demand in proportion.

This means that elasticity is not small at all. So we can predict and we expect and increase of 10% in prices. So when we only consider substitution effect and walrasian demand (uncompensated demand). In this case we already have the estimati which is -0.6. So if price increase 1% the demand for housing decrease for 0.6%.

We can compute compensated demand in price change.
First of all we get the substitution elasticity that we can get from the parameter.

Application of IE and SE

- Other useful lessons from the previous expression

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta$$

- Price-elasticities very close $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$ if
  - Share of budget spent on this particular good, θ, is very small (Example: garlic).
  - The income-elasticity is really small (Example: pizza).

- Advantages if $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$:
  - The Walrasian and Hicksian demand are very close to each other. Hence, $CV \simeq EV \simeq CS$.

Application of IE and SE

- You can read sometimes "in this study we use the change in CS to measure welfare changes due to a price increase given that income effects are negligible"
  - What the authors are referring to is:
    - Share of budget spent on the good is relatively small and/or
    - The income-elasticity of the good is small
- Remember that our results are not only applicable to price changes, but also to changes in the sales taxes.
- For which preference relations a price change induces no income effect? Quasilinear.

Application of IE and SE

- In 1981 the US negotiated voluntary automobile export restrictions with the Japanese government.
- Clifford Winston (1987) studied the effects of these export restrictions:
  - Car prices: p_{Jap} was 20% higher with restrictions that without. p_{US} was 8% higher with restrictions than without.
  - What is the effect of these higher prices on consumer's welfare?
  - Would you use CS? Probably not, since both θ and $\varepsilon_{w,Q}$ are relatively high.

Imaging import tax. So what happen to the consumer? The demand decreases since the prices increases and we are going to replace with internally goods.
On average we tend to replace internal good instead of abroad good but prices will increase.

We can evaluate in advance to evaluate the introduction of import tax.

Application of IE and SE

- Winston did not use CS. Instead, he focused on the CV. He found that CV = -$14 billion.
  - *Intuition*: The wealth compensation that domestic car owners would need after the price change (after setting the export restrictions) in order to be as well off as they were before the price change is $14 billion.
- This implies that, considering that in 1987 there were 179 million car owners in the US, the wealth compensation per car owner should have been $14,000/$179 = $78.
- Of course, this is an underestimation, since we should divide over the new number of car owners (lower) during the period of export restriction was active (not the number of all current car owners).

Application of IE and SE

- Jerry Hausmann (MIT) measures the welfare gain consumers obtain from the price decrease they experience after a Walmart store locates in their locality/country.
- He used CV. Why? Low-income families spend a non-negligible part of their budget in Wal-Mart.
- Result: welfare improvement of 3.75%.

Advanced Microeconomic Theory

Chapter 3: Gross and net complements and substitutes, and substitutability across goods

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

Gross/Net Complements and Gross/Net Substitutes

Perfect substitute we are looking for the crossite

Demand Relationships among Goods

- So far, we were focusing on the SE and IE of varying the price of good k on the demand for good k.

- Now, we analyze the SE and IE of varying the price of good k on the demand for other good j.

Demand Relationships among Goods

- For simplicity, let us start our analysis with the two-good case.
  - This will help us graphically illustrate the main intuitions.

- Later on we generalize our analysis to $N > 2$ goods.

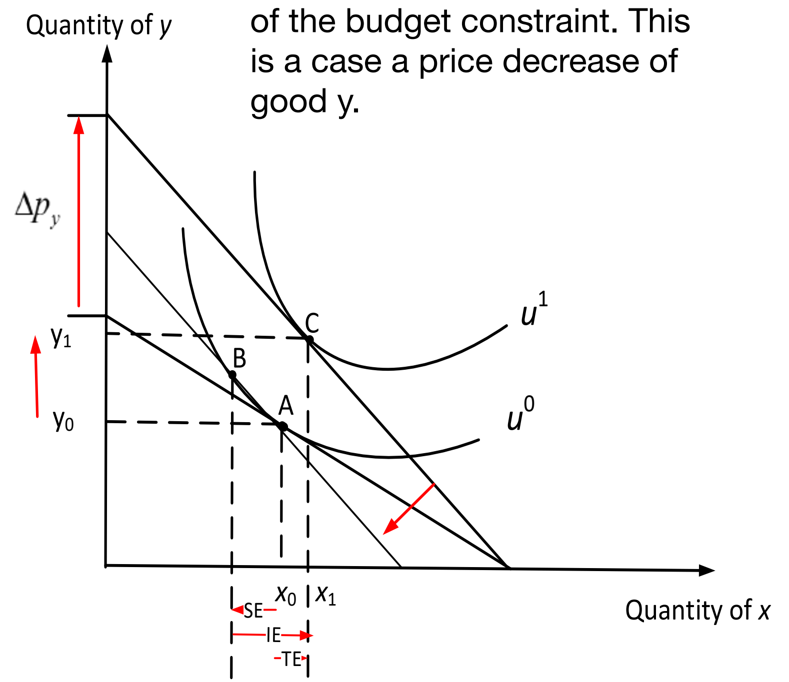
Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *small* that the consumer purchases more x and more y.
  - In this case, we call x and y ***gross complements***.

$$\boxed{\frac{\partial x}{\partial p_y} < 0}$$

NEGATIVE DERIVATIVE

If y increase BC shift up or down. So there is a rotation of the budget constraint. This is a case a price decrease of good y.



C is the new walrasian demand and account for total effect. Moving A to C. The price of Py decrease and quantity of x increase so TE is positive. What about demand for y? Increases. Demand of both increase due to decrease in price of y.

Are the two good complement or substitute?
So see the walrasian or Hicksian? WALRASIAN and they are complements. If der negative they are moving in opposite direction: if py decrease the demand for x increases.

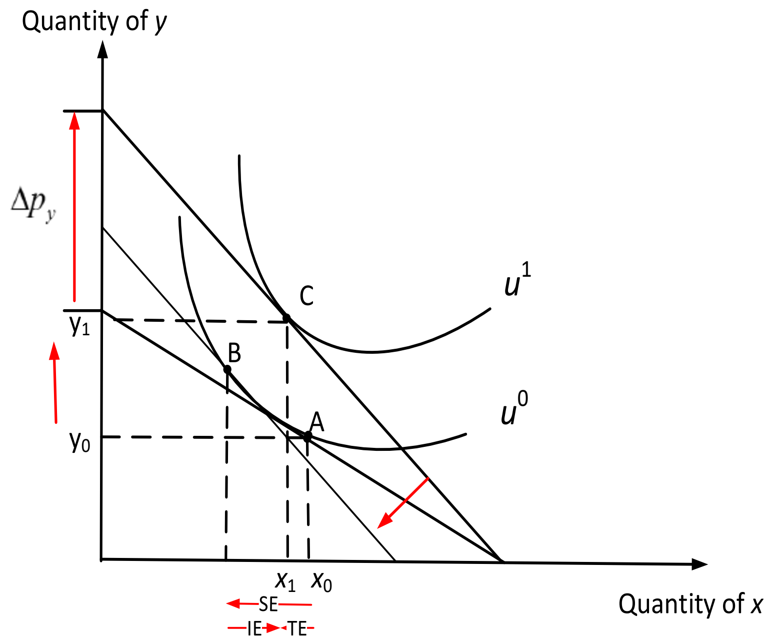
From A to B demand for X decrease, but demand of y increases. So this is the SE for good y.
From B to C we can find income effect: are the two good normal or inferior? Demand for X increase so x is normal. Demand for Y increase so also Y is normal.

Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *large* that the consumer purchases less x and more y.
  - In this case, we call x and y ***gross substitutes***.

$$\boxed{\frac{\partial x}{\partial p_y} > 0}$$

POSITIVE DERIVATIVE → SAME DIRECTION



Demand Relationships among Goods: The Two-Good Case

- A mathematical treatment
  - The change in x caused by changes in p_y can be shown by a Slutsky-type equation:

$$\underbrace{\frac{\partial x}{\partial p_y} = \underbrace{\frac{\partial h_x}{\partial p_y}}_{SE\ (+)} - \underbrace{y\frac{\partial x}{\partial w}}_{\substack{IE: \\ (-)\text{ if } x \text{ is normal} \\ (+)\text{ if } x \text{ is inferior}}}}_{\text{Combined effect (ambiguous)}}$$

  $SE > 0$ is not a typo: Δp_y induces the consumer to buy more of good x, if his utility level is kept constant. Graphically, we are moving along the same indifference curve.

Demand Relationships among Goods: The Two-Good Case

- Or, in elasticity terms

$$\varepsilon_{x,p_y} = \underbrace{\tilde{\varepsilon}_{x,p_y}}_{SE\ (+)} - \underbrace{\theta_y \varepsilon_{x,w}}_{\substack{IE: \\ (-) \text{ if } x \text{ is normal} \\ (+) \text{ if } x \text{ is inferior}}}$$

  where θ_y denotes the share of income spent on good y. The combined effect of Δp_y on the observable Walrasian demand, $x(p,w)$, is ambiguous.

Demand Relationships among Goods: The Two-Good Case

- ***Example***: Let's show the SE and IE across different goods for a Cobb-Douglas utility function $\underline{u(x, y) = x^{0.5} y^{0.5}}$.
  - The Walrasian demand for good x is $\rightarrow$ FOR X

    OPT. DEMAND

$$x(p, w) = \frac{1}{2} \frac{w}{p_x} \qquad = \emptyset$$

  - The Hicksian demand for good x is $\rightarrow$ HICKSIAN DEMAND

$$h_x(p, u) = \frac{\sqrt{p_y}}{\sqrt{p_x}} \cdot u$$

Is X gross complement or substitute with respect to y?
What we have to do?
WE CAN USE DERIVATIVE OF x with respect to Py $\longrightarrow \dfrac{\partial x}{\partial P_y} = 0$

By looking at the walrasian demand the consumption of x and y is independent.
Let's see the income and the substitution effect for these cobb Douglas.

If we look at the Hicksian demand: What would you conclude between the relationship between X or Y (are they net complement or substitutes?)
The derivative here is

$$\frac{\partial h_x}{\partial P_y} = \frac{1}{2} P_y^{-1/2} P_x^{-1/2} u > 0$$

THEY ARE NOT SUBSTITUTES
SINCE DER IS > 0

Effect Walrasian demand is 0
What about income effect? Is the same as the substitution effect since TE = 0 of increasing Py. So IE = SE and opposite.
So the effect on walrasian demand is 0 and we can prove this if we compute the SE of the derivative here (sopra).

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - First, not that differentiating $x(p, w)$ with respect to p_y, we obtain

$$\frac{\partial x(p, w)}{\partial p_y} = 0$$

    i.e., variations in the price of good y do not affect consumer's Walrasian demand.
  - But,

$$\frac{\partial h_x(p, u)}{\partial p_y} = \frac{1}{2} \frac{u}{\sqrt{p_x p_y}} \neq 0$$

  - How can these two (seemingly contradictory) results arise?

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Answer: the SE and IE completely offset each other.
  - **Substitution Effect**: Given

$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}},$$

plug Walrasian demands for x and y in u(x,y) to get the indirect utility function $u = \frac{1}{2}\frac{w}{\sqrt{p_x p_y}}$, and replace it in the expression above to obtain a SE of $\frac{1}{4}\frac{w}{p_x p_y}$.

  - **Income Effect**: –> Right side of slushy equation

at Walrasian   Walrasian

$$-y\frac{\partial x}{\partial w} = -\left(\frac{1}{2}\frac{w}{p_y}\right)\left(\frac{1}{2}\frac{1}{p_x}\right) = -\frac{1}{4}\frac{w}{p_x p_y}$$

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  – Therefore, the total effect is

$$\overbrace{\frac{\partial x(p,w)}{\partial p_y}}^{TE} = \overbrace{\frac{\partial h_x}{\partial p_y}}^{SE} - \overbrace{y\frac{\partial x}{\partial w}}^{IE}$$
$$= \frac{1}{4}\frac{w}{p_x p_y} - \frac{1}{4}\frac{w}{p_x p_y} = 0$$

  – Intuitively, this implies that the substitution and income effect completely offset each other.

Demand Relationships among Goods: The Two-Good Case

- Common mistake:
  - “ $\frac{\partial x(p,w)}{\partial p_y} = 0$ means that good x and y cannot be substituted in consumption. That is, they must be consumed in fixed proportions (perfect complents). Hence, this consumer’s utility function is a Leontief type. ”

- No! We just showed that

$$\frac{\partial x(p,w)}{\partial p_y} = 0 \implies \frac{\partial h_x}{\partial p_y} = y\frac{\partial x}{\partial w}$$

i.e., the SE and IE completely offset each other.

Demand Relationships among Goods: The N-Good Case

- We can, hence, generalize the Slutsky equation to the case of $N > 2$ goods as follows:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial w}$$

for any i and j.

- The change in the price of good j induces IE and SE on good i.

Asymmetry of the Gross Substitute and Complement

$\frac{\partial x}{\partial p_y}$ $\frac{\partial Y}{\partial p_x}$

- Two goods are ***substitutes*** if one good may replace the other in use.
  - Example: tea and coffee, butter and margarine
- Two goods are ***complements*** if they are used together.
  - Example: coffee and cream, fish and chips.
- The concepts of gross substitutes and complements include both SE and IE.
  - Two goods are gross substitutes if $\frac{\partial x_i}{\partial p_j} > 0$.
  - Two goods are gross complements if $\frac{\partial x_i}{\partial p_j} < 0$.

Asymmetry of the Gross Substitute and Complement

- The definitions of gross substitutes and complements are *not* necessarily symmetric.
  - It is possible for x_1 to be a substitute for x_2 and at the same time for x_2 to be a complement of x_1.

- Let us see this potential asymmetry with an example.

We can get that x Perfect Comp to y but
not the contrary.

Asymmetry of the Gross Substitute and Complement

EXAMPLE

- Suppose that the utility function for two goods is given by

$$\underline{U(x,y) = \ln x + y}$$

QUASILINEAR! UTILITY

- The Lagrangian of the UMP is

$$L = \underline{\ln x + y + \lambda(w - p_x x - p_y y)}$$

- The first order conditions are

$$\left\{\begin{array}{l} \dfrac{\partial L}{\partial x} = \dfrac{1}{x} - \lambda p_x = 0 \\ \dfrac{\partial L}{\partial y} = \overset{1}{\cancel{x}} - \lambda p_y = 0 \\ \dfrac{\partial L}{\partial \lambda} = w - p_x x - p_y y = 0 \end{array}\right.$$

OPT DEMAND OF X

$$x^* = \frac{p_y}{p_x}$$

$$w - \cancel{p_x}\,\frac{p_y}{\cancel{p_x}} - p_y\, y = 0$$

OPT DEMAND OF Y

$$y^* = \frac{w - p_y}{p_y}$$

Asymmetry of the Gross Substitute and Complement

- Manipulating the first two equations, we get

$$\frac{1}{p_x x} = \frac{1}{p_y} \implies p_x x = p_y$$

- Inserting this into the budget constraint, we can find the Marshallian demand for y

$$\underbrace{p_x x}_{p_y} + p_y y = w \implies p_y y = w - p_y \implies$$

$$y = \frac{w - p_y}{p_y}$$

THEN CHECK ACROSS COMP OR SUBSTITUTE

$$\frac{\partial x}{\partial p_y} = \frac{\lambda}{p_x} > 0 \quad G.S$$

$\boxed{\text{ASYMMETRIC!!}}$

$$\frac{\partial y}{\partial p_x} = 0$$ INDEPENDENT!

Asymmetry of the Gross Substitute and Complement

- An increase in p_y causes a decline in spending on y
  - Since p_x and w are unchanged, spending on x must rise $\left(\frac{\partial x}{\partial p_y} > 0\right)$.
    - Hence, x and y are gross substitutes.
  - But spending on y is independent of p_x $\left(\frac{\partial y}{\partial p_x} = 0\right)$.
    - Thus, x and y are neither gross substitutes nor gross complements.
  - This shows the asymmetry of gross substitute and complement definitions.
    - While good y is a gross substitute of x, good x is neither a gross substitute or complement of y.

EX 7. MID TERM

$U(x, y) = xy \qquad P_x = 1 \quad P_y = 1 \quad P_x' = 4$

FIND WALRASIAN DEMAND AND INDIRECT UTILITY FUNCTION

$$p_1 x_1 + p_2 x_2 \leq w$$

$$L = xy + \lambda (w - p_x x_1 - p_y x_2)$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = y - \lambda p_x = 0 \\ \dfrac{\partial L}{\partial y} = x - \lambda p_y = 0 \\ \dfrac{\partial L}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \end{cases}$$

$$\frac{x}{y} = \frac{p_x}{p_y} \rightarrow y = \frac{p_x}{p_y} x \rightarrow w - p_1 x - p_y \frac{p_x}{p_y} x = 0$$

$$\boxed{x^* = \frac{w}{2p_x} \qquad y^* = \frac{w}{2p_y}}$$

WALRASIAN DEMAND FUNCTION

WHAT HAPPEN IF PRICE INCREASE? I JUST REPLACE IT IN WALRAS DEMAND INSTEAD OF REDOING THE UMP SOLUTION FOR INCREASED PRICE!

$x^* = \frac{48}{2} = 24$ $\quad A(24, 24)$

$y^* = \frac{48}{2} = 24$

VALUE
INDIRECT UTILITY → VALUE OF THE MAXIMUM

SO REPLACE A IN UTILITY

$U(24, 24) = 24^2$

INDIRECT UTILITY FUNCTION? REPLACE WALRAS DEMAND INTO UTILITY FUNCTION!

$$U(x, y) = x \cdot y = \frac{w}{2p_x} \cdot \frac{w}{2p_y} = \frac{w^2}{2p_x p_y}$$

FIND HICKSIAN DEMAND AND EXPENDITURE FUNCTION

IN GENERAL TO FIND HICKSIAN DEMAND?

MINIMISATION PROBLEM $\rightarrow$ MINIMISE EXPENDITURE

$$\min_{x,y\geq 0} \; p_x X + p_y Y \quad \text{s.t. } x \cdot y = U = 2u^2$$

WE CAN AVOID DOING THIS!

REWRITE IND U. F. AS $v(x,y)$

$$v(x,y) = \frac{w^2}{4p_x p_y}$$

HOW MUCH GUY SPENT?

REPLACE e INTO w

$$v(x,y) = \frac{e^2(p_x, p_y, w)}{4p_x p_y}$$

THIS FUNCTION CAN BE INVERTED

$$e(p_x, p_y, w) = \left(v(x,y) \cdot 4\, p_x p_y\right)^{1/2} =$$

$$= u^{1/2} \cdot 2\,(p_x p_y)^{1/2}$$

SOLVING EMP WE CAN GET SAME RESULT

INVERTING $v(x,y) = \frac{e^2(p_x, p_y, w)}{4p_x p_y}$

IF WE SOLVE EMP WE GET BEFORE ARGUMENT AND THEN VALUE OF THE FUNCTION

HERE WE ARE DOING THE OPPOSITE

$$\frac{\partial e}{\partial p_x} = u^{1/2} \cdot 2\left(\frac{1}{2}\right) p_x^{-1/2} p_y^{1/2} = \underline{u^{1/2}\left(\frac{p_y}{p_x}\right)^{1/2}} = h_x$$

$$\frac{\partial e}{\partial p_y} = u^{1/2}\, 2 \cdot \left(\tfrac{1}{2} p_y^{-1/2} p_x^{1/2}\right) = \underline{u^{1/2}}\left(\frac{p_x}{p_y}\right)^{1/2} = h_y$$

TO GET EXACT VALUE REPLACE VALUES

GET WALRASIAN DEMAND

$p_x = 4 \quad p_y = 1$

UMP? WE CAN REPLACE WITH PREVIOUS

$w = 48$

$x^* = \frac{w}{2p_x} = \frac{48}{8} = 6 \quad \Rightarrow \quad ((6, 24)$

$y^* = \frac{w}{2p_y} = 24$

SMALL WHAT HAPPEN TO BC BECAUSE OF PRICE CHANGE

(USUALLY EXACT SCALE IS NOT IMPORTANT)

TOTAL, SUBSTITUTION AND INCOME EFFECT

TOTAL => A TO C

$TE \; x = \overset{Cx}{6} - \overset{Ax}{24} = -18$

$TE \; y = \overset{Cy}{24} - \overset{Ay}{24} = 0$

SUBSTITUTION => NEW MAXIMISATION PROBLEM

COMPUTING HICKSIAN FOR COMPENSATED DEMAND

UTILITY IS SAME AFTER CHANGING PRICE

$$\begin{cases} \\ x \cdot y = 24^2 \end{cases}$$

-> SAME U BEFORE PRICE CHANGE

SLOPE OF IC TANG SLOPE OF B.C. ??

JUST EXPLOIT TANG CONDITION ON

$$\frac{y}{x} = \frac{P_x}{P_y}$$

MRS = SLOPE OF BC

$$\begin{cases} \frac{y}{x} = \frac{P'_x}{P_y} \\ x \cdot y = 24^2 \end{cases} \rightarrow \quad \frac{y}{x} = \frac{P_x}{P_y} \rightarrow y = \frac{P_x x}{y}$$

$$x \cdot y = u \rightarrow x \cdot \left(\frac{P_x x}{y}\right) = u$$

$$\to \quad x^2 = \frac{p_y}{p_x} u$$

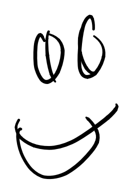
$$h_x = \left(\frac{p_y}{p_x}\right)^{1/2} u^{1/2} \qquad h_y = \left(\frac{p_x}{p_y}\right)\left(\frac{p_y}{p_x}\right)^{1/2} u^{1/2}$$

$$= \left(\frac{p_x}{p_y}\right)^{1/2} u^{1/2}$$

NOT REPLACING BECAUSE COMPARING THIS SOLUTION WITH HICKSIAN WITH SHEPPARD (THEY ARE THE SAME)

! CAN GET HICKSIAN DEMAND WITHOUT SOLVING UMP

PICK ONE



Asymmetry of the Gross Substitute and Complement

- Depending on how we check for gross substitutability or complementarities between two goods, there is potential to obtain different results.

- Can we use an alternative approach to check if two goods are complements or substitutes in consumption?
  – Yes. We next present such approach.

Net Substitutes and Net Complements

- The concepts of net substitutes and complements focus solely on SE.
  - Two goods are ***net (or Hicksian) substitutes*** if
$$\frac{\partial h_i}{\partial p_j} > 0$$
  - Two goods are ***net (or Hicksian) complements*** if
$$\frac{\partial h_i}{\partial p_j} < 0$$

  where $h_i(p_i, p_j, u)$ is the Hicksian demand of good i.

Net Substitutes and Net Complements

- This definition looks only at the shape of the indifference curve.
- This definition is unambiguous because the definitions are perfectly symmetric

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

  - This implies that every element above the main diagonal in the Slutsky matrix is symmetric with respect to the corresponding element below the main diagonal.

Net Substitutes and Net Complements

$$S(p,w) = \begin{pmatrix} \dfrac{\partial h_1(p,u)}{\partial p_1} & \dfrac{\partial h_1(p,u)}{\partial p_2} & \dfrac{\partial h_1(p,u)}{\partial p_3} \\[2ex] \dfrac{\partial h_2(p,u)}{\partial p_1} & \dfrac{\partial h_2(p,u)}{\partial p_2} & \dfrac{\partial h_2(p,u)}{\partial p_3} \\[2ex] \dfrac{\partial h_3(p,u)}{\partial p_1} & \dfrac{\partial h_3(p,u)}{\partial p_2} & \dfrac{\partial h_3(p,u)}{\partial p_3} \end{pmatrix}$$

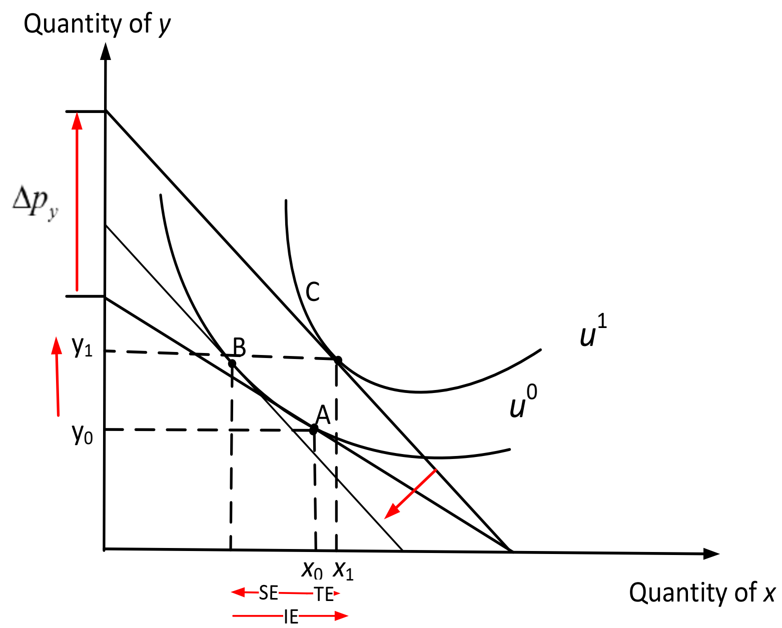
Net Substitutes and Net Complements

- Proof:
  - Recall that, from Shephard's lemma, $h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}$. Hence,
$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_k \partial p_j}$$
  - Using Young's theorem, we obtain
$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j \partial p_k}$$
which implies
$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_k}$$

Net Substitutes and Net Complements

- Even though x and y are gross complements, they are net substitutes.

- Since MRS is diminishing, the own-price SE must be negative $(SE < 0)$ so the cross-price SE must be positive $(TE > 0)$.



A Note on the Euler's Theorem

- We say that a function $f(x_1, x_2)$ is homogeneous of degree k if
$$f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$$
- Differentiating this expression with respect to x_1, we obtain
$$\frac{\partial f(tx_1, tx_2)}{\partial x_1} \cdot t = t^k \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$
or, rearranging,
$$\frac{\partial f(tx_1, tx_2)}{\partial x_1} = t^{k-1} \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$

A Note on the Euler's Theorem

- Last, denoting $f_1 \equiv \frac{\partial f}{\partial x_1}$, we obtain

$$f_1(tx_1, tx_2) = t^{k-1} \cdot f_1(x_1, x_2)$$

- Hence, if a function is homogeneous of degree k, its first-order derivative must be homogeneous of degree $k - 1$.

A Note on the Euler's Theorem

- Differentiating the left-hand side of the definition of homogeneity, $f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$, with respect to t yields

$$\frac{\partial(tx_1, tx_2)}{\partial t} = f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$

- Differentiating the right-hand side produces

$$\frac{\partial(t^k \cdot f(x_1, x_2)}{\partial t} = k \cdot t^{k-1} f(x_1, x_2)$$

A Note on the Euler's Theorem

- Combining the differentiation of LHS and RHS,

$$f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$
$$= k \cdot t^{k-1} f(x_1, x_2)$$

- Setting $t = 1$, we obtain

$$f_1(x_1, x_2)x_1 + f_2(x_1, x_2)x_2 = k \cdot f(x_1, x_2)$$

where k is the homogeneity order of the original function $f(x_1, x_2)$.
  - If $k = 0$, the above expression becomes 0.
  - If $k = 1$, the above expression is $f(x_1, x_2)$.

A Note on the Euler's Theorem

- ***Application***:
  - The Hicksian demand is homogeneous of degree zero in prices, that is,
$$h_k(tp_1, tp_2, \dots, tp_n, u) = h_k(p_1, p_2, \dots, p_n, u)$$
  - Hence, multiplying all prices by t does not affect the value of the Hicksian demand.
  - By Euler's theorem,
$$\frac{\partial h_i}{\partial p_1} p_1 + \frac{\partial h_i}{\partial p_2} p_2 + \cdots + \frac{\partial h_i}{\partial p_n} p_n$$
$$= 0 \cdot t^{0-1} h_i(p_1, p_2, \dots, p_n, u) = 0$$

Substitutability with Many Goods

- ***Question:*** Is net substitutability or complementarity more prevalent in real life?
- To answer this question, we can start with the compensated demand function

$$h_k(p_1, p_2, \dots, p_n, u)$$

- Applying Euler's theorem yields

$$\frac{\partial h_k}{\partial p_1} p_1 + \frac{\partial h_k}{\partial p_2} p_2 + \cdots + \frac{\partial h_k}{\partial p_n} p_n = 0$$

- Dividing both sides by h_k, we can alternatively express the above result using compensated elasticities

$$\tilde{\varepsilon}_{i1} + \tilde{\varepsilon}_{i2} + \cdots + \tilde{\varepsilon}_{in} \equiv 0$$

Substitutability with Many Goods

- Since the negative sign of the SE implies that $\tilde{\varepsilon}_{ii} \leq 0$, then the sum of Hicksian cross-price elasticities for all other $j \neq i$ goods should satisfy

$$\sum_{j \neq i} \tilde{\varepsilon}_{ij} \geq 0$$

- Hence, "most" goods must be substitutes.
- This is referred to as ***Hick's second law of demand***.

$$h_x = \left(\frac{P_y}{P_x}\right)^{1/2} u^{1/2} = \left(\frac{1}{4}\right)^{1/2} (24^2)^{1/2} = \frac{1}{2}\cdot 24 = 12$$

$$h_y = \left(\frac{P_x}{P_y}\right)^{1/2} u^{1/2} = 4^{1/2} (24^2)^{1/2} = 2\cdot 24 = 48$$

$B(12, 48)$

C ? $x^* = \frac{w}{2P_x} = \frac{48}{8} = 6$ $C(6, 24)$

$y^* = \frac{w}{2P_y} = \frac{48}{2} = 24$

SE => FROM (A) TO (B) → B - A

SE x = $12 - 24 = -12$

SE y = $48 - 24 = +24$

IE => FROM (B) TO (C) → C - B

IE $x = 6 - 12 = -6$

IE $y = 24 - 48 = -24$

TE => SE + IE

$TE_x = -12 - 6 = -18$

TE $y = 0$

C.V. A B → AFTER BEFORE $P_x \uparrow$

$$CV = e(P_x^1, u_0) - \boxed{e(P_x^1, u_1)} = P_x^1 \cdot h_x + P_y h_y - 48 =$$

$w = 48$

POSITIVE INCOME TRANSFER

SOLUTION OF EMP → WE GET BUNDLE CALLED HICKSIAN DEMAND

$$= \overset{\text{NEW PRICE}}{4} \cdot 12 + 1 \cdot 48 - (48) = 96 - 48 = 48$$

TO KEEP THE GUY AT THE SAME LEVEL AFTER PRICE CHANGE WE HAVE TO TRANSFER 48 OF WEALTH

EV BA → BEFORE AFTER

$$EV = \underbrace{e(p_x^0, u_0)}_{w = 48} - e(p_x^0, u_1) =$$

$$h_x(p_x^0, p_y, u_1) = h_x^*$$
$$h_y(p_x^0, p_y, u_1) = h_y^*$$

$$e(p_x, p_y, u_1) = h_x^*$$
$$e(p_x, p_y, u_1) = h_y^*$$

HICKSIAN
$$\Rightarrow \left(\frac{p_x}{p_y}\right)^{1/2} u^{1/2}$$

So $u_1 = u(\overset{(6,\,24)}{c}) = u(6, 24) = 144$

$$w - 24 = 48 - 24 = 24$$

BEF PRICE CHANGE, TO GET SAME VALUE WE HAVE TO TAKE AWAY SOME INCOME

IF $p\uparrow \Rightarrow u\downarrow$ → SO TAKE INCOME TO IT THAT IS 24

NEGATIVE INCOME TRANSFER

PRICE OF CO SUITE IS W

THE MORE YOU PAY A GUY, MORE HE WILL WORK

IF YOU BECOME A MILLIONAIRE, 1C €

Advanced Microeconomic Theory

Chapter 3: Aggregate demand

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

Aggregate Demand

Aggregate Demand

WALRAS

- We now move from individual demand, $x_i(p, w_i)$, to aggregate demand,

$$\boxed{\sum_{i=1}^{I} x_i(p, w_i)}$$

AGG. OF WALRAS OF EACH INDIVIDUAL

which denotes the total demand of a group of I consumers.

- Individual i's demand $x_i(p, w_i)$ still represents a vector of L components, describing his demand for L different goods.

IS CONVENIENT TO HAVE AGGREGATE DEMAND THAT DEPENDS ON AGGREGATE WEALTH

Aggregate Demand

- We know individual demand depends on prices and individual's wealth.
  - When can we express aggregate demand as a function of prices and aggregate wealth?
  - In other words, when can we guarantee that aggregate demand defined as

$$x(p, w_1, w_2, \dots, w_I) = \sum_{i=1}^{I} x_i(p, w_i)$$

satisfies SUM OF IND DEMAND FUNCTION OF P AND

$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

SUM OF WEALTH

(AND NOT HOW WEALTH IS DISTRIBUTED)

Aggregate Demand

- This is satisfied if, for any two distributions of wealth, $(w_1, w_2, \dots, w_I)$ and $(w_1', w_2', \dots, w_I')$ such that $\sum_{i=1}^{I} w_i = \sum_{i=1}^{I} w_i'$, we have

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} x_i(p, w_i')$$

- For such condition to be satisfied, let's start with an initial distribution $(w_1, w_2, \dots, w_I)$ and apply a differential change in wealth $(dw_1, dw_2, \dots, dw_I)$ such that the aggregate wealth is unchanged, $\sum_{i=1}^{I} dw_i = 0$.

Aggregate Demand

- If aggregate demand is just a function of aggregate wealth, then we must have that

$$\sum_{i=1}^{I} \frac{\partial x_i(p,w_i)}{\partial w_i} dw_i = 0 \text{ for every good } k$$

In words, the wealth effects of different individuals are compensated in the aggregate. That is, in the case of two individuals i and j,

$$\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$$

for every good k.

Aggregate Demand

- This result *does not* imply that $IE_i > 0$ while $IE_j < 0$.
- In addition, it indicates that its absolute values coincide, i.e., $|IE_i| = \left|IE_j\right|$, which means that any redistribution of wealth from consumer i to j yields

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} dw_i + \frac{\partial x_{kj}(p, w_j)}{\partial w_j} dw_j = 0$$

which can be rearranged as

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} \underbrace{dw_i}_{-} = -\frac{\partial x_{kj}(p, w_j)}{\partial w_j} \underbrace{dw_j}_{+}$$

- Hence, $\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$, since $|dw_i| = \left|dw_j\right|$.

Aggregate Demand

- In summary, for any
  - fixed price vector p,
  - good k, and
  - wealth level any two individuals i and j

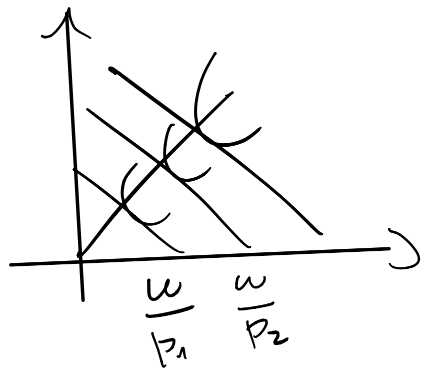
  the wealth effect is the same across individuals.
- In other words, the wealth effects arising from the distribution of wealth across consumers cancel out.
- This means that we can express aggregate demand as a function of aggregate wealth

$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

Aggregate Demand

- Graphically, this condition entails that all consumers exhibit *parallel*, *straight* wealth expansion paths.
  - ***Straight***: wealth effects do not depend on the individuals' wealth level.
  - ***Parallel***: individuals' wealth effects must coincide across individuals.
    - Recall that wealth expansion paths just represent how an individual demand changes as he becomes richer.

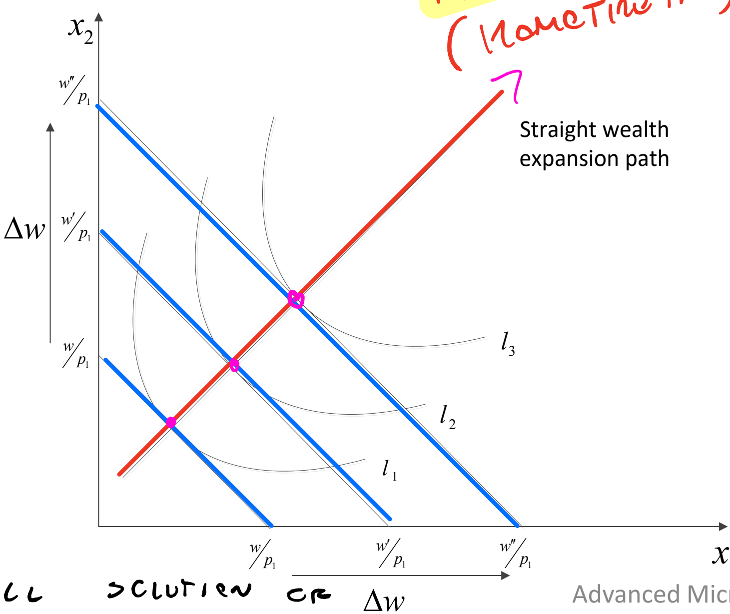
w 也 p



Aggregate Demand

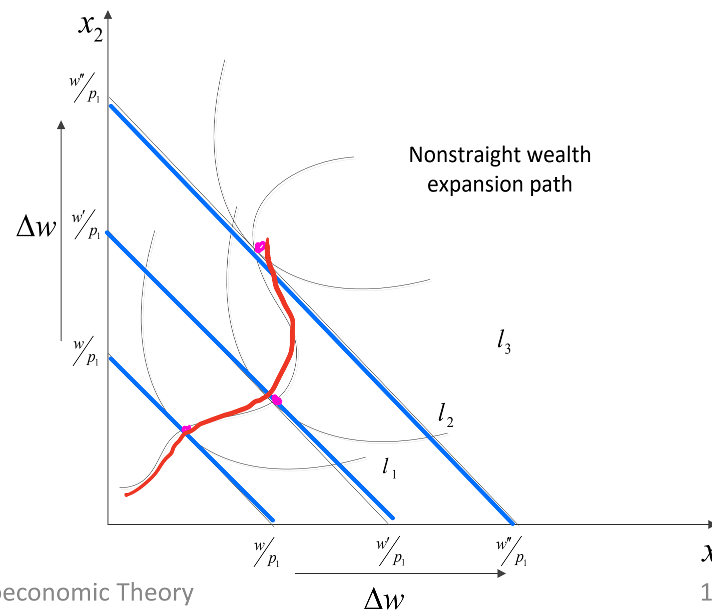
A given increase in wealth leads the same change in the consumption of good x_i, *regardless* of the initial wealth of the individual

HOMOGENEOUS !!
(HOMOTHETIC)



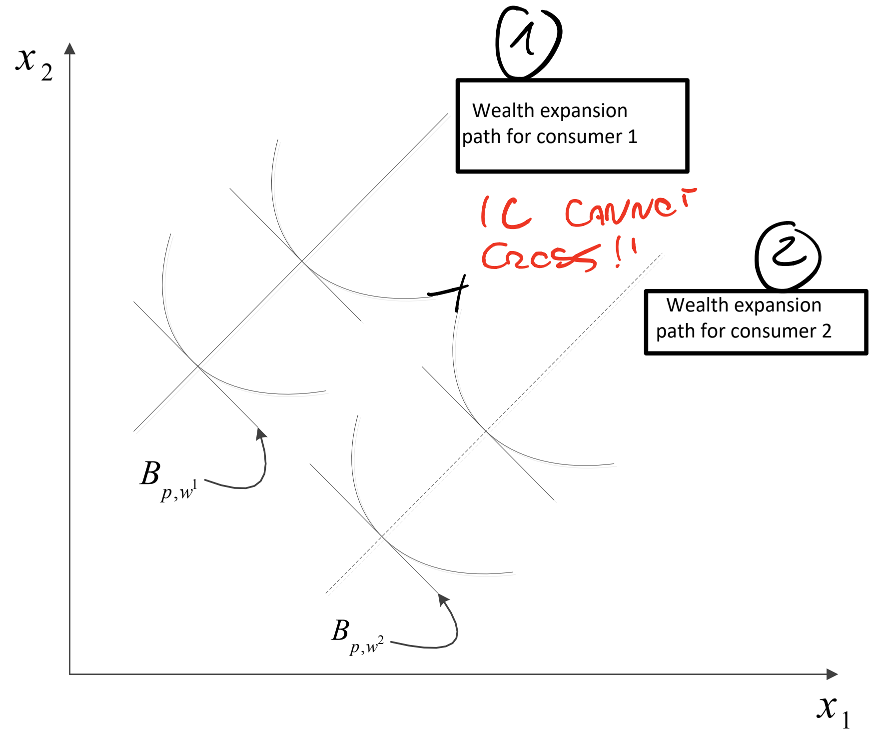
ALL SOLUTION OF UMP ARE IN THE SAME LINE

A given increase in wealth leads to changes in the consumption of good x_i that are *dependent* on the individual's wealth level



Aggregate Demand

- Individuals' wealth effects coincide.
- The wealth expansion path for consumers 1 and 2 are parallel to each other
  - both individuals' demands change similarly as they become richer.



Aggregate Demand

- Preference relations that yield *straight* wealth expansion paths:
  - Homothetic preferences
  - Quasilinear preferences
- Can we embody all these cases as special cases of a particular type of preferences?
  – Yes. We next present such cases.

Aggregate Demand: Gorman Form

- ***Gorman form.*** A necessary and sufficient condition for consumers to exhibit parallel, straight wealth expansion paths is that every consumer's indirect utility function can be expressed as:

PRICE AND WEALTH

$$v_i(p, w_i) = a_i(p) + b(p)w_i$$

→ EACH INDIVIDUAL

This indirect utility function is referred to as the *Gorman form*. (IND UTILITY IS LINEAR IN w_i)

- Indeed, in case of quasilinear preferences

$$v_i(p, w_i) = a_i(p) + \frac{1}{p_k} w_i \quad \text{so that} \quad b(p) = \frac{1}{p_k}$$

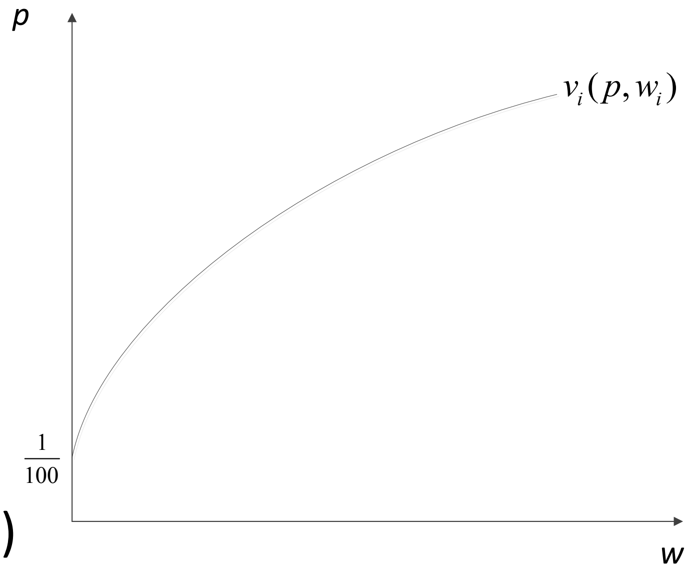
Aggregate Demand: Gorman Form

- ***Example*** (continued):
  - The vertical intercept of this function is $p(0) = \frac{1}{100}$.
  - The slope of this function is

$$\frac{\partial p(w_i)}{\partial w_i} = \frac{1}{10} + \frac{1}{10\sqrt{1 + 40w_i}} > 0$$

and it is decreasing in w_i (concavity)

$$\frac{\partial^2 p(w_i)}{\partial w_i^2} = \frac{2}{(1 + 40w_i)^{3/2}}$$



$x_i = a_i(p) + b(p)\, w_i$ IF LINEAR IN W, WHEN WE TAKE SUMMATION

Aggregate Demand: Gorman Form

$\sum x_i = \sum a_i(p) + \sum b(p)\, w_i$

- Let’s show that, for indirect utility functions of the Gorman form, we obtain

$\sum a_i(p) + b(p) \underbrace{\sum w_i}_{\text{TOTAL WEALTH}}$

SINCE b CONSTANT CAN BE CUT OF SUMMATION

$$\sum_{i=1}^{I} x_i(p, w_i) = x(p, \sum_{i=1}^{I} w_i)$$

SO DO NOT DEP ON HOW W. IS DISTRIBUTED SINCE IS A LINEAR FUNCTION

- First, use Roy’s identity to find the Walrasian demand associated with this indirect utility function

APPLY ROY WE CAN FIND WALRAS DEMAND

$$-\frac{\frac{\partial v_i(p, w_i)}{\partial p}}{\frac{\partial v_i(p, w_i)}{\partial w}} = x_i(p, w_i)$$

NO PROOF

Aggregate Demand: Gorman Form

- In particular, for good j,

$$-\frac{\frac{\partial v_i(p,w_i)}{\partial p_j}}{\frac{\partial v_i(p,w_i)}{\partial w}} = -\frac{\frac{\partial a_i(p)}{\partial p_j}}{b(p)} - \frac{\frac{\partial b(p)}{\partial p_j}}{b(p)} w_i = x_i^j(p,w_i)$$

Walras linear

- In matrix notation,

$$-\frac{\nabla_p v_i(p,w_i)}{\nabla_w v_i(p,w_i)} = -\frac{\nabla_p a_i(p)}{b(p)} - \frac{\nabla_p b(p)}{b(p)} w_i = x_i(p,w_i)$$

for all goods.

Aggregate Demand: Gorman Form

- We can compactly express $x_i(p, w_i)$ as follows

$$-\frac{\nabla_p v_i(p, w_i)}{\nabla_w v_i(p, w_i)} = \alpha_i(p) + \beta(p) w_i = x_i(p, w_i)$$

where $-\frac{\nabla_p a_i(p)}{b(p)} \equiv \alpha_i(p)$ and $-\frac{\nabla_p b(p)}{b(p)} \equiv \beta(p)$.

Aggregate Demand: Gorman Form

- Hence, aggregate demand can be obtained by summing individual demands

$$\alpha_i(p) + \beta(p)w_i = x_i(p, w_i)$$

across all I consumers, which yields

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} \alpha_i(p) + \beta(p) \sum_{i=1}^{I} w_i$$

$$= \sum_{i=1}^{I} \alpha_i(p) + \beta(p)w = x(p, \sum_{i=1}^{I} w_i)$$

where $\sum_{i=1}^{I} w_i = w$.

QUASI LINEAR UTILITY FUNCTION CAN BE WRITTEN IN GORMAN FORM?

QUASILINEAR SINCE LINEAR IN Y

$$u(x,y) = \ln x + y$$

1 BEFORE WE HAVE TO FIND INDIRECT UTILITY 2 AND THEN CHECK LINEAR IN w_i

S.T. $p_x \cdot x + p_y \cdot y \leq w$

$$L = \ln x + y + \lambda(w - p_x \cdot x - p_y \cdot y)$$

INT. SOLUTION

$$\begin{cases} \frac{\partial L}{\partial x} = \frac{1}{x} - \lambda p_x = 0 \\ \frac{\partial L}{\partial y} = 1 - \lambda p_y = 0 \\ \frac{\partial L}{\partial \lambda} = w - p_x x - p_y \cdot y = 0 \end{cases}$$

$x = 0$ $\quad$ $x = \frac{w}{p_x}$ $\quad$ CHECK CORNER SOLUTIONS

$y = \frac{w}{p_y}$ $\quad$ $y = 0$

$$x^* = \frac{p_y}{p_x}$$

$$y^* = w - \cancel{p_x} \frac{p_y}{\cancel{p_x}} + p_y \cdot y \rightarrow y^* = \frac{w - p_y}{p_y}$$

FIND IND. UTILITY

$$u(x,y) = \ln\left(\frac{p_y}{p_x}\right) + \frac{w - p_y}{p_x} \rightarrow y$$

$$= \underbrace{\ln\left(\frac{p_y}{p_x}\right) - 1}_{a} + \underbrace{\frac{1}{p_y}}_{b} w_i$$

SATISFY GORMAN FORM WHERE INTERCEPT IS

$$a_i(p) = \ln\left(\frac{p_y}{p_x}\right) - 1$$

$$b(p) = \frac{1}{p_y}$$

DISTRIBUTION ON INCOME DOES NOT DEPENDS ON THE TOTAL WEALTH

EXERCISE 3.2 P. 176 GARCIA

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$$

FIND CV, CE, CS

UMP (SOLUTIONS, C'È SVOLGIMENTO SUL LIBRO)

$$x_1(p,w) = \frac{\alpha \cdot w}{(\alpha+\beta) p_1} \qquad x_2(p,w) = \frac{\beta \cdot w}{(\alpha+\beta) p_2}$$

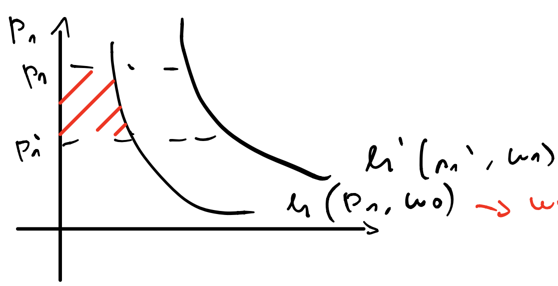
$$h_1(p,w) = \left(\frac{\alpha}{\beta} \frac{p_2}{p_1}\right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

$$h_2(p,w) = \left(\frac{\beta}{\alpha} \frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

CV $\quad p_1 = p_2 = 2 \qquad \alpha = \beta = \frac{1}{2} \qquad w = 10$

$p_1' = 1 \Rightarrow p \downarrow$

AB AFTER BEFORE



$p \downarrow \Rightarrow u \uparrow \Rightarrow h_1$ SHIFT RIGHT

$\rightarrow$ WE ARE USING THIS

2. WAY TO APPLY C.V.

$$CV = \int^{p_1} h_1(p, u_0)\, dp_1 = \int_1^2 \left(\frac{\alpha}{\beta} \frac{p_2}{p_1}\right)^{\frac{\beta}{\alpha+\beta}} u_0^{\frac{1}{\alpha+\beta}}$$

$\cdot\ p_1'$

$p_1' < p_1$ SINCE $p \downarrow$

$$= \int_1^2 \left(\frac{2}{p_1}\right)^{1/2} \frac{\cancel{\frac{1}{2}}\ 1}{\cancel{\frac{1}{2}}\ 1}\ 2.5\ dp_1$$

WE MAINTAIN p_1 AS VARIABLE

$$u_1^{\frac{1}{\alpha+\beta}} = u_0^{1} \rightarrow \text{OLD LEVEL OF UTILITY}$$

$w = 2.5$

WALRAS

$x_1^* = 2.5$

$x_2^* = 2.5$

$$2^{1/2}\ \frac{1}{2}\ (p_1)^{-1/2}\ 2.5\ dp_1 = \int p_1^{-1/2}\ dp_1$$

$$\sqrt{2}\ 2.5 \qquad = \sqrt{2}.25\ (2.\sqrt{2} - 2)$$

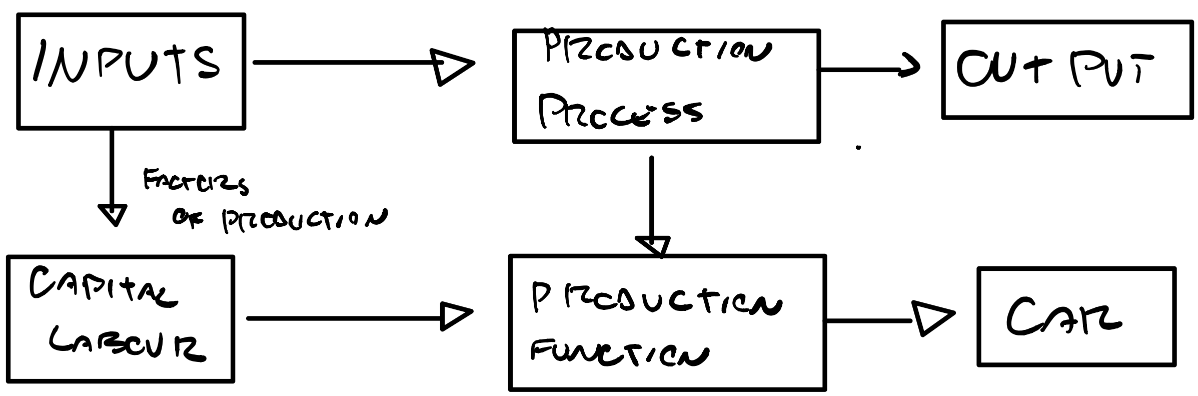
$$\frac{p_1^{-\frac{1}{2}+1}}{\frac{1}{2}} \qquad 2\ p_1^{1/2}$$

Firm to produce use some technologies.
They use inputs that are factors of production that are combine in a production function (production process). And then after the input are combine in the production process they give and output.
To produce a car we will use capital (machinery) and Labor and then there will be a production process that give an output that is car.
Production process can be approximated by a production function

FIRMS

TECHNOLOGIES

—> process from input to output



PRODUCTION FUNCTION

Maximum amount of output possible from input bundle

Advanced Microeconomic Theory

Chapter 4: Production function and Profit Maximization Problem (PMP)

Outline

- Production sets and production functions
- Profit maximization and cost minimization
- Cost functions
- Aggregate supply
- Efficiency (1st and 2nd FTWE)

Production Functions

Technology

- A technology is a process by which inputs are converted to an output. → Given quantity of output
- *E.g.* labor, a computer, a projector, electricity, and software are being combined to produce this lecture.
- Usually several technologies will produce the same product -- a blackboard and chalk can be used instead of a computer and a projector.
- Which technology is "best"?
- How do we compare technologies?

Inputs

When we have technologies we have inputs bundles. It is similar to consumption bundle but refers to the firm to produce certain output.

- x_i denotes the amount used of input i; *i.e.* the level of input i.
- An input bundle is a vector of the input levels; $(x_1, x_2, \ldots, x_n)$.
- *E.g.* $(x_1, x_2, x_3) = (6, 0, 9)$.

Output

- y denotes the output level.
- The technology's production function states the maximum amount of output possible from an input bundle.

PRODUCTION FUNCTION

$$y = f(x_1, x_2, \ldots\ldots x_n)$$

This is a scalar since we are considering only one good as output.

You will have many technologies and production will give the most efficient way of producing y given x1, x2 ... xn

Technology set

- A production plan is an input bundle and an output level; $(x_1, \ldots, x_n, y)$.
- A production plan is feasible if

$$y \leq f(x_1, x_2, \ldots \ldots x_n)$$

- The collection of all feasible production plans is the technology set.

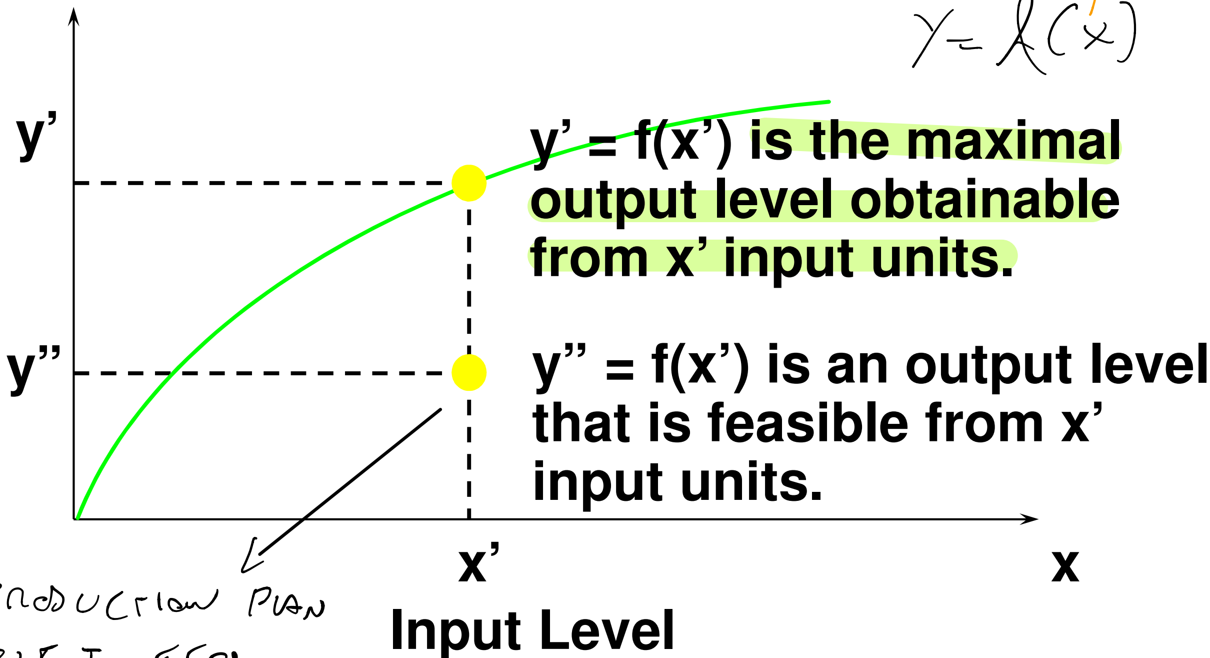
Feasible if this tech logic produce at least y.
So collection of this feasible production plan is called technology set.

Technology set - I

ONLY 1 INPUT

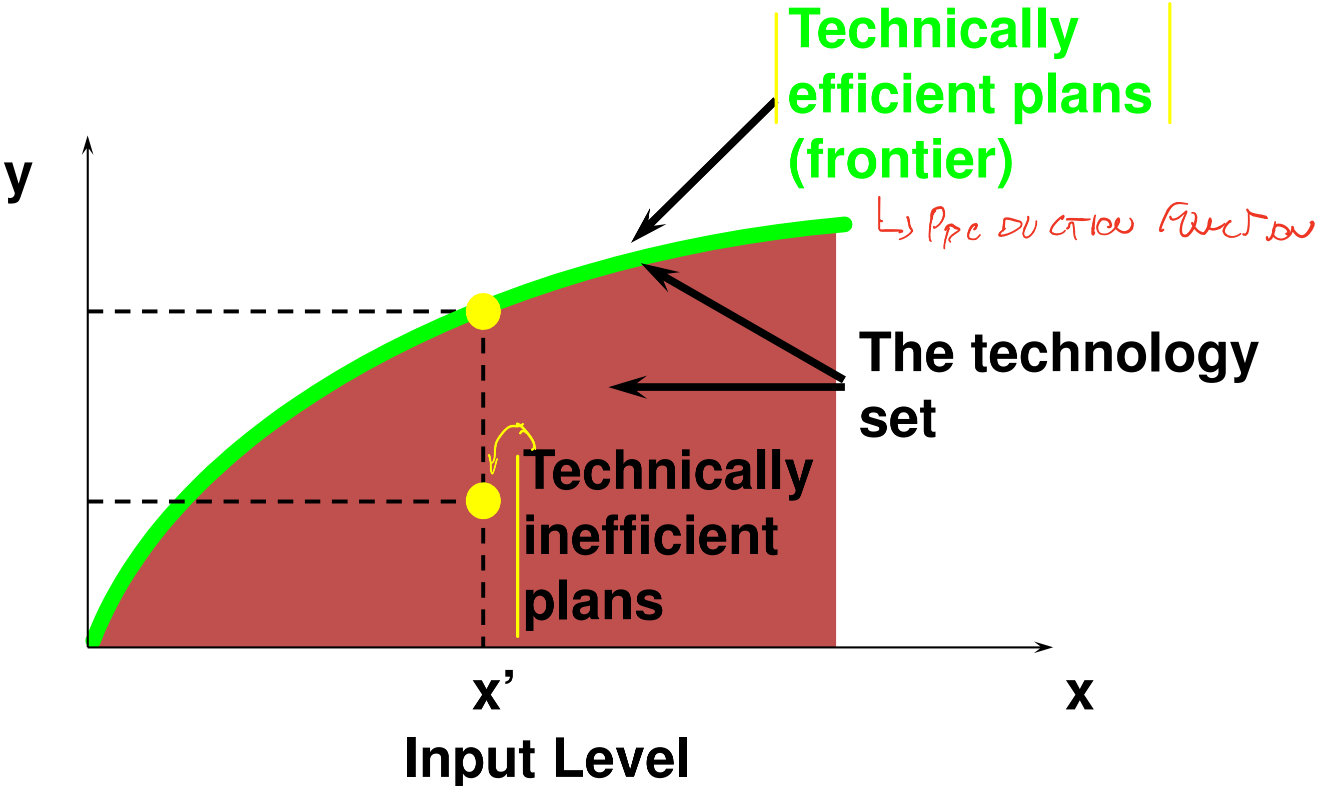
- One input one output (simpler case)

$y = f(x)$

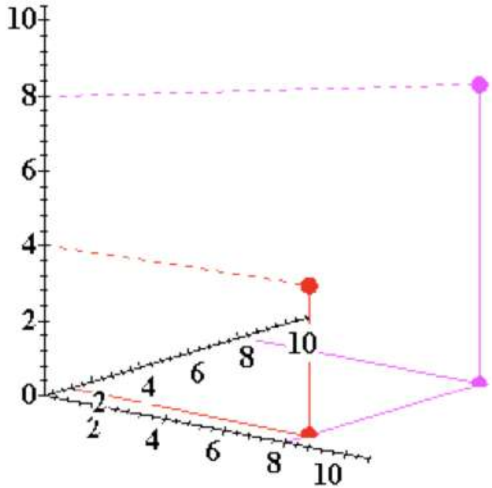


THIS PRODUCTION PLAN IS FEASIBLE TO FIRM BUT INEFFICIENT SINCE y' GIVE MORE EFFICIENT OUTPUT

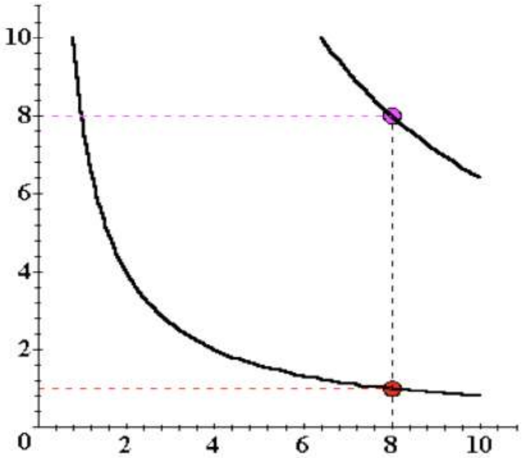
Technology set - II



Multiple inputs, one output



Multiple inputs, one output

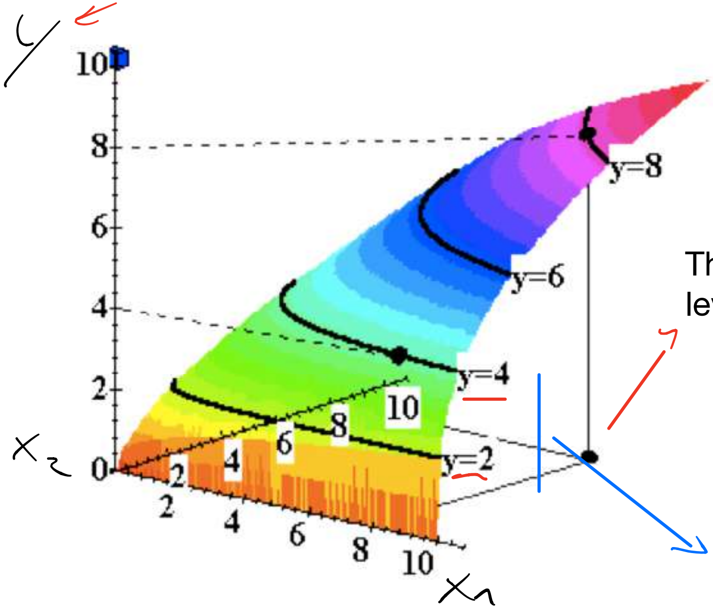


Isoquant: the set of all input bundles that yield at most the same output level y.

EXAMPLE

Multiple inputs, one output

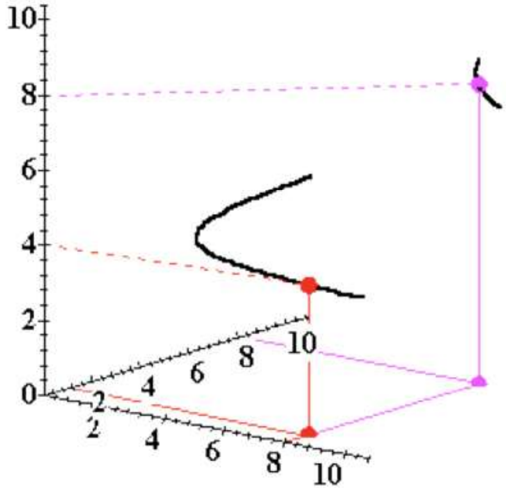
LEVEL OF OUTPUT



This production plan gives this level of production.

This sections give the same level of production output.

Multiple inputs, one output

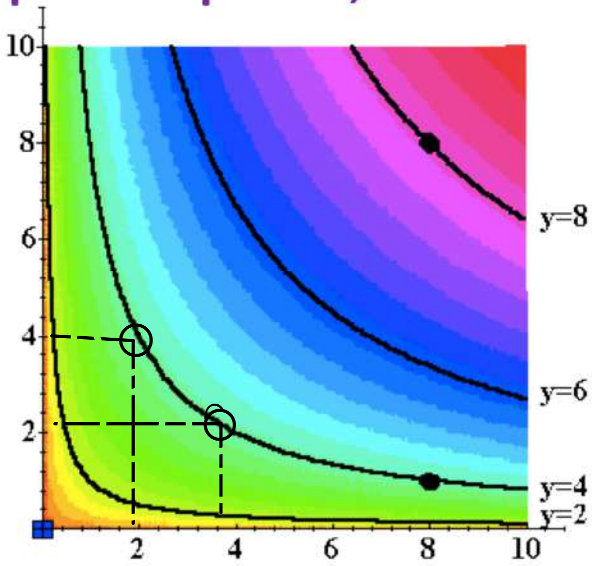


Isoquant: How is it obtained?

Multiple inputs, one output

SORT OF

SUBSTITUTION!

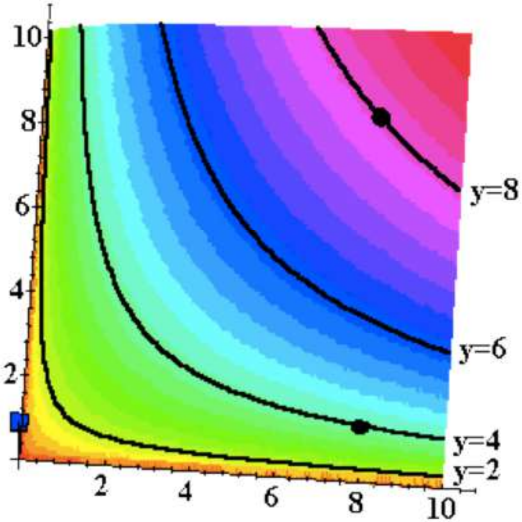


Combination of factors that give the same level of output. We can notice that as the IC for the consumer were representing combination of good that gave the same level of utility.

Isoquant represent the combination of inputs that give the same level of output(or production)

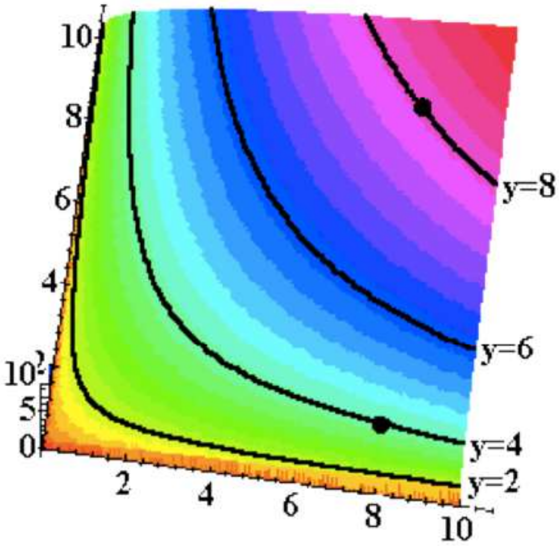
Isoquant: level map (like indifference curve for utility) – combination of inputs that give same output level

Multiple inputs, one output

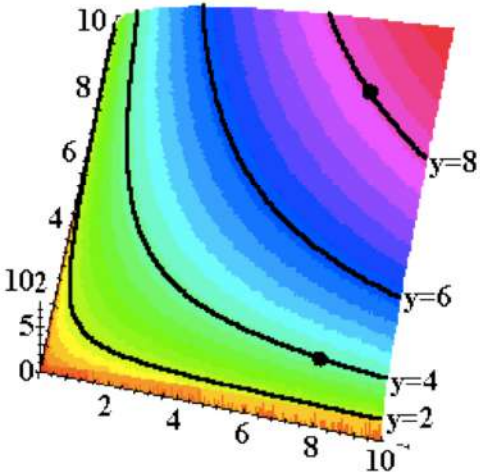


K

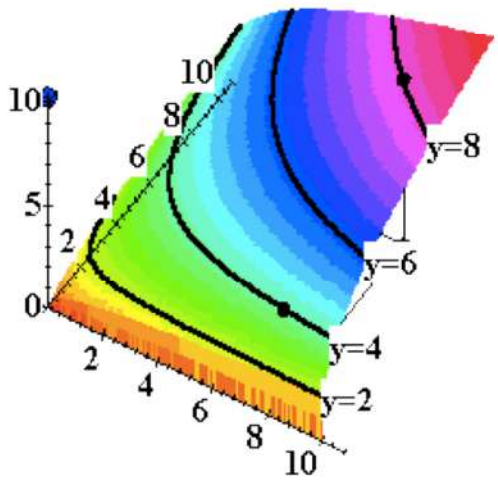
Multiple inputs, one output



Multiple inputs, one output



Multiple inputs, one output



Slope of Isoquant is called the Marginal rate of technical substitution

EXAMPLE WITH 2 INPUTS

A simple production function

- We consider the following production function in which output depends on physical capital (k), such as machinery, and labour (l)

N. Of workers or hours of works, It depends on the model

Inputs cannot be negative: positive capital and labour

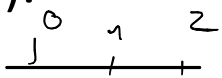
$$\boxed{y = f(k, l)}$$

Derivative: If i increase small amount of capital how much production will increase?

with $k, l \geq 0$, $\frac{\partial f(y)}{\partial x} > 0$ and decreasing, i.e. $\frac{\partial^2 f(y)}{\partial x \partial x} < 0$, where x is the generic input.

GENERIC INPUT

The first derivative is called the marginal productivity of input x (= k, l).



Half worker is consider like a part-time worker. So we are not considering discrete case but continuous

$$0 \quad \underbrace{1 \quad 2}_{\Delta Y} \quad 3 \; \dots \; \underbrace{100 \quad 101}_{\Delta Y'}$$

Same increase of the two firms but delta y' < delta y. ==> marginal productivity is decreasing.

In agriculture you have an amount of land: initially production will increase if i put 2 worker instead of 1 but if i put more worker in the same instance of land then worker will get a decreasing production since there is a lot of persons.

"A firm uses intermediate goods before reach the production in reality"

Now define the MRTS.

Marginal rate of technical substitution (MRTS)

Is the slope of the isoquant —> isoquant is the combination of inputs giving the same output level.

To find the MRTS we compute the total differential of the production function.

$$y = f(k, l)$$

This is a production function in two variables. The total differential now is:

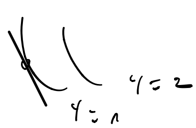
VARIATION OF Y

$$dy = \frac{\partial f}{\partial k} \cdot dk + \frac{\partial f}{\partial l} dl = 0$$

VARIATION SHOULD BE 0!

TOTAL VARIATION → MARG PRODUCTIVITY OF K how MANY UNIT OF CAPITAL + MARG PRODUCTIVITY OF L = how MANY UNIT OF l

$$dy = 0$$



↳ SLOPE OF THIS ISOQUANT

SLOPE OF ISOQUANT = $\dfrac{\partial K}{\partial \ell}$

Y IS NO MORE A VARIABLE THEN WE ONLY CONSIDER ONE VARIABLES TO COMPUTE THE SLOPE SO

$$\frac{\partial K}{\partial \ell}$$

$$\frac{\partial f}{\partial k} dk = -\frac{\partial f}{\partial \ell} d\ell \rightarrow \frac{dk}{d\ell} = -\frac{\dfrac{\partial f}{\partial \ell}}{\dfrac{\partial f}{\partial k}} \begin{array}{l} MP_\ell \\ \\ MP_k \end{array}$$

ISOQUANT = RATIO MARG PRODUCTIVITY OF ℓ AND K

M. PROD ARE BOTH POSITIVE SO SLOPE IS NEGATIVE (DECREASING SLOPE)

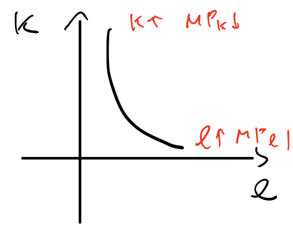
MRTS is given by the ratio of the Marginal productivity. The MRTS is how much you have to substitute the two good to maintain the same level of production.

According to the example the MRTS is increasing or decreasing moving to the right?
Increasing I the MRTS is decreasing.

$$\frac{\dfrac{\partial f}{\partial \ell}}{\dfrac{\partial f}{\partial k}}$$

1° DER IS POSITIVE BUT 2° DER IF IS NEGATIVE IS DECREASING

IF $\ell \uparrow \Rightarrow MP_\ell \downarrow \rightarrow K \downarrow \Rightarrow MP_k \uparrow \Rightarrow$ MRTS $\downarrow$ SINCE DENOMINATOR $\uparrow$ AND NUMERATOR $\downarrow$



K
K↑ MPK↓
l↑ MPl↓
l

Production function

- Along an **isoquant** y is constant, therefore totally differentiating the production function

$$(dy =) \quad \frac{\partial f(\bar{y})}{\partial k} dk + \frac{\partial f(\bar{y})}{\partial l} dl = 0$$

solving

$$\frac{dl}{dk} = -\frac{\frac{\partial f(\bar{y})}{\partial k}}{\frac{\partial f(\bar{y})}{\partial l}}, \text{ where } -\frac{\frac{\partial f(\bar{y})}{\partial k}}{\frac{\partial f(\bar{y})}{\partial l}} = MRTS_{l,k}(\bar{y})$$

  - $MRTS_{l,k}(\bar{y})$ is the **Marginal Rate of Technical Substitution** measures how much k must decrease (increase) if l increases (decreases) so as the maintain the same output [the book defines MRTS without the minus sign]

Diminishing MRTS

- The slope of the firm's isoquants is

$$MRTS_{l,k} = \frac{dk}{dl}, \text{ where } MRTS_{l,k} = -\frac{f_l}{f_k}$$

(NB. K is in the vertical axes ointhe isoquant graph)

- Where $f_l = \frac{\partial f(y)}{\partial l}$ is the marginal productivity of labour and $f_l = \frac{\partial f(y)}{\partial l}$ is the marginal productivity of capital
- Differentiating $MRTS_{l,k}$ with respect to labor and taking into account that along an isoquant $k = k(l)$ i.e. capital is a function $k(.)$ of labour yields

$$\boxed{\frac{\partial |MRTS_{l,k}|}{\partial l} = \frac{f_k\left(f_{ll}+f_{lk}\cdot\frac{dk}{dl}\right)-f_l\left(f_{kl}+f_{kk}\cdot\frac{dk}{dl}\right)}{(f_k)^2}}$$

(we apply the rule of a composite function)

$$y = \frac{f(x)}{g(x)}$$

allora

$$y' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

SOSTITUISCO

WE WANT TO CHECK HOW THE SLOPE CHANGE

Diminishing MRTS

- Using the fact that $\boxed{\frac{dk}{dl} = -\frac{f_l}{f_k}}$ (slope an isoquant) along an isoquant and Young's theorem $f_{lk} = f_{kl}$ (if f double differentiable than cross derivatives are symmetric),

$$\frac{\partial |MRTS_{l,k}|}{\partial l} = \frac{f_k\left(f_{ll} - f_{lk} \cdot \frac{f_l}{f_k}\right) - f_l\left(f_{kl} - f_{kk} \cdot \frac{f_l}{f_k}\right)}{(f_k)^2}$$

$$= \frac{f_k f_{ll} - f_{lk} f_l - f_l f_{kl} + f_{kk} \cdot \frac{f_l^2}{f_k}}{(f_k)^2}$$

Along and isoquant k is a function of l. There is a relationship between k and l. So computing derivative we have to keep in mind that k is function of l

$$|MRTS| = \left|\frac{dk}{dl}\right| = \frac{\boxed{f_l(l, k(l))}}{f_k(l, k(l))}$$

DERIVE

K AND l ARE NOT FREE TO MOVE ALONG THE ISOQUANT

DER OF A RELATION

MRTS → I HAVE TO PUT "−" SIGN

$$\frac{\delta|MRTS|}{\delta l} = \frac{f_k\left(f_{ll} - \underline{f_{lk}\frac{\delta k}{\delta l}}\right) - f_l\left(f_{kl} - f_{kk}\frac{\delta k}{\delta l}\right)}{f_k^2} =$$

COMPOSITE FUNCTION DERIVATIVE

$l \to k(l) \to f(k(l))$

f_l WITH RESPECT TO K. DER OF TERMS

$f_l(l) \to$ DER IS f_{ll}

$f_l(k_l) \to$ DER $f_{lk} \cdot \frac{\delta k}{\delta l}$

$$= \frac{f_k\left(f_{ll} - f_{lk}\cdot\frac{f_l}{f_k}\right) - f_l\left(f_{kl} - f_{kk}\cdot\frac{f_l}{f_k}\right)}{f_{k}^2} =$$

× PRODUCTS

THIS TWO DERIVATIVE ARE THE SAME

$$= \frac{f_k \cdot f_{ll} - f_{lk}\cdot f_l - f_l\, f_{kl} + f_{kk}\, f_l^2/f_k}{f_k^2} =$$

$$\simeq \frac{\left| f_k \, \overset{\text{NEG}}{f_{ll}} \right| - 2 \, f_{lk} \, f_l + \left| \overset{\text{NEG}}{f_{kk}} \, \frac{f_l^2}{f_k} \right|}{\underset{\text{POS}}{f_k^2}} = \frac{\partial |MRS|}{\partial l}$$

SIGN OF THIS ?? DEPENDS ON SIGN OF 2° DER OF PRODUCTION FUNCTION

SUFFICIENT CONDITION FOR THE DERIVATIVE TO BE NEGATIVE ?

IF $f_{lk} > 0 \Rightarrow$ DERIVATIVE < 0

BUT NOT NECESSARY!

IF $f_{lk} < 0$ NOT FOR SURE DERIVATIVE > 0

Given a fix amount of workers if you increase capital the Marginal productivity of the worker will increase!

Diminishing MRTS

- Multiplying numerator and denominator by f_k

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{\overbrace{f_k^2}^{+} \overbrace{f_{ll}}^{-} + \overbrace{f_{kk}}^{-} \overbrace{f_l^2}^{+} - \overbrace{2 f_l f_k}^{+} \overbrace{f_{lk}}^{-\text{ or}+}}{(f_k)^3}$$

No

(I have used f_{lk}=f_{kl} by Young's theorem, if f twice differentiable, i.e. second derivatives exist.)

- Thus,
  - If $\boxed{f_{lk} > 0}$ (i.e., $\uparrow k \Longrightarrow \uparrow MP_l$), then $\boxed{\frac{\partial MRTS_{l,k}}{\partial l} < 0}$
  - If $\boxed{f_{lk} < 0}$, then we have

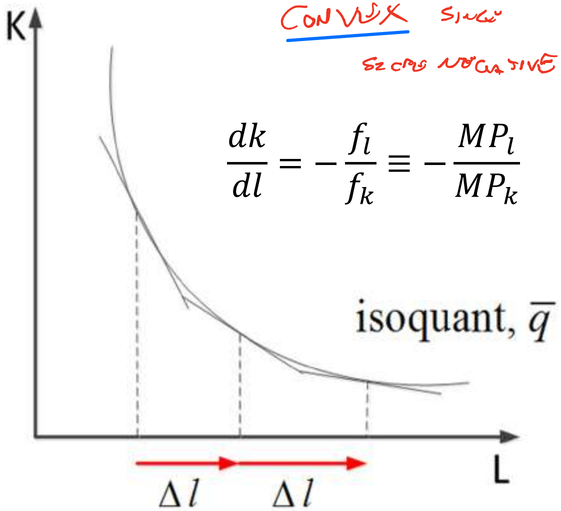
$$\left|f_k^2 f_{ll} + f_{kk} f_l^2\right| \begin{Bmatrix} > \\ < \end{Bmatrix} \left|2 f_l f_k f_{lk}\right| \Longrightarrow \boxed{\frac{\partial MRTS_{l,k}}{\partial l} \begin{Bmatrix} < \\ > \end{Bmatrix} 0}$$

If folk < 0 is like stundyting by your self give you a greater grade than also follow lectures.
If folk >0 following lectures and studying by your self gives you a greater grade

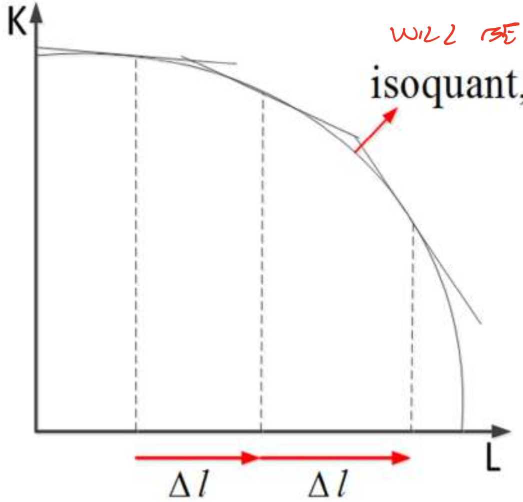
Diminishing MRTS

ISOQUANT CONVEX SINCE SLOPE NEGATIVE

$$\frac{dk}{dl} = -\frac{f_l}{f_k} \equiv -\frac{MP_l}{MP_k}$$



SLOPE of ISOQUANT WILL BE CONCAVE



$f_{lk} > 0$ ($\uparrow k \Longrightarrow \uparrow MP_l$), or
$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow MP_l$) but small $\downarrow$ in MP_l

$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow\downarrow MP_l$)

We will use convex to be able to use the maximisation problem

Diminishing MRTS

- ***Example***: Let us check if the production function $f(k, l) = kl$ yields convex isoquants (i.e. decreasing MRTS).
- Use the generic equation of an isoquant, i.e.

$$kl = \bar{q}; \text{ i.e. } k = \frac{\bar{q}}{l}$$

- $MRTS_{l,k} = \frac{\partial k}{\partial l} = -\frac{\bar{q}}{l^2} = -\bar{q}l^{-2}$, to check is if convex I compute the

second derivative of the MRTS, i.e.

- $\frac{\partial MRTS_{l,k}}{\partial l} = \frac{\partial^2 k}{\partial l \partial l} = \frac{2\bar{q}}{l^3} > 0$

Thus isoquant is convex. MRTS $\Downarrow$

Constant Returns to Scale

- If production function $f(k, l)$ exhibits CRS, then increasing all inputs by a common factor t yields

$$\boxed{f(tk, tl) = tf(k, l)}$$

I can exactly replicate a technology. Double amount of capital and labour i also duplicate the production.

- Hence, $f(k, l)$ is homogenous of degree 1, thus implying that its first-order derivatives

$$f_k(k, l) \text{ and } f_l(k, l)$$

are homogenous of degree zero.

So this is like homogeneous of degree 1 when production function exhibit constant return to scale

Constant Returns to Scale

NO

- Therefore,

$$MP_l = \frac{\partial f(k,l)}{\partial l} = \frac{\partial f(tk,tl)}{\partial l}$$
$$= f_l(k,l) = f_l(tk,tl)$$

- Setting $t = \frac{1}{l}$, we obtain

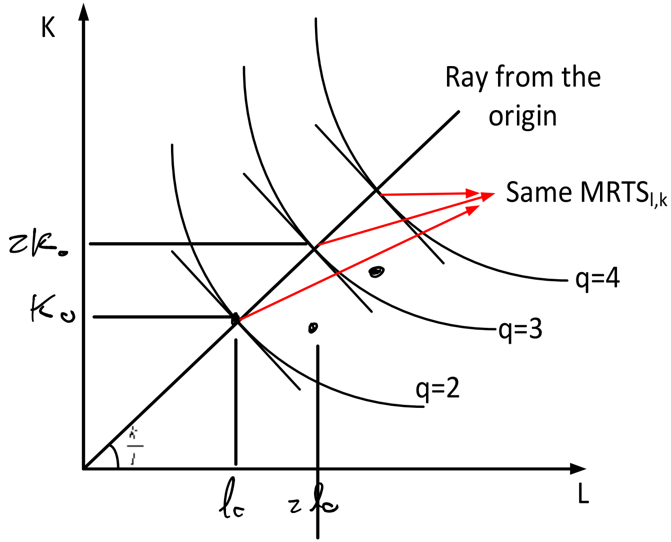
$$MP_l = f_l(k,l) = f_l\left(\frac{1}{l}k, \frac{l}{l}\right) = f_l\left(\frac{k}{l}, 1\right)$$

- Hence, $\boldsymbol{MP_l}$ **only depends on the ratio** $\boldsymbol{\frac{k}{l}}$**, but not on the absolute levels of** $\boldsymbol{k}$ **and** $\boldsymbol{l}$ **that firm uses.**
- A similar argument applies to MP_k.

Constant Returns to Scale

- Thus, $MRTS = -\frac{MP_l}{MP_k}$ only depends on the ratio of capital to labor.
- The slope of a firm's isoquants coincides at any point along a ray from the origin.
- Firm's production function is, hence, **homothetic**.

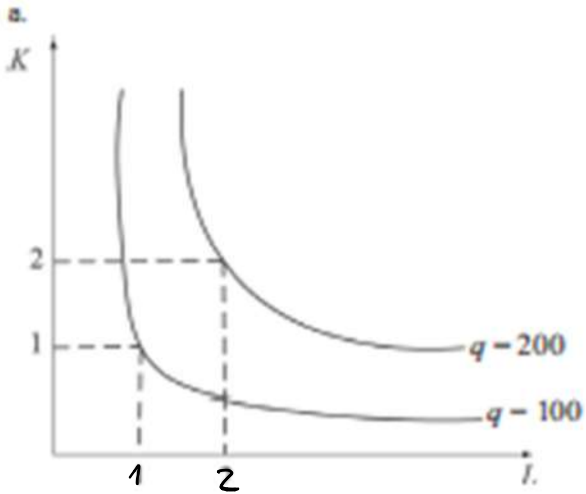
Fl fk do not depends on q (scale of production) so MRTS does not depend on q



Doubling the input also doubling the production

Constant Returns to Scale

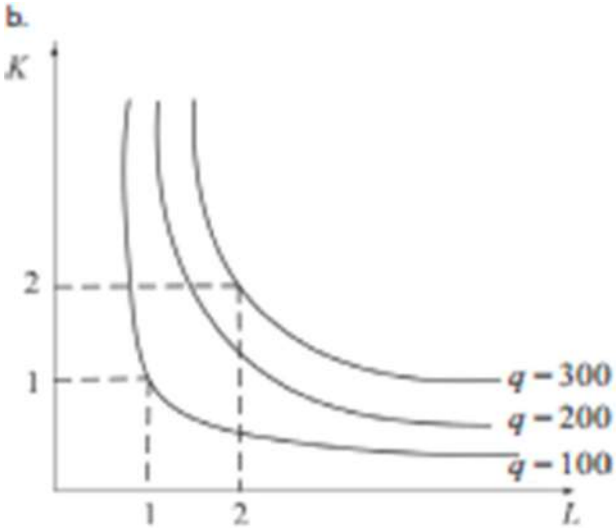
- $f(tk, tl) = tf(k, l)$



Increasing Returns to Scale

- $f(tk, tl) > tf(k, l)$

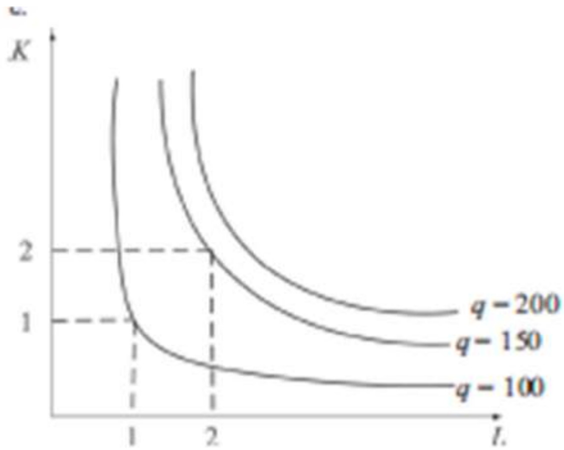
Increasing inputs by same proportion the amount of production increase more than proportion



Decreasing Returns to Scale

- $f(tk, tl) < tf(k, l)$

Increasing input by same proportion the amount of production increase less than proportion



Buying inputs is costly so we have some cost to achieve a certain amount of production.
Increasing return to scale: doubling the size of your plan you will receive a larger production than splitting the plan in half and double them by the same proportion.

100 K

50K 50K TOTAL COST SAME BUT ACHIEVE MORE PRODUCTION

IN MONOPOLY THERE ARE SOME INDUSTRIES IN WHICH IS CONVENIENT TO INCREASE PLAN (LIKE TRANSPORT)

INCREASING SCALE YOU WILL DECREASE SOME COST

Elasticity of Substitution

Elasticity of Substitution

- ***Elasticity of substitution (σ)*** measures the proportionate change in the k/l ratio relative to the proportionate change in the $MRTS_{l,k}$ along an isoquant:

$$\boxed{\sigma = \frac{\%\Delta(k/l)}{\%|\Delta MRTS|}} = \frac{d(k/l)}{d|MRTS|} \cdot \frac{|MRTS|}{k/l} = \frac{\partial \ln(k/l)}{\partial \ln(|MRTS|)}$$

where $\sigma > 0$ since ratio k/l and $|MRTS|$ move in the same direction.

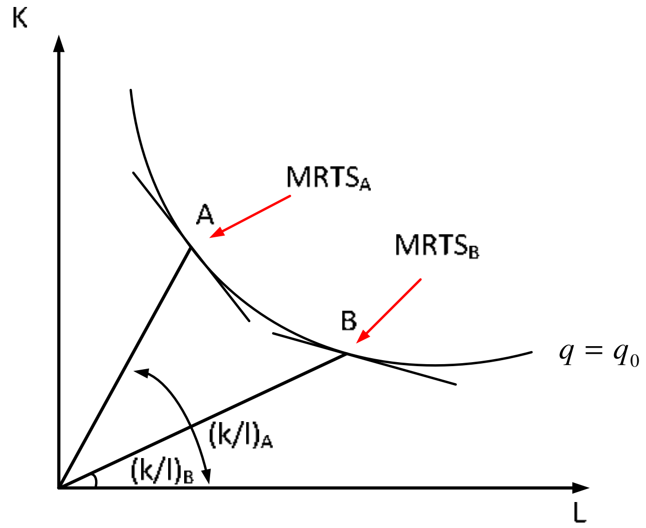
$$\frac{\frac{d\,k/l}{k/l}}{d\,|MRTS|/}$$

$$\frac{\frac{d\,k/l}{k/l}}{\frac{d\,|MRTS|}{|MRTS|}}$$

|MRTS|

Elasticity of Substitution

- Both $MRTS$ and k/l will change as we move from point A to point B.
- σ is the ratio of these changes.
- σ measures the **curvature of the isoquant**.



Elasticity of Substitution

Elasticity of Substitution

$\frac{\partial \ln\left(\frac{k}{l}\right)}{\partial \frac{k}{l}}$ $d\,\frac{k}{l}$

- ***Elasticity of substitution (σ)*** measures the proportionate change in the k/l ratio relative to the proportionate change in the $MRTS_{l,k}$ along an isoquant:

$1/\frac{k}{l}$

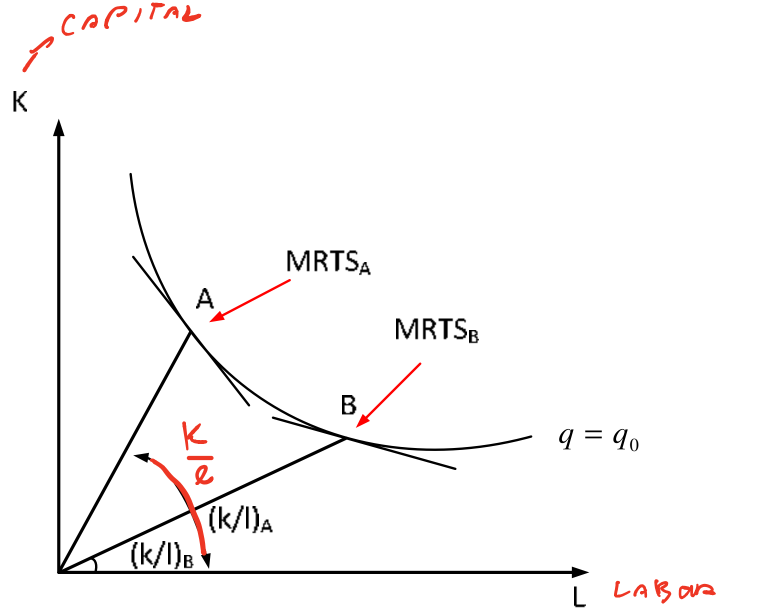
$$\sigma = \frac{\%\Delta(k/l)}{\%|\Delta MRTS|} = \frac{d(k/l)}{d|MRTS|} \cdot \frac{|MRTS|}{k/l} = \frac{\boxed{\partial \ln(k/l)}}{\partial \ln(|MRTS|)}$$

where $\sigma > 0$ since ratio k/l and $|MRTS|$ move in the same direction.

Elasticity of Substitution

CURVATURE OF THE ISOQUANT

- Both $MRTS$ and k/l will change as we move from point A to point B.
- σ is the ratio of these changes.
- σ measures the **curvature of the isoquant**.



BIG CHANGES MEANS THAT THE RATIO WILL BE LOW

ELASTICITY OF SUBSTITUTION IS LOW

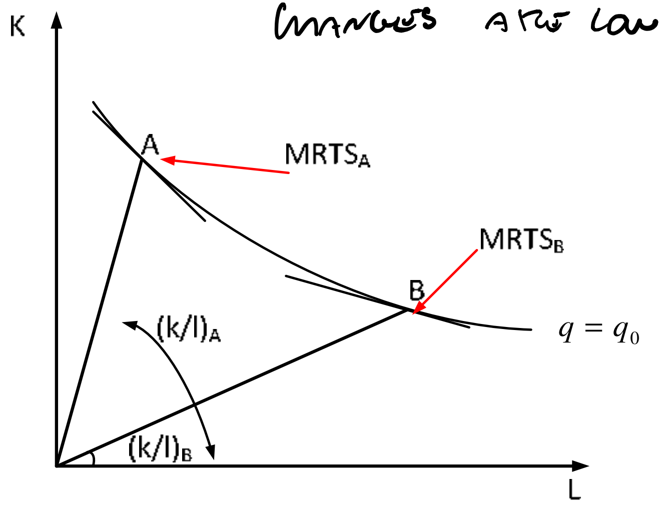
Elasticity of Substitution

- If we define the elasticity of substitution between two inputs to be proportionate change in the ratio of the two inputs to the proportionate change in $MRTS$, we need to hold:
  - output constant (so we move along the same isoquant), and
  - the levels of other inputs constant (in case we have more than two inputs). For instance, we fix the amount of other inputs, such as land.

Elasticity of Substitution

- **High elasticity of substitution (σ):**
  - $MRTS$ does not change substantially relative to k/l.
  - **Isoquant is relatively flat.**

CHANGES ARE LOW

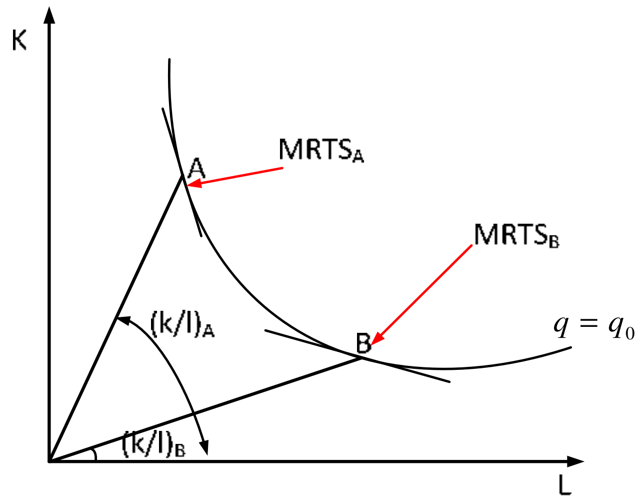


IMPLY AN HIGH LEVEL OF ELASTICITY OF SUBSTITUTION

THE EXTREME CASE IS WHEN THE ISOQUANT IS A STRAIGHT LINE

Elasticity of Substitution

- **Low elasticity of substitution (σ):**
  - $MRTS$ changes substantially relative to k/l.
  - **Isoquant is relatively sharply curved**.



Elasticity of Substitution: Linear Production Function

WE WANT MARGINAL PRODUCTION POSITIVE!

$a, b > 0$

- Suppose that the production function is

$$q = f(k, l) = \underbrace{ak + bl}_{\text{LINEAR FUNCTION}}$$

- This production function exhibits constant returns to scale

$$f(tk, tl) = atk + btl = t(ak + bl)$$
$$= tf(k, l)$$

- Solving for k in q, we get $k = \frac{f(k,l)}{a} - \frac{b}{a} l$.
  - All isoquants are straight lines
  - k and l are perfect substitutes

ELASTICITY OF SUBSTITUTION IS INFINITE

CHECK CONSTANT RETURN TO SCALE

$$f(tk, tl) = a(tk) + b(tl) =$$

$$= t(ak + bl) = t f(k, l) \Rightarrow$$ HOMOGENEOUS OF DEGREE 1

SO CONSTANT RETURN TO SCALE

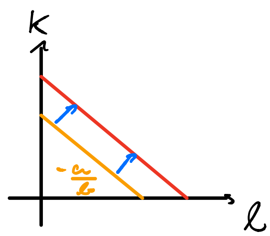
$t > 1$

$> t f(k, l) \rightarrow$ INCREASE

$< t f(k, l) \rightarrow$ DECREASE

$\bar{q} = ak + bl$

$k = \frac{\bar{q}}{a} - \frac{bl}{a}$



CURVE IS NEGATIVELY SLOPE

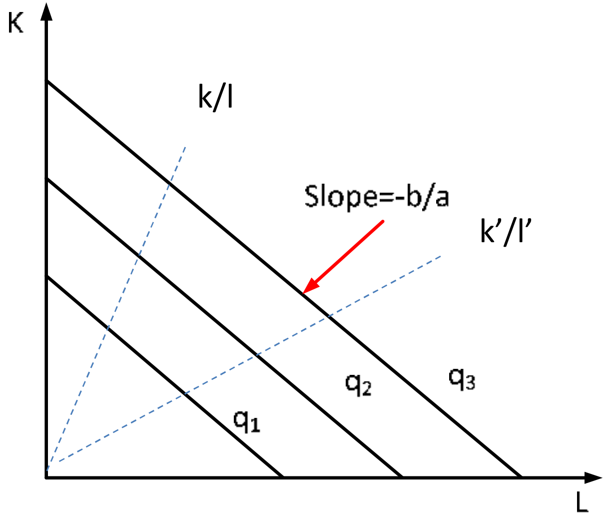
Elasticity of Substitution: Linear Production Function

- $MRTS$ (slope of the isoquant) is constant as k/l changes.

$$\sigma = \underbrace{\frac{\%\Delta(k/l)}{\%\Delta MRTS}}_{0} = \infty$$

SLOPE IS CONSTANT

- **Perfect substitutes**
- This production function satisfies homotheticity.



Elasticity of Substitution: Fixed Proportions Production Function

- Suppose that the production function is
$$q = \min(ak, bl) \quad a, b > 0$$
- Capital and labor must always be used in a fixed ratio (**perfect complements**)
  - No substitution between k and l
  - The firm will always operate along a ray where k/l is constant (i.e., at the kink!).
- Because k/l is constant (b/a),
$$\sigma = \frac{\%\Delta(k/l)}{\underbrace{\%\Delta MRTS}_{\infty}} = 0$$

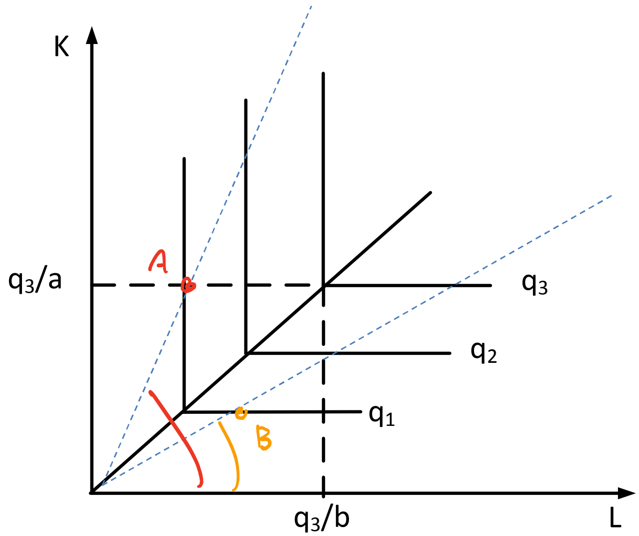
$\boxed{ak = bl}$

this hold at the optimal

so, $\frac{k}{l} = \frac{b}{a}$

Elasticity of Substitution: Fixed Proportions Production Function

- $MRTS = \infty$ for l before the kink of the isoquant.
- $MRTS = 0$ for l after the kink.
- The change in MRTS is infinite (**perfect complements**)
- This production function also satisfies homotheticity.



Elasticity of Substitution: Cobb-Douglas Production Function

- Suppose that the production function is

$$q = f(k, l) = Ak^a l^b \quad \text{where } A, a, b > 0$$

(A is sometimes called the "efficiency" parameter)

WE CAN USE THIS TO MEASURE THE EFFICIENT PRODUCTIVITY

- This production function can exhibit any returns to scale

$$f(tk, tl) = A(tk)^a (tl)^b = At^{a+b} k^a l^b = t^{a+b} f(k, l)$$

  - If $a + b = 1 \Longrightarrow$ **constant returns to scale**, $f(tk, tl) = tf(k, l)$
  - If $a + b > 1 \Longrightarrow$ **increasing returns to scale** $f(tk, tl) > tf(k, l)$
  - If $a + b < 1 \Longrightarrow$ **decreasing returns to scale** $f(tk, tl) < tf(k, l)$

$$\frac{\partial q}{\partial k} = At \cdot a k^{a-1} l^b$$

$At\uparrow$

CHECK

$$q = f(tk, tl) = A(kt)^a (lt)^b = A k^a l^b (t^{a+b})$$

HOMOGENEOUS OF DEGREE $a+b$

RETURN TO SCALE

- CONSTANT : $a+b=1$ $\left(q = f(tk,tl) = Ak^a l^b t\right)$
- INCREASING : $a+b>1$ $\left(q = f(tk,tl) > Ak^a l^b t\right)$
- DECREASING : $a+b<1$ $\left(q = f(tk,tl) < Ak^a l^b t\right)$

$$q = A k^a l^b$$

$$\underbrace{\ln}_{\tilde q} = \ln A + a \underbrace{\ln k}_{\tilde k} + b \underbrace{\ln l}_{\tilde l}$$

$\rightarrow$ IS LINEAR IN THE NEW TRANSFORMED VARIABLE

$$\varepsilon_{q,k} = \frac{\frac{dq}{q}}{\frac{dk}{k}} \rightsquigarrow \frac{d\ln q}{d\ln k} \rightsquigarrow \frac{\partial \ln q}{\partial q} \cdot dq = \frac{1}{q}\, dq$$

$$\rightsquigarrow \frac{d\tilde q}{d\tilde k} = a$$

$$\varepsilon_{q,l} = \frac{\frac{dq}{q}}{\frac{dl}{l}} \rightsquigarrow \frac{d\ln q}{d\ln l} = b$$

Elasticity of Substitution: Cobb-Douglas Production Function

- The Cobb-Douglass production function is linear in logarithms

$$\ln(q) = \ln(A) + a\ln(k) + b\ln(l)$$

  - a is the elasticity of output with respect to k

$$\varepsilon_{q,k} = \frac{\partial \ln(q)}{\partial \ln(k)}$$

  - b is the elasticity of output with respect to l

$$\varepsilon_{q,l} = \frac{\partial \ln(q)}{\partial \ln(l)}$$

Elasticity of Substitution: Cobb-Douglas Production Function

- The elasticity of substitution (σ) for the Cobb-Douglas production function:
  - First,

$$MRTS = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{bAk^a l^{b-1}}{aAk^{a-1} l^b} = \frac{b}{a} \cdot \frac{k}{l}$$

  - Hence,

$$\ln(|MRTS|) = \ln\left(\frac{b}{a}\right) + \ln\left(\frac{k}{l}\right)$$

MRTS

$$q = A k^a l^b$$

$$|MRTS| = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{A k^a \cdot b l^{b-1}}{A l^b a k^{a-1}} =$$

$$= \frac{b}{a} \frac{k}{l}$$

$$\frac{\partial q}{\partial l} = A k^a b l^{b-1} \qquad \frac{\partial q}{\partial k} = A l^b a k^{a-1}$$

$$\sigma = \frac{\Delta \% \frac{k}{l}}{\Delta \% \, MRTS} = \frac{d \ln\left(\frac{k}{l}\right)}{d \ln |MRTS|} = 1$$

ELASTICITY OF SUBSTITUTION IS 1

$$\frac{\frac{\frac{\partial l}{l}}{\frac{k}{l}}}{\frac{\partial |MRTS|}{|MRTS|}} = \frac{\frac{\frac{\partial k}{l}}{l}}{(\partial |MRTS|)} \cdot \frac{|MRTS|}{\frac{k}{l}}$$

$$\ln \underbrace{|MRTS|}_{MRTS} = \ln \frac{b}{a} + \ln \underbrace{\frac{k}{l}}_{\frac{k}{l}} \rightsquigarrow \ln \underbrace{\left(\frac{k}{l}\right)}_{\frac{k}{l}} =$$

$$= -\ln \frac{b}{a} + \ln |MRTS|$$

Elasticity of Substitution: Cobb-Douglas Production Function

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\ln\left(\frac{k}{l}\right) = \ln(|MRTS|) - \ln\left(\frac{b}{a}\right)$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(|MRTS|)} = 1$$

Transformations of a degree 1 homogenous Function

- Assume $y = f(k, l)$ is homogeneous of degree one, i.e. $f(tk, tl) = tf(k, l)$ i.e CRS. $\longrightarrow$ Constant Return to scale
- Then define the new production function

$$F(k, l) = [f(k, l)]^{\gamma}$$

Then the Returns to Scale (RTS) of this new function depend on γ. Indeed,

$$F(tk, tl) = [f(tk, tl)]^{\gamma} = [tf(k, l)]^{\gamma} = t^{\gamma}[f(k, l)]^{\gamma}$$
$$= t^{\gamma}F(k, l)$$

That is the new function is homogenous of degree γ, which also determines the RTS. *If* $\gamma > 1$ *IRS; if* $\gamma = 1$ *CRS; if* $\gamma < 1$ *DRS.*

Elasticity of Substitution: CES Production Function

- Suppose that the production function is
$$q = f(k,l) = (k^\rho + l^\rho)^{\gamma/\rho}$$
where $\rho \leq 1, \rho \neq 0, \gamma > 0$. Applying what we just said:
  - $\gamma = 1 \Longrightarrow$ constant returns to scale
  - $\gamma > 1 \Longrightarrow$ increasing returns to scale
  - $\gamma < 1 \Longrightarrow$ decreasing returns to scale

  This happens because $f(k,l) = (k^\rho + l^\rho)^{1/\rho}$ is homogeneous of degree 1, i.e. $f(tk,tl) = t(k^\rho + l^\rho)^{1/\rho}$ [prove it!]

$\rightsquigarrow f(f(k,\ell))^\gamma = \left[(k^\rho + \ell^\rho)^{\frac{1}{\rho}}\right]^\gamma$

Homogeneous of degree 1

$f(k,\ell)^\gamma$

ex $(k^{1/3} + \ell^{1/3})^3$

$\sigma = \frac{1}{1-1/3}$

- Alternative representation of the CES function
$$\boxed{f(k,l) = \left(k^{\frac{\sigma-1}{\sigma}} + l^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma-1}{\sigma}}}$$
where σ is the elasticity of substitution.

$\rho = \frac{\sigma - 1}{\sigma}$

$\sigma = \frac{1}{1-\rho}$

$\sigma \rightarrow$ elasticity of substitution

$$q = f(k, \ell) = (k^{\rho} + \ell^{\rho})^{\frac{1}{\rho}}$$

$$f(tk, t\ell) = \left((tk)^{\rho} + (t\ell)^{\rho}\right)^{\frac{1}{\rho}} = \left[t^{\rho} \cdot (k^{\rho} + \ell^{\rho})\right]^{\frac{1}{\rho}} =$$

$$= t \cdot (k^{\rho} + \ell^{\rho})^{\frac{1}{\rho}} = t \cdot f(k, \ell)$$

Elasticity of Substitution: CES Production Function

- The elasticity of substitution (σ) for the CES production function: $\rightsquigarrow (k^{\rho} + \ell^{\rho})^{\gamma/\rho}$
  - First,

$$|MRTS| = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{\cancel{\frac{\gamma}{\rho}} \cancel{[k^{\rho} + l^{\rho}]^{\frac{\gamma}{\rho}-1}} (\cancel{\rho} l^{\rho-1})}{\cancel{\frac{\gamma}{\rho}} \cancel{[k^{\rho} + l^{\rho}]^{\frac{\gamma}{\rho}-1}} (\cancel{\rho} k^{\rho-1})}$$

$$= \left(\frac{l}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{\underline{1-\rho}}$$

Elasticity of Substitution: CES Production Function

$$\frac{d \ln \frac{k}{l}}{d \ln |MRTS|}$$

– Hence,

$$\ln(|MRTS|) = (1-\rho)\ln\left(\frac{k}{l}\right)$$

$$y = \frac{1}{1-\rho} x$$

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\overbrace{\ln\left(\frac{k}{l}\right)}^{y} = \overbrace{\frac{1}{1-\rho}\ln(|MRTS|)}^{x}$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(|MRTS|)} = \frac{1}{1-\rho}$$

Elasticity of Substitution: CES Production Function

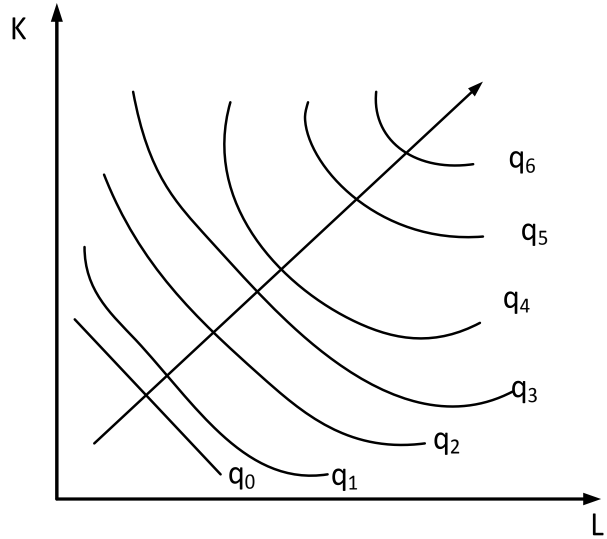
- Elasticity of Substitution in German Industries (Source: Kemfert, 1998):

Industry	σ
Food	0.66
Iron	0.50
Chemicals	0.37
Motor Vehicles	0.10

Elasticity of Substitution

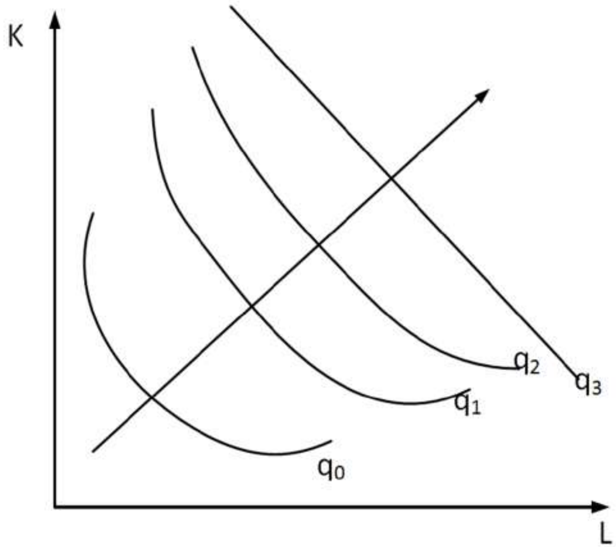
$q \uparrow \quad \sigma \uparrow$

- The elasticity of substitution σ between k and l is *decreasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very high σ
  - q_5 and q_6 have very low σ



Elasticity of Substitution

- The elasticity of substitution σ between k and l is *increasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very low σ
  - q_2 and q_3 have very high σ



Elasticity of scale

ELASTICITY OF OUTPUT IN RESPECT TO FACTOR

- The elasticity of scale is the elasticity of output q to increasing the scale of production (λ), i.e.

% CHANGE IN PRODUCTION

$$\epsilon_{q,\lambda} \equiv \frac{\dfrac{\partial f(\lambda k, \lambda l)}{f(k,l)}}{\dfrac{\partial \lambda}{\lambda}} = \frac{\partial f(\lambda k, \lambda l)}{\partial \lambda} \frac{\lambda}{f(k,l)}$$

% CHANGE

SCALE

$\varepsilon_{q,k}$

$\varepsilon_{q,l}$

Relation btw returns to scale and elasticity of scale

- We have the production function $q = f(l,k)$ and assume that is homogeneous of degree α.
- We take the total differential

$$dq = f_l dl + f_k dk$$

- Divide both sides by q

$$\frac{dq}{q} = \frac{f_l}{q}dl + \frac{f_k}{q}dk$$

- Then multiply the first term of the RHS by $\frac{l}{l}$ and the second term by $\frac{k}{k}$

$$\frac{dq}{q} = \frac{f_l l}{q}\frac{dl}{l} + \frac{f_k k}{q}\frac{dk}{k}$$

$$\frac{dl}{l} = \frac{dk}{k} = \frac{d\lambda}{\lambda}$$

Relation btw returns to scale and elasticity of scale - II

- Since we are considering a change in scale, all inputs increase by the same proportion, i.e. $\frac{dl}{l} = \frac{dk}{k} = \frac{d\lambda}{\lambda}$ and substituting in the previous equation

(Nerlove Sato)

$\circledast$

$\alpha q \to \alpha f(k, l)$

$$\frac{dq}{q} = \left(\frac{f_l l}{q} + \frac{f_k k}{q}\right)\frac{d\lambda}{\lambda} = \frac{(f_l l + f_k k)}{q}\frac{d\lambda}{\lambda}$$

- But by the Euler's theorem, if f homogeneous of degree α, then $f_l l + f_k k = \alpha q$

- Thus $\frac{dq}{q} = \frac{\alpha\, q}{q}\frac{d\lambda}{\lambda} = \alpha\frac{d\lambda}{\lambda}$, or

$$\frac{dq}{q} = \alpha\frac{d\lambda}{\lambda} \rightsquigarrow \frac{\frac{dq}{q}}{\frac{d\lambda}{\lambda}} = \alpha$$

$$\epsilon_{q,\lambda} \equiv \frac{\frac{dq}{q}}{\frac{d\lambda}{\lambda}} = \alpha$$

- **NB. Scale elasticity coincides with the production function degree of homogeneity.**

$K^a l^b \to$ elasticity of scale if $a+b$

EULER THEOREM

X, Y FACTOR

f HOMOGENEOUS OF DEGREE α

$$f(tx, ty) = t^{\alpha} f(x,y)$$

WE CAN DIFFERENTIATE

t IS THE SCALE OF PRODUCTION

$$\frac{\partial f}{\partial tx} \cdot x + \frac{\partial f}{\partial ty} \cdot y = \alpha \cdot t^{\alpha - 1} f(x,y)$$

EVALUATE THIS EQUALITY WITH $t = 1$

$$\frac{\partial f}{\partial tx} \cdot x + \frac{\partial f}{\partial ty} \cdot y = \alpha f(x,y)$$

$$\underline{PM_x \cdot X + PM_y \cdot Y = \alpha f(x,y)}$$

THIS RESULT IS USEFULL IN THIS CASE ⊛

$$\underset{\text{PROFIT}}{\pi} = \underset{\text{TOTAL REVENUE}}{TR} - \underset{\text{TOTAL COST}}{TC} = P \cdot Q - TC(Q)$$

FUNTION OF QUANTITY

QUANTITY OF PRODUCTION FACTOR THAT YOU BUY TO PRODUCE Q

K, L

$$Q = f(K, L)$$

$$\pi = \underset{\text{GIVEN}}{P} \cdot \overbrace{f(\underbrace{K, L}_{Q})}^{TR} - (rK + wL)$$

PERFECT COMPETITION

r = INTEREST RATE UNIT PRICE OF K

w = WAGE RATE UNIT PRICE OF L

- MANY CONSUMER AND FIRMS
- PERFECT INFORMATION
- HOMOGENEOUS GOODS

TWO PRODUCERS $P_A > P_C$

p IS EXOGENOUS (NOT A CHOICE)

NO CAPITAL

$\max \pi = p \cdot f(L) - wL$

$L \geq 0$

OPT QUANTITY OF LABOUR THAT MAXIMISE THE PROFIT

FOC $\quad \dfrac{\partial \pi}{\partial L} = 0 : \; p f'_L - w = 0 \qquad \dfrac{\partial f}{\partial L}$

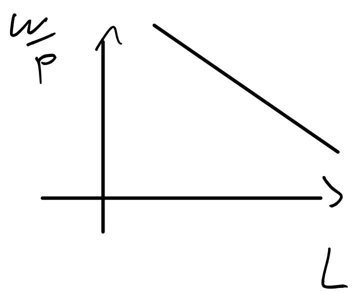
AT $L^* \rightarrow f'_L = \dfrac{w}{p}$

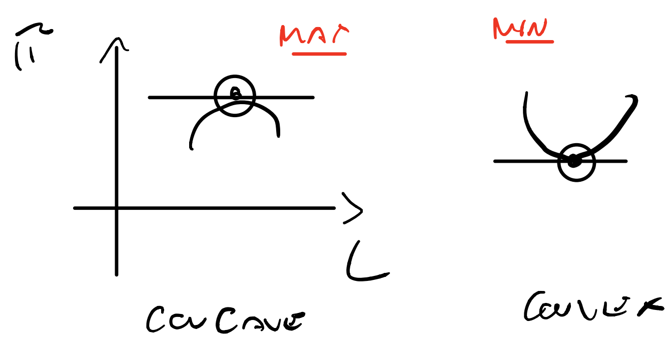
w NOMINAL WAGE

p PRICE LEVEL

$\dfrac{w}{p} \rightarrow$ REAL WAGE

$f'_L \rightarrow$ DECREASE





WE HAVE TO CHECK CONCAVITY $\rightarrow$ 2° ORDER CONDITION

SOC $\rightarrow \dfrac{\partial^2 \pi (L^*)}{\partial L \, \partial L} < 0$ IS CONCAVE IN L^*

$$L = f_L^{\prime -1}\left(\frac{w}{p}\right)$$

IN MOST CASE WE WORK WITH FUNCTIONS ALWAYS NEGATIVE IN 2° DERIVATIVE

$\boxed{\text{GLOBALLY CONCAVE FUNCTION}}$

$$\frac{\partial \pi}{\partial L} = p f_L' - w$$

$$\frac{\partial^2 \pi}{\partial L \partial L} = p \cdot f_{LL}'' \longrightarrow f_{LL}'' < 0$$

PRODUCTION FUNCTION IS GLOBALLY CONCAVE

$$\max_{K,L} \pi = \underbrace{p \cdot f(K,L)}_{\text{T.R}} - \underbrace{wL - rK}_{\text{T. COST}}$$

FOR ONLY FUNCTION OF K, BOTH VARIABLE

FOC

$$\frac{\partial \pi}{\partial K} = p \cdot f_K' - r = 0 \rightarrow f_K' = \frac{r}{p}$$

$f'_K(K,L)$

$$\frac{\partial \pi}{\partial L} = p \cdot f_L' - w = 0 \rightarrow f_L' = \frac{w}{p}$$

$\downarrow$

$f'_K(K,L)$

$i \Rightarrow$ NOMINAL INTEREST RATE

$\frac{r}{p}$ REAL INTEREST RATE

$\to$ NOMINAL INTEREST RATE

$f'_L > \frac{w}{p}$ $\quad f'_L$ DECREASING $\Rightarrow$ $f''_{KK} < 0$, $f''_{LL} < 0$

INCREASE LABOUR $\to$ IF $f'_L \downarrow \to \frac{w}{p}$

RATIO BETWEEN f'_K AND f'_L

$$-\frac{f'_L}{f'_K} = \frac{w}{r} \to \frac{f'_L}{w} = \frac{f'_K}{r}$$

PRICE RATIO

$\boxed{|MRTS|}$

USUALLY WE WORK WITH GLOBALLY CONCAVE FUNCTIONS

SOC

HESSIAN → NEGATIVE DEFINITE

$$H = \begin{bmatrix} \quad \end{bmatrix}$$

↓

$|H|$

$-wL - rK$ DERIVE WITH RESPECT TO L

WILL BE $-w$

TO K WILL BE $\varnothing$

(CROSS DERIVATIVE)

SO WE DON'T CARE ABOUT THIS

SO TAKE FIRST PART $p \cdot f(K, L)$

$$H = \begin{bmatrix} \overset{<0}{f''_{LL}} & f''_{LK} \\ f''_{KL} & f''_{KK} \end{bmatrix}$$

→ DEFINITE NEGATIVE ?

CHECK

$f''_{LL} < 0$

DETERMINANT POSITIVE ← $|H| > 0$

$$|H| = f_{LL} f_{KK} - (f_{KL})^2 > 0$$

$\Downarrow$

WE CAN CHECK IF GLOBALLY CONCAVE

IF $f''_{KK} > 0 \rightarrow f_{LL} f_{KK} - (f_{KL})^2$

IF $f''_{KK} < 0 \rightarrow$ NEGATIVE $-$

SO MUST BE NEGATIVE

EXERCISES

$$\max_{z_1, z_2 \geq 0} \quad p \cdot \underbrace{f(z_1, z_2)}_{q} - w_1 z_1 - w_2 z_2$$

$q = f(z_1, z_2) \rightarrow$ PRODUCTION FUNCTION

COBB-DOUGLAS

$q = A z^a z^b$

$$p\left(A z_1^a z_2^b\right) - w_1 z_1 - w_2 z_2$$

FOC

$$\begin{cases} \dfrac{\partial \pi}{\partial z_1} = p \cdot A a z_1^{a-1} z_2^b - w_1 = 0 \\ \dfrac{\partial \pi}{\partial z_2} = p \cdot A b z_1^a z_2^{b-1} - w_2 = 0 \\ \dfrac{\partial \pi}{\partial \lambda} \end{cases}$$

$$\overbrace{A a z_1^{a-1}}^{f'_{z_1}} z_2^b = \frac{w_1}{p} \quad \text{REAL PRICE OF } z_1 \qquad (1)$$

$$\underbrace{A b z_2^{b-1}}_{f'_{z_2}} z_1^a = \frac{w_2}{p} \quad \text{REAL PRICE OF } z_2 \qquad (2)$$

OPT CONDITION AS RATIO BETWEEN THE TWO

$$\frac{A a z_1^{a-1} z_2^{b} = \frac{w_1}{p}}{A b z_2^{b-1} z_1^{a} = \frac{w_2}{p}} = \underbrace{\frac{a}{b}\frac{z_2}{z_1}}_{|MRTS|} = \underbrace{\frac{w_1}{w_2}}_{\text{PRICE RATIO}}$$

$$z_2^* = \frac{b}{a}\frac{w_1}{w_2} z_1^* \qquad \frac{z_2}{z_1} = \frac{b}{a}\frac{w_1}{w_2}$$

I REPLACE IN 1 $\rightarrow A a z_1^{a-1} z_2^{b} = \frac{w_1}{p}$

$$A a z_1^{a-1}\left(\frac{\beta}{a}\right)^{\beta}\left(\frac{w_1}{w_2}\right)^{\beta} z_1^{\beta} = w_1$$

$$A a z_1^{a-1+\beta} = w_1\left(\frac{b}{a}\right)^{-\beta}\left(\frac{w_1}{w_2}\right)^{-\beta}\frac{1}{Aa}$$

IN THE END WE FIND

PRODUCTION INTHE BOOK IS $y \rightarrow y = q$

$$\left.\begin{aligned} z_1^* &= \frac{a p}{w_1}\cdot y \\ z_2^* &= \frac{\beta p}{w_2}\cdot y \end{aligned}\right\}$$ CONDITIONAL DEMAND FACTOR (DEPEND WHICH ON PRODUCTION)

$$\left.\begin{aligned} z_1^* &= A^{\frac{1}{1-a-\beta}}\left(\frac{a p}{w_1}\right)^{\frac{1-b}{a-b}}\left(\frac{b p}{w_2}\right)^{\frac{b}{1-a-b}} \\ z_2^* &= A^{\frac{1}{1-a-b}}\left(\frac{a p}{w_1}\right)^{\frac{a}{1-a-b}}\left(\frac{b p}{w_2}\right)^{\frac{1-a}{1-a-b}} \end{aligned}\right\}$$ UNCONDITIONAL DEMAND FACTORS

IF PRICE ↑ (w_1), FIRM WILL BUY MORE OR LESS?
OF THAT FACTOR

COMPUTE DERIVATIVE

$$\frac{\partial z_1^*}{\partial w_1} = \underbrace{\theta_1 \theta_2}_{>0} \underbrace{\left(\frac{1-b}{1-a-b}\right)}_{X} \underbrace{\left(\frac{ap}{w_1}\right)^{\frac{1-b}{1-a-b}-1}}_{>0} \cdot \underbrace{\left(-\frac{ap}{w_1^2}\right)}_{<0}$$

$A > 0 \quad b > 0 \quad p > 0 \quad w_2 > 0 \quad \theta_2 > 0$

So $\theta_1 > 0$

$+ \quad + \quad -$ BUT WE WILL RETURN TO SIGN OF a, b

COBB DOUGLAS WAS DECREASING RETURN TO SCALE?

$X > 0$

WITH DRS $\dfrac{\partial z_1^*}{\partial w_1} = 0$

HOW DO WE FIND THE SUPPLY OF THE FIRM?

(QUANTITY THAT MAXIMISE PROFITS)

q

RELATION BETWEEN z_1, z_2 TO q?

THE PRODUCTION FUNCTION!

$$q^* = f(z_1^*, z_2^*) \rightarrow q = A(z_1^*)^{a}(z_2^*)^{b}$$

$$q^* = \underbrace{A^{\frac{1}{1-a-b}}}_{\substack{>0 \\ \theta_1}} \, p^{\frac{a+b}{1-a-b}} \underbrace{\left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}}}_{\substack{>0 \\ \theta_2}} \underbrace{\left(\frac{b}{w_2}\right)^{\frac{\beta}{1-a-b}}}_{\substack{>0 \\ \theta_3}}$$

FIRM SUPPLY FUNCTION

HOW QUANTITY DEPENDS ON PRICE?

$$\frac{\partial q^*}{\partial p^*} = \theta_1 \theta_2 \theta_3 \cdot \frac{a+b}{1-a-b} \, p^{\frac{a+b}{1-a-b} - 1}$$

IF DRS SIGN WILL BE POSITIVE

LAW OF SUPPLY HOLDS!!

CONCAVE? CHECK SOC AND VERIFY IF HOLDS

$f(z) = 2^{3/4} z_1^{1/4} z_2^{1/4}$ $\quad z_1, z_2 \geq 0$

VECTOR

$t > 1$

CHECK IF NON INCR., NON DECR, CONSTANT? R.T.S.

$f(tz_1, tz_2) = 2^{3/4} (tz_1)^{1/4} (tz_2)^{1/4} = 2^{3/4} t^{2/4} z_1^{1/4} z_2^{1/4} =$

$= t^{1/2} (2^{3/4} z_1^{1/4} z_2^{1/4}) = t^{1/2} f(z)$

IF $> t f(z)$ $\rightarrow$ THIS WILL BE INCREASING R.T.S.

ELSE $\leq t f(z)$

$(t^{1/2} - t) \leq 0 \quad t^{1/2} \leq t \quad \sqrt{t} \leq t$? YES

SO NON INCREASING

$\Downarrow$

NOT CONSTANT R.T.S

CHECK NON DECREASING R.T.S.

TRY EXERCISE SET N. 1 (EX SECTION CH. 4)

Profit maximisation

Optimal demand factor in the production we got the optimal factor.

Profit Maximization

Profit Maximization

- Assumptions:
  - Firms are price takers: the production plans of an individual firm do not alter price levels $p = (p_1, p_2, \dots, p_L) \gg 0$.
  - The production set satisfies: non-emptiness, closedness, and free-disposal (if a production plan is in the production set a production plan with whose elements are lower than the original is also in the production set).

Profit Maximization: Single Output

$$\boxed{\begin{array}{c}\max_{z \geq 0}\ p \cdot y - wz \\ s.\ t.\ y = f(z)\end{array}}$$

- where y is the output, z a vector of inputs, p is the output price and w the vector of input prices, and $y = f(z)$ is a constraint given by technology

- By Kunh-Tucker conditions, FOCS are

$$p\frac{\partial f(z^*)}{\partial z_k} \leq w_k$$

with complementary slackness (i.e. if z_k>0, condition holds with equality)

Profit Maximization: Single Output

- Note that for any two input, this implies for internal solutions

$$p = \frac{w_k}{\frac{\partial f(y^*)}{\partial z_k}} \text{ for every input } k$$

Hence, for inputs z_1 and z_2

$$\frac{w_1}{w_2} = \frac{\frac{\partial f(z^*)}{\partial z_1}}{\frac{\partial f(z^*)}{\partial z_2}} = \frac{MP_{z_1}}{MP_{z_2}} \ (= |MRTS_{z_1,z_2}(z^*)|)$$

or

$$\frac{MP_{z_1}}{w_1} = \frac{MP_{z_2}}{w_2}$$

Intuition: Marginal productivity per dollar spent on input z_1 is equal to that spent on input z_2.

1- the solution of the PMP gives the optimal **unconditional factor demands** (i.e. unconditional on output level)

2- by replacing z_1^* and z_2^* in the production function we obtain **the firm supply** (i.e. amount of output produced).

It is important to check concavity.

Profit Maximization: Single Output

- Second-order condition (SOC):

STRICTLY CONCAVE IF HESSIAN NEGATIVE DEFINITE

- The matrix of second-derivatives of the production function is **negative semidefinite** at the optimal point, i.e. Matrix of second derivatives

$$D^2 f(z^*) = \left(\frac{\partial^2 f(z^*)}{\partial x_i \partial x_j} \right)$$

To prove matrix negative and semi definite.
We don't prove SOC usually

- i.e. Must satisfy $\mathrm{h}\, D^2 f(z^*) \mathrm{h}^{\mathrm{T}} \leq 0$ for all vectors h.
- **We generally use globally concave production functions (in many inputs) – i.e. the Hessian matrix H is negative definite - and the SOC automatically holds. So FOCs are sufficient conditions.**

Profit Maximization: Single Output

- ***Example:*** Cobb-Douglas production function

$$y = f(z_1, z_2) = Az_1^{\alpha} z_2^{\beta}$$

- On your own:
  - Solve PMP (differentiating with respect to z_1 and z_2.
  - Find optimal input usage $z_1(w, q)$ and $z_2(w, q)$.
    - These are referred to as "conditional factor demand correspondences"
  - Show how demand for inputs depend on input prices, and how supply depends on output price.
  - Plug them into the production function to obtain the the output level when the firm uses its profit-maximizing input combination. This is firm supply function (if solution of PMP is unique)

Properties of Profit Function

- Assume that the production set Y is closed and satisfies the free disposal property.

1) Homog(1) in prices

INPUT

$$\pi(\lambda p) = \lambda\pi(p)$$

- Increasing the prices of all inputs and outputs by a common factor λ produces a proportional increase in the firm's profits.

$$\pi(p) = pq - w_1 z_1 - \cdots - w_n z_n$$

Scaling all prices by a common factor, we obtain

$$\pi(\lambda p) = \lambda pq - \lambda w_1 z_1 - \cdots - \lambda w_n z_n$$
$$= \lambda(pq - w_1 z_1 - \cdots - w_n z_n) = \lambda\pi(p)$$

$$\frac{\partial \pi}{\partial p} = y^*(p)$$

$$\frac{\partial \pi}{\partial w_1} = -z_1^*$$ OPPOSITE OF OPTIMAL DEMAND FOR z_1

FACTOR PRICE OF z_1

$\boxed{\text{ENVELOPE THEOREM}}$ => CASE ONE VARIABLE

$$\max_x f(x, a)$$ $x_1^a x_2^b$

PARAMETERS

CHOICE VARIABLE

OPT x DEPENDS ON a, SO a ENTERS TWICE

$$f(x^*(a), a)$$

how max changes if a changes?

$$\frac{\partial \max}{\partial a} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial a} + \frac{\partial f}{\partial a}$$

IF a CHANGES, WE HAVE THE EFFECT ON f

F.O.C.

$$\frac{\partial f}{\partial x} = 0$$

WE ARE AT MAX, SO MUST HOLDS

$\boxed{\text{ENVELOPE THEOREM}}$ $\Rightarrow$ CASE TWO VARIABLE

$\pi = p \cdot y - w \cdot z$ $\qquad (w_1, w_2 \dots)$

SCALAR $\qquad$ VECTORS $\qquad (z_1, z_2 \dots)$

ENVELOPE ON 2 FACTORS

$\pi^* = p \cdot y^*(z_1^*, z_2^*) - w_1 z_1^* - w_2 z_2^*$

VALUE FUNCTION $\qquad z_1^*(p, w_1, w_2) \qquad z_2(p, w_1, w_2)$

HOW PROFIT CHANGES AS P CHANGE?

$$\frac{\partial \pi^*}{\partial p} = y^*$$

$$\frac{\partial \pi}{\partial w_1} = -z_1^* \longrightarrow z_1$$

$$y^* + p\left[\frac{\partial y^*}{\partial z_1} \cdot \frac{\partial z_2}{\partial p} + \frac{\partial y^*}{\partial z_2} \cdot \frac{\partial z_2}{\partial p}\right] \dots$$

AT THE END WITH FOC WE GET z_1

Remarks on Profit Function

- ***Remark 1***: the profit function is a value function, measuring firm profits *only* for the profit-maximizing vector y^*.

Properties of Supply Correspondence (study after cost-minimization)

2) Hotelling's Lemma: If $y(\bar{p})$ consists of a single point, then $\pi(\cdot)$ is differentiable at $\bar{p}$. Moreover, $\nabla_p \pi(\bar{p}) = y(\bar{p})$.

– This is an application of the duality theorem from consumer theory.

- If $y(\cdot)$ is a function differentiable at $\bar{p}$, then $D_p y(\bar{p}) = D_p^2 \pi(\bar{p})$ is a symmetric and positive semidefinite matrix, with $D_p \pi(\bar{p})\bar{p} = 0$.
  - This is a direct consequence of the **law of supply (if the good's price increases supply increases)**.

Properties of Supply Correspondence

– Since $D_p \pi(\bar{p})\bar{p} = 0, D_p\, y(\bar{p})$ must satisfy:

- *Own substitution effects* (main diagonal elements in $D_p y(\bar{p})$) are non-negative, i.e.,

$$\frac{\partial y_k(p)}{\partial p_k} \geq 0 \text{ for all } k$$

- *Cross substitution effects* (off diagonal elements in $D_p y(\bar{p})$) are symmetric, i.e.,

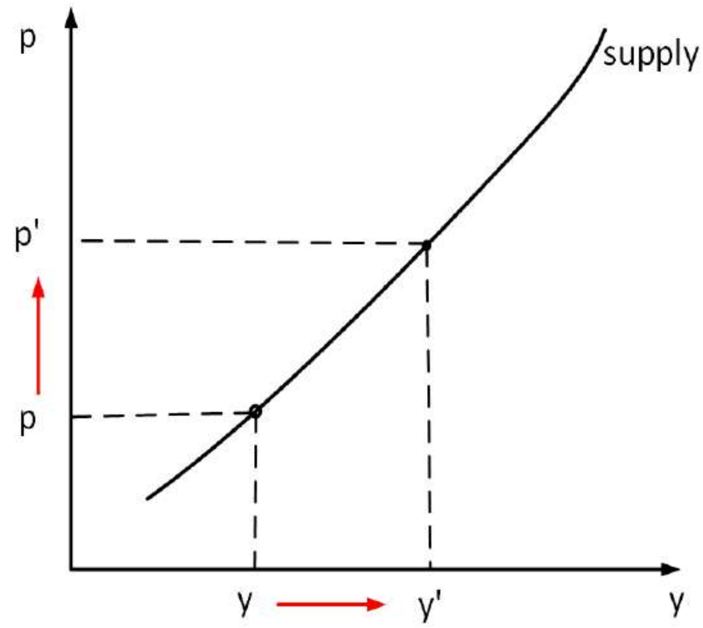
$$\frac{\partial y_l(p)}{\partial p_k} = \frac{\partial y_k(p)}{\partial p_l} \text{ for all } l \text{ and } k$$

Properties of Supply Correspondence

- $\frac{\partial y_k(p)}{\partial p_k} \geq 0$, which implies that quantities and prices move in the *same* direction,

  $(p - p')(y - y') \geq 0$

  – The **law of supply** holds!



Conditional demand factors, now we start **cost minimization**

Advanced Microeconomic Theory

Chapter 4: Cost minimization problem (CMP), factor demand functions, cost functions

Cost Minimization

Suppose we have not to choose what to produce (like government want at least production at minimum cost)

Cost Minimization

- We focus on the single output case, where
  - z is the input vector
  - $f(z)$ is the production function
  - q are the units of the (single) output
  - $w \gg 0$ is the vector of input prices
- The cost minimization problem (CMP) is

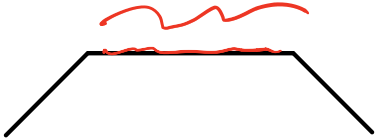
$$\min_{z \geq 0} \boxed{w \cdot z} \longrightarrow \text{cost}$$

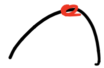
$w_1 z_1 + w_2 z_2$

$$\text{s.t. } f(z) \geq q$$

AT LEAST q UNITS

More solutions





→ one solution

Cost Minimization

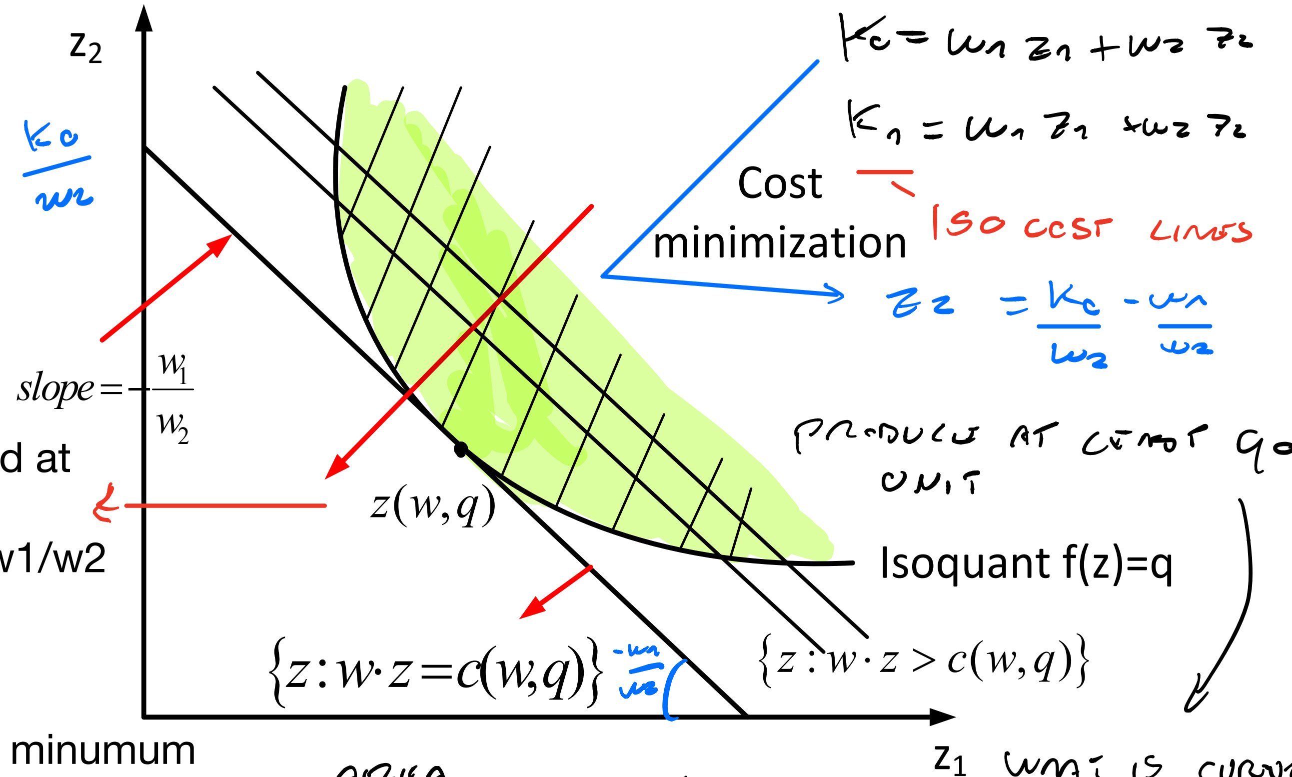
- The optimal vector of input (or factor) choices is $z(w, q)$, and is known as the ***conditional factor demand correspondence***.
  - If single-valued, $z(w, q)$ is a function (not a correspondence)
  - Why "conditional"? Because it represents the firm's demand for inputs, conditional on reaching output level q.
- The value function of this CMP $c(w, q)$ is the ***cost function***.

How to represent graphically?

TWO FACTORS

Cost Minimization

$TC = w_1 z_1 + w_2 z_2$



We will find at this point |MRTS|= w1/w2

PRODUCE AT LEAST q_0 UNIT

WHAT IS CURVE OF QUANTITY?

Point with minumum cost stay in the lowest possible isoquant line

AREA ABOVE ISOQUANT

Cost Minimization

- Graphically,
  - For a given isoquant $f(z) = q$, choose the **isocost** line associated with the lowest cost $w \cdot z$.
  - The tangency point is $z(w, q)$.
  - The isocost line associated with that combination of inputs is

$$\{z: w \cdot z = c(w, q)\}$$

    where the cost function $c(w, q)$ represents the lowest cost of producing output level q when input prices are w.
  - Other isocost lines are associated with either:
    - output levels higher than q (with costs exceeding $c(w, q)$), or
    - output levels lower than q (with costs below $c(w, q)$).

We call cost minimization as Dual of profit maximisation problem since the FOC are the same

Cost Minimization

$f(z) \geq q$

$f(z) - q \geq 0$

$\Downarrow$

$q - f(z) \boxed{\leq 0}$

to have ≤ 0

- The Lagrangian of the CMP is

$$\mathcal{L}(z;\lambda) = wz + \lambda[q - f(z)]$$

- Differentiating with respect to z_k

In which situation corner solution will be relevant? PERFECT SUBSTITUTES

$$w_k - \lambda \frac{\partial f(z^*)}{\partial z_k} \overset{=}{\cancel{\geq}} 0$$

We should check corner solution. Most case we will not consider it, so we will have equal condition

$$(= 0 \text{ if interior solution, } z_k^*)$$

or in matrix notation

$$w - \lambda \nabla f(z^*) \geq 0$$

IF YOU PRODUCE MORE THAN CONSTRAINT COST WILL GROW

Cost Minimization

$$\frac{w_1}{w_2} = \frac{\cancel{\lambda}\,\frac{\partial f}{\partial z_1}}{\cancel{\lambda}\,\frac{\partial f}{\partial z_2}}$$

- From the above FOCs,

$$\frac{w_k}{\frac{\partial f(z^*)}{\partial z_k}} = \lambda \implies \frac{w_k}{w_l} = \frac{\frac{\partial f(z^*)}{\partial z_k}}{\frac{\partial f(z^*)}{\partial z_l}} \quad (= \underline{MRTS(z^*)})$$

MARGINAL PRODUCTIVITY IS THE SAME

- Alternatively,

$$\boxed{\frac{\frac{\partial f(z^*)}{\partial z_k}}{w_k} = \frac{\frac{\partial f(z^*)}{\partial z_l}}{w_l}}$$

at the cost-minimizing input combination, the marginal product per euro spent on input k must be equal that of input l.

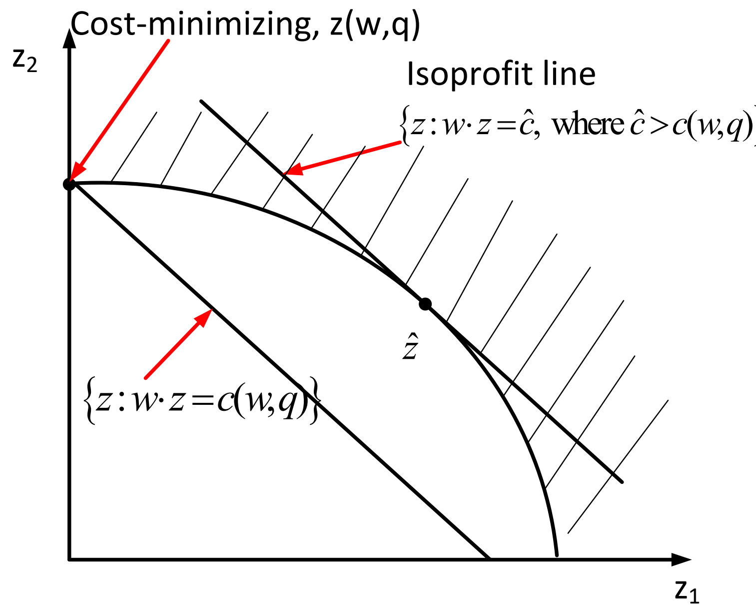
NB. PROFIT IS LIKE THE BUDGET CONSTRAINT IN THE CONSUMER THEORY

Cost Minimization

We will consider this cases

- ***Sufficiency***: If the production set is *convex* (i.e. quasi-concave production function), then the FOCs are also sufficient.
- A non-convex production set:
  - The input combinations satisfying the FOCs are NOT a cost-minimizing input combination $z(w,q)$.
  - The cost-minimizing combination of inputs $z(w,q)$ occurs at the corner.

In this case to reach q0 is the corner solution. So we don't have sufficient condition.



$\lambda = \frac{\partial c(q)}{\partial q} \Rightarrow$ Marginal cost of production

Cost Minimization

- ***Lagrange multiplier***: λ can be interpreted as the cost increase that the firm experiences when it needs to produce a higher q. Constraint stricter = relaxing
  – Recall that, generally, the Lagrange multiplier represents the variation in the objective function that we obtain if we relax the constraint (e.g., wealth in UMP, utility level we must reach in the EMP).
- Therefore, **λ is the marginal cost of production**: the marginal increase in the firm's costs (objective function) from producing additional output units (i.e. relaxing the q constraint).

If you change q (the constrain) lambda will give how the min change by changing q. So this can be also define as marginal cost of production

Marginal since the change are low

If you want to demonstrate this we could use the envelope

$$C(z_1, z_2, \lambda, q) = \underbrace{wt + \lambda[q - f(z)]}_{L(z_1, z_2, \lambda, q)}$$

$q \downarrow$ constraint

BY ENVELOPE THEOREM

$$\frac{\partial C}{\partial q} = \lambda \rightarrow \text{MARGINAL COST PRODUCTION}$$

MIN COST CHANGE IF WE CHANGE PRODUCTION ?? [55.00]

Properties of Cost Function

(I show some proofs) → DOESN'T ASK PROOF

- Assume that the production set Y is closed and satisfies the free disposal property.

  1) *$c(w,q)$ is Homog(1) in w*

  - That is, increasing all input prices by a common factor λ yields a proportional increase in the minimal costs of production:

$$\boxed{c(\lambda w, q) = \lambda c(w, q)}$$

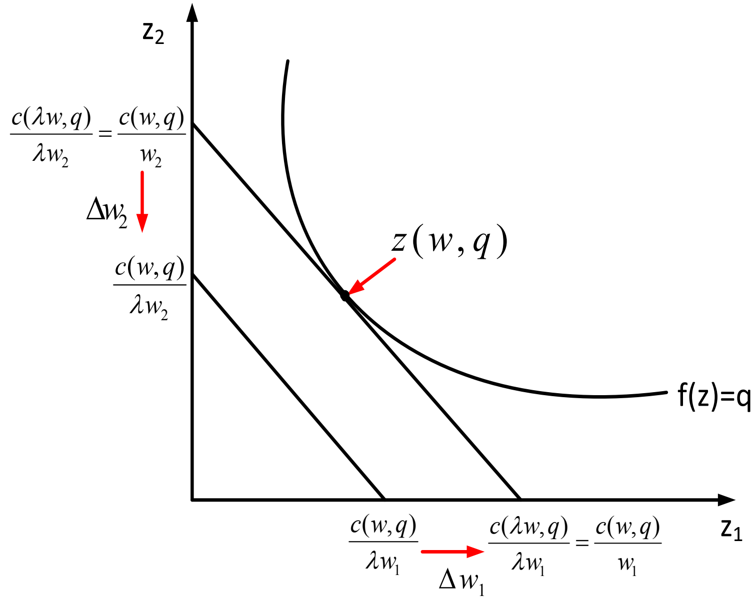
Graphically, the optimal solution (conditional factor demand $z^c(\lambda w, q)$) does not change when all prices change by the same proportion (same tangency condition) and the constraint does not change. So conditional demand is the same. Thus

$$c(\lambda w, q) = \lambda w z^c(\lambda w, q) = \lambda w z^c(w, q) = \lambda\big(w z^c(w, q)\big)$$
$$= \lambda c(w, q)$$

$w_1 z_1 + w_2 z_2$

Properties of Cost Function

- An increase in all input prices (w_1, w_2) by the same proportion λ, produces a parallel downward shift in the firm's isocost line.



Properties of Cost Function

2) *$c(w,q)$ is non-decreasing in w_l* (i.e. ~~cost~~ price of each factor).

$w_l \uparrow$

- Consider w' and w'' such that $w_l'' \geq w_l'$ and $w_k'' = w_k'$ for every $k \neq l$. (i.e. price larger for one factor and the same for the others.)

w' $\quad$ w''

- Let z' and z'' be the solutions of the CMP with w' and w'', respectively. Then by definition of cost function $c(w,q)$:

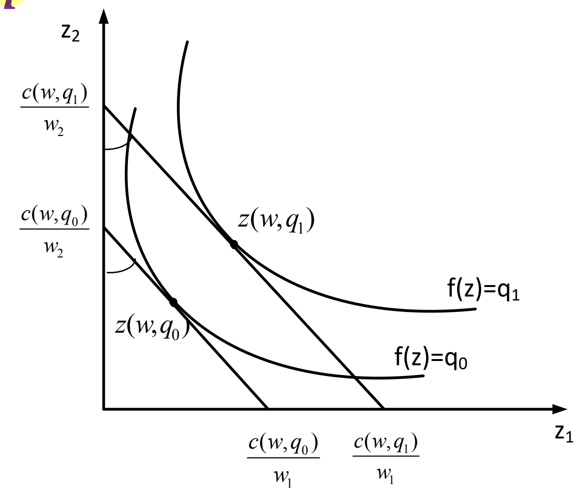
$$c(w'',q) = w''z'' \geq w'z'' \geq w'z' \geq c(w',q)$$

If price of one factor increases, then the cost will increases or remain the same

Properties of Cost Function

3) $c(w, q)$ *is non-decreasing in* q.

- Producing higher output levels implies a weakly higher minimal cost of production
- Suppose not. Then there exist q' < q'' such that (denote z' and z'' the corresponding solution to the cost minimization problem)
- $wz' \geq wz''$. If the latter inequality is strict we have an immediate contradiction of z' solving the cost minimization problem.
- That is, it must be $wz' \leq wz''$ (i.e. if q increases costs increases or remains the same)



Properties of Cost Function

4) $c(w, q)$ is concave in w

- Let $\widehat{w} = tw + (1 - t)w'$ with $t \in [0,1]$.
- Let $\hat{x}$ be the solution of the CMP with $\widehat{w}$. Then
- $c(\widehat{w}, q) = \widehat{w}\hat{x} = tw\hat{x} + (1 - t)w'\,\hat{x} \;\geq$
$$tc(w, q) + (1 - t)c(w', q)$$
- By the definition of the cost functions, as $w\hat{x} \geq c(w, q)$ and $w'\hat{x} \geq c(w', q)$.

Properties of Cost Function

- ***5) Shephard's lemma***
- If $z(w, q)$ is single valued with respect to w, then $c(w, q)$ is differentiable with respect to w and

$$\boxed{\frac{\partial c(w,q)}{\partial w_l} = z_l(w, q)}$$

In practice you can get the **conditional factor demand** of z_1 by differentiating the cost function with respect to w_1.

APPLICATION OF

Proof. By the constrained Envelope theorem (ET), i.e. ET considering the Lagrangian:

$$c(w, q) = wz(w, q) - \lambda(w, q)[f(z(w, q) - q].$$

Compute the derivative of this and use the FOC:

$$\frac{\partial c(w,q)}{\partial w_l} = w\frac{\partial z}{\partial w_l} + z_l - \frac{\partial \lambda}{\partial w_l}[f(.) - q] - \lambda \frac{\partial f}{\partial z_l}\frac{\partial z_l}{\partial w_l} \text{ that is}$$

$$\frac{\partial c(w, q)}{\partial w_l} = z_l + \left(w - \lambda\frac{\partial f}{\partial z_l}\right)\frac{\partial z}{\partial w_l} - \frac{\partial \lambda}{\partial w_l}[f(.) - q]$$

But second and third terms on the RHS are zero by the FOCs (for an interior optimum).

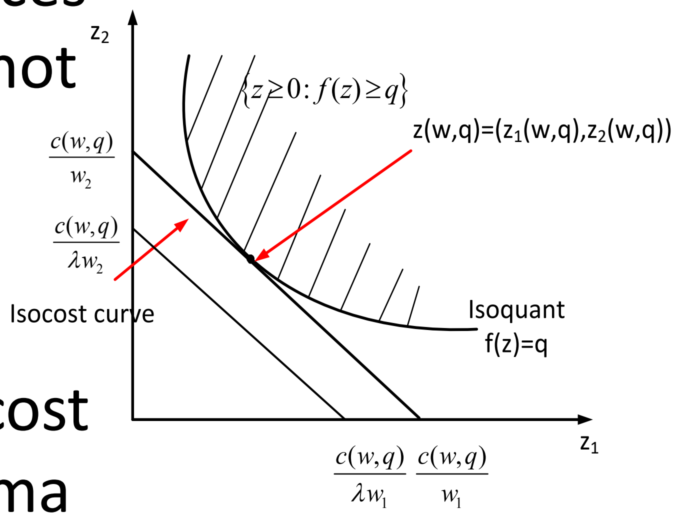
$$\begin{vmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{vmatrix} = \lambda \begin{vmatrix} \frac{\partial f}{\partial z_1} \\ \frac{\partial f}{\partial z_2} \\ \vdots \end{vmatrix}$$

Properties of Conditional Factor Demand Correspondence

1) *$z(w,q)$ is Homog(0) in w.*

- That is, increasing input prices by the same factor λ does not alter the firm's demand for inputs at all,

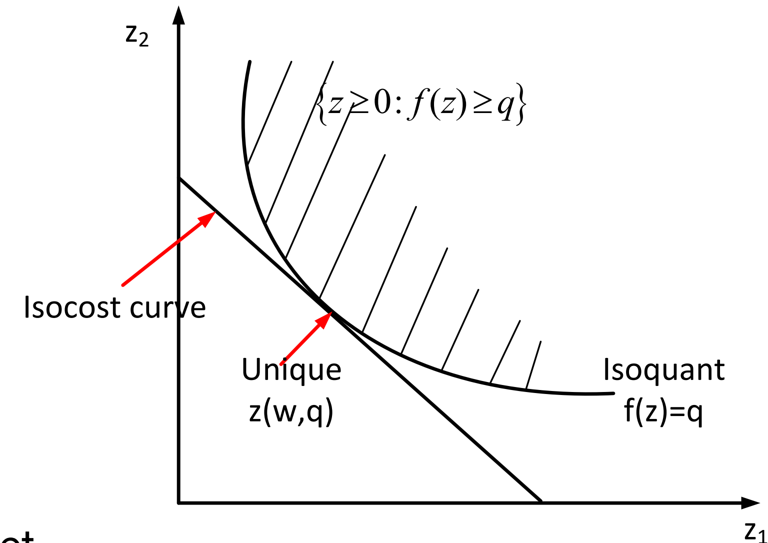
$$\boxed{z(\lambda w,q)=z(w,q).}$$



Proof: homogeneity of the cost function and Shepard's lemma

Properties of Conditional Factor Demand Correspondence

2) If the set $\{z \geq 0: f(z) \geq q\}$ *is strictly convex, then the firm's demand correspondence* $z(w, q)$ *is single valued.*

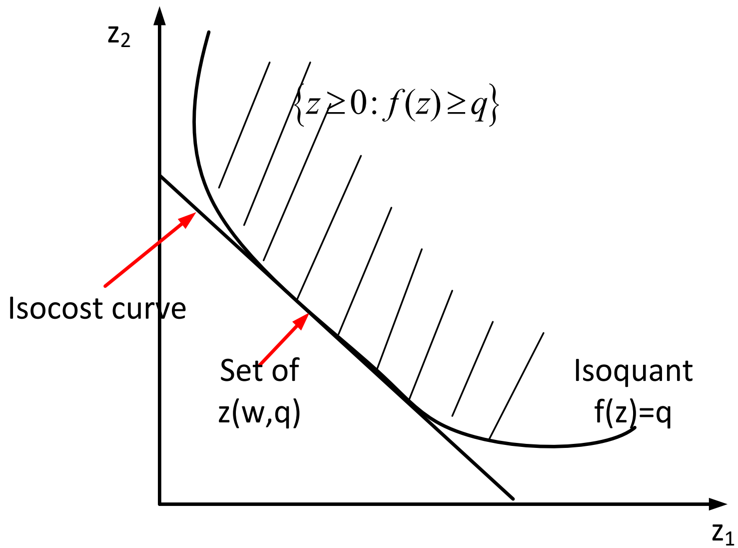


Production function is quasi concave, set is strictly convex so solution is unique and OPT demand is in the tangency point

Properties of Conditional Factor Demand Correspondence

2) *(continued)*

If the set $\{z \geq 0: f(z) \geq q\}$ *is weakly convex, then the demand correspondence* $z(w, q)$ *is not a single-valued, but a convex set.*



Properties of Conditional Factor Demand Correspondence

3) *If $z(w,q)$ is differentiable at $\bar{w}$, then $D_w^2 c(\bar{w},q) = D_w z(\bar{w},q)$ is a **symmetric** and **negative semidefinite matrix**, with $D_w z(\bar{w},q) \cdot \bar{w} = 0$.*

- Proof. Symmetry derives from Shephard's lemma and Young's theorem:

$$\frac{\partial z_\ell}{\partial w_i} = \frac{\partial}{\partial w_i}\left(\frac{\partial c(w,y)}{\partial w_\ell}\right) = \frac{\partial}{\partial w_\ell}\left(\frac{\partial c(w,y)}{\partial w_i}\right) = \frac{\partial z_i}{\partial w_\ell}$$

- While negative semi-definiteness by concavity in w of the cost function.
- $D_w z(\bar{w},q)$ is a matrix representing how the firm's demand for every input responds to changes in the price of such input, or in the price of the other inputs.

Properties of Conditional Factor Demand Correspondence

4) *Negative semi-definiteness, in turn, entails that*

- Own substitution effects are non-positive,

  $$\frac{\partial z_k(w,q)}{\partial w_k} \leq 0 \text{ for every input } k$$

  i.e., if the price of input k increases, the firm's factor demand for this input decreases.

- Cross substitution effects are symmetric,

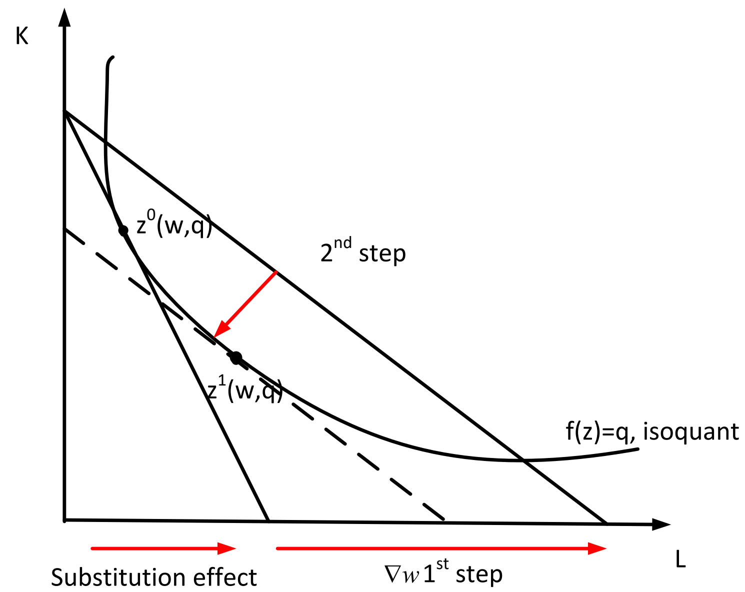
  $$\frac{\partial z_k(w,q)}{\partial w_l} = \frac{\partial z_l(w,q)}{\partial w_k} \text{ for all inputs } k \text{ and } l$$

Cost Minimization: SE and OE Effects

- ***Comparative statics of $z(w, q)$***: Let us analyze the effects of an input price change. Consider two inputs, e.g., labor and capital. When the price of labor, w, falls, two effects occur:
  - *Substitution effect*: if output is held constant, there will be a tendency for the firm to substitute l for k.
  - *Output effect*: a reduction in firm's costs allows the firm to produce larger amounts of output (i.e., to reach a higher isoquant), which entails the use of more units of both l for k.

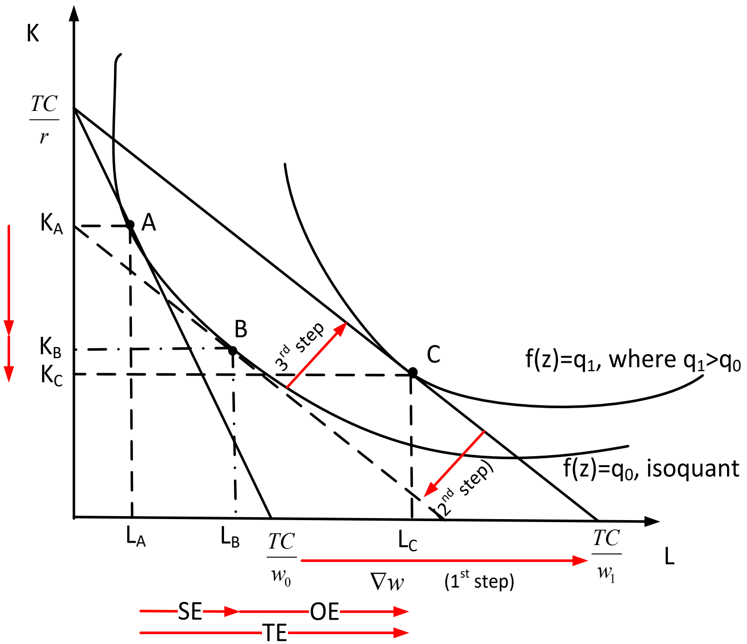
Cost Minimization: SE and OE Effects

- Substitution effect:
  - $z^0(w,q)$ solves CMP at the initial prices.
  - $\downarrow$ in wages $\Longrightarrow$ isocost line pivots outwards.
  - To reach q, push the new isocost inwards in a parallel fashion.
  - $z^1(w,q)$ solves CMP at the new input prices (for output level q).
  - At $z^1(w,q)$, firm uses more l and less k.



Cost Minimization: SE and OE Effects

- Substitution effect (SE):
  - increase in labor demand from L_A to L_B.
  - same output as before the input price change.
- Output effect (OE):
  - increase in labor demand from L_B to L_C.
  - output level increases, **total cost is the same as before the input price change**.



Cost Minimization: Own-Price Effect

- We have two concepts of demand for each input. E.g. for labor
  - the **conditional demand for labor**, $l^c(r, w, q)$
    - $l^c(r, w, q)$ solves the CMP
  - **the unconditional demand for labor**, $l(p, r, w)$
    - $l(p, r, w)$ solves the PMP

    where $w_l = w$ (i.e. wage) and $w_k = r$ (i.e. interest rate)
- At the profit-maximizing level of output, i.e., $q(p, r, w)$, the two must coincide

$$l(p, r, w) = l^c(r, w, q) = l^c(r, w, q(p, r, w))$$

Cost Minimization: Own-Price Effect

- Differentiating with respect to w yields

$$\frac{\partial l(p,r,w)}{\partial w} = \underbrace{\underbrace{\frac{\partial l^c(r,w,q)}{\partial w}}_{SE\ (-)} + \underbrace{\overbrace{\frac{\partial l^c(r,w,q)}{\partial q}}^{(+)} \cdot \overbrace{\frac{\partial q}{\partial w}}^{(-)}}_{OE\ (-)}}_{TE\ (-)}$$

From negative semi-definiteness of H of cost function $D_w^2 c(\overline{w},q) = D_w z(\overline{w},q)$. That is nonincreasing conditional factor demand in w.

From nondecreasing conditional factor demand in q

From nonincreasing supply function in w.

Formal proof that OE<0

We have to check the sign of

$$\frac{\partial l^c(r,w,q)}{\partial q}\frac{\partial q}{\partial w} = \frac{\partial l^c(r,w,q)}{\partial q}\frac{\partial \frac{\partial \pi}{\partial p}}{\partial w} = \frac{\partial l^c(r,w,q)}{\partial q}\frac{\partial \frac{\partial \pi}{\partial p}}{\partial w} =$$

$$\frac{\partial l^c(r,w,q)}{\partial q}\frac{\partial^2 \pi}{\partial p \partial w} = \frac{\partial l^c(r,w,q)}{\partial q}\frac{\partial(-l)}{\partial p} =$$

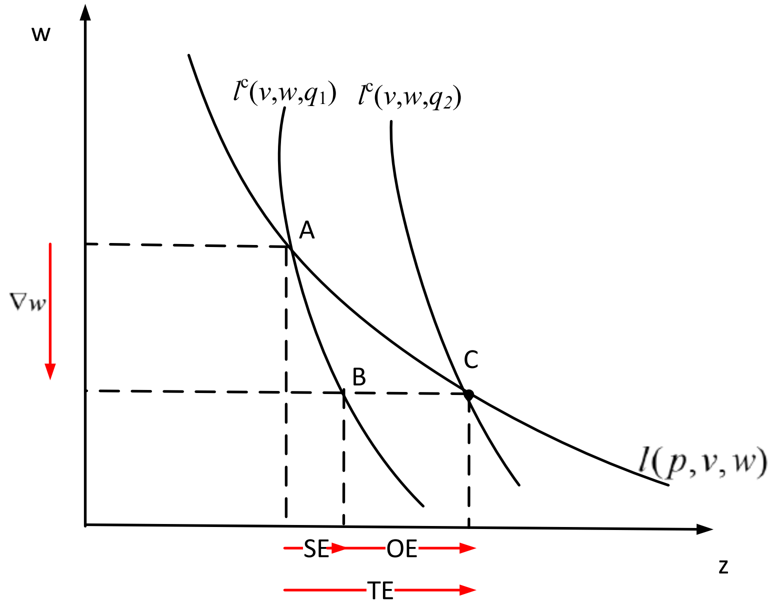
Then since $l(r,w,q) = l^c(w,q(p,w))$ at the optimum q, then $\frac{\partial(-l)}{\partial p} = \frac{\partial l^c}{\partial q}\frac{\partial q}{\partial p}$ and

$$= -\left(\frac{\partial l^c(r,w,q)}{\partial q}\right)^2 \frac{\partial q}{\partial p} = -\left(\frac{\partial l^c(r,w,q)}{\partial q}\right)^2 \frac{\partial^2 \pi}{\partial p \partial p}$$

Where the first factor is negative and the second positive by convexity of profit function in p (i.e. profit function semi-definite positive). Hence OE<0.

Cost Minimization: Own-Price Effect

- Since $TE > SE$, the unconditional labor demand is flatter than the conditional labor demand.
- Both SE and OE are negative.
  - Giffen paradox from consumer theory cannot arise in production theory.



Cost Minimization: Cross-Price Effect

- No definite statement can be made about cross-price (CP) effects.
  - A fall in the wage will lead the firm to substitute away from capital.
  - The output effect will cause more capital to be demanded as the firm expands production.

$$\underbrace{\frac{\partial k(p,r,w)}{\partial w}}_{CP\ TE\ (+)\text{ or }(-)} = \underbrace{\frac{\partial k^c(r,w,q)}{\partial w}}_{CP\ SE\ (+)} + \underbrace{\overbrace{\frac{\partial k^c(r,w,q)}{\partial q}}^{(+)} \cdot \frac{\overbrace{\partial q}^{(-)}}{\partial w}}_{CP\ OE\ (-)}$$

SE (+) because along an isoquant (conditional demand, if w increases I demand less labour, and I have to demand more capital to keep q fixed), OE (-) you can do a proof similar to that at p. 29. You end up with:

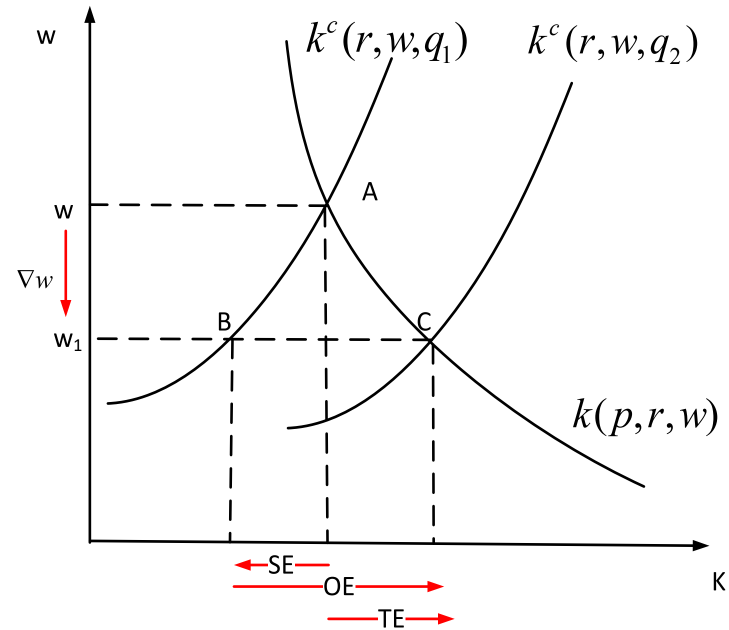
$$\frac{\partial k^c(r,w,q)}{\partial q}\frac{\partial q}{\partial w} = -\left(\frac{\partial l^c(r,w,q)}{\partial q}\frac{\partial k^c(r,w,q)}{\partial q}\frac{\partial^2 \pi}{\partial p \partial p}\right) < 0$$

where first two terms on the RHS in parentheses positive by conditional factor demands non decreasing in quantity and the third by convexity of the profit function in p.

Cost Minimization: Cross-Price Effect

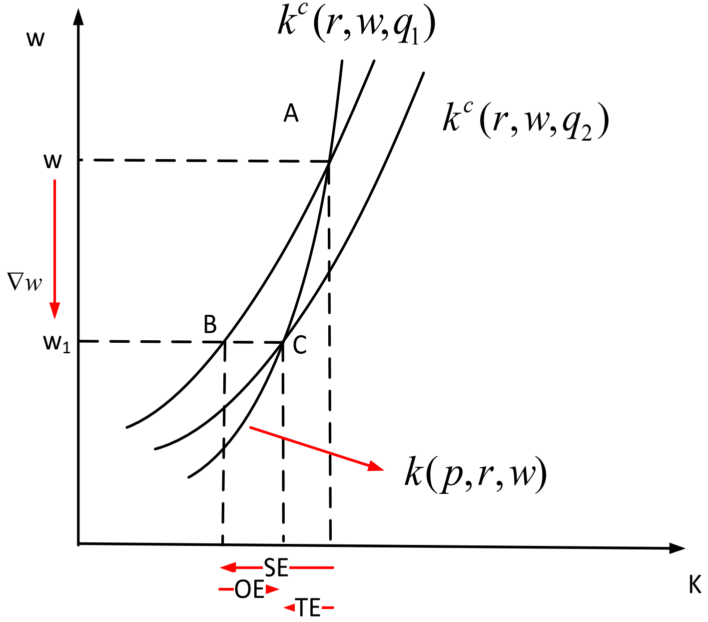
Example 1:

- The $(+)$ cross-price OE completely offsets the $(-)$ cross-price SE, **leading to a positive cross-price TE**.
- **Unconditional capital demand negatively sloped w.r.t to w (i.e. the price of labour)**



Cost Minimization: Cross-Price Effect

- Example 2:
- The $(+)$ cross-price OE only partially offsets the $(-)$ cross-price SE, leading to a negative cross-price TE.
- **Unconditional capital demand positively sloped w.r.t to w (i.e. the price of labour)**



Properties of Production Function and of C and Z

1) *If $f(z)$ is Homog(1) (i.e., if $f(z)$ exhibits constant returns to scale), then $c(w,q)$ and $z(w,q)$ are Homog(1) in q. [I skip the proofs]*

- Intuitively, if $f(z)$ exhibits CRS, then an increase in the output level we seek to reach induces an increase of the same proportion in the cost function and in the demand for inputs. That is,

$$c(w,\lambda q) = \lambda c(w,q)$$

and

$$z(w,\lambda q) = \lambda z(w,q)$$

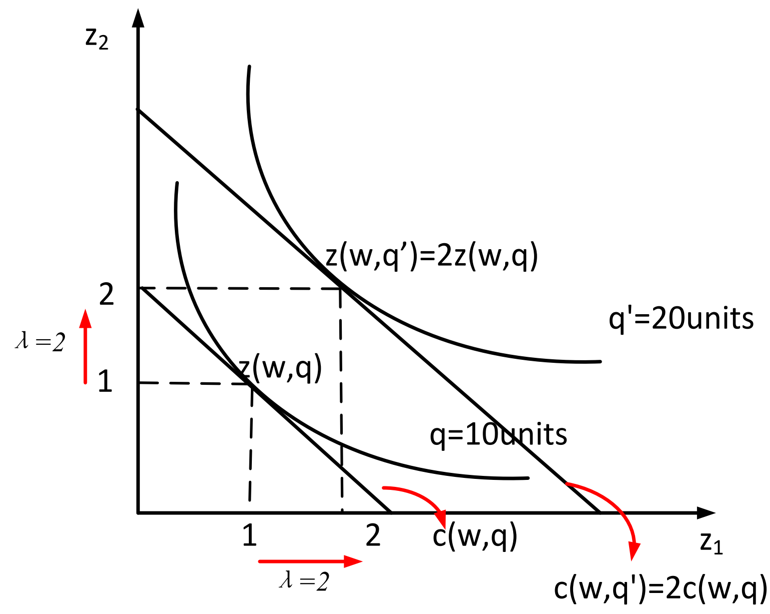
Properties of Production Function

- $\lambda = 2$ implies that demand for inputs doubles

  $$z(w, 2q) = 2z(w, q)$$

and that minimal costs also double

$$c(w, 2q) = 2c(w, q)$$



Properties of Production Function

2) *If $f(z)$ is concave, then $c(w, q)$ is convex function of q (i.e., marginal costs are non-decreasing in q).*

- More compactly,

$$\frac{\partial^2 c(w, q)}{\partial q \partial q} \geq 0$$

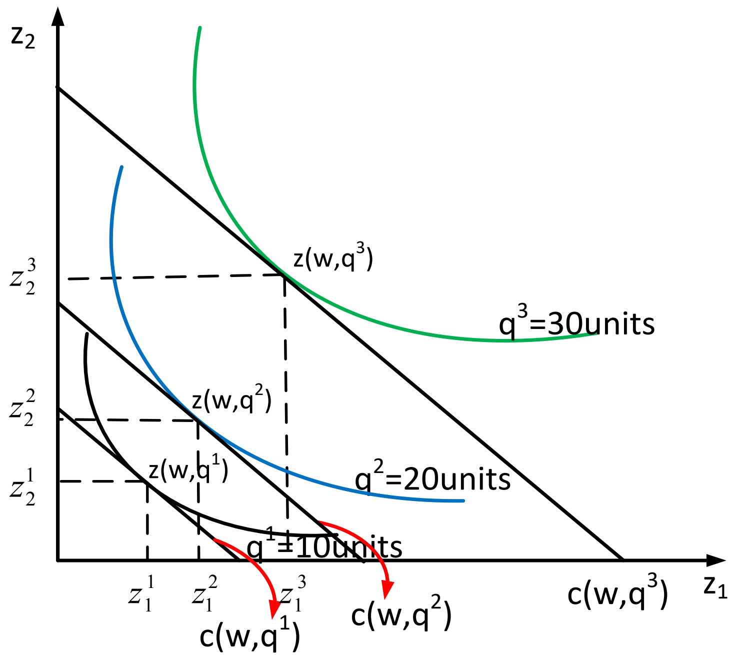
  or, in other words, marginal costs $\frac{\partial c(w,q)}{\partial q}$ are weakly increasing in q.

  Intuition: if marginal productivity is decreasing, then to further increase production (at higher levels of q) requires increasingly more amounts of factors, so marginal cost increases.

Properties of Production Function

2) (continued)

- Firm uses more inputs when raising output from q^2 to q^3 than from q^1 to q^2.
- Hence,
  $c(w,q^3) - c(w,q^2) > c(w,q^2) - c(w,q^1)$
- This reflects the convexity of the cost function $c(w,q)$ with respect to q.



EXERCISES

1, 2 RETURN TO SCALE

3

AVERAGE PRODUCT

$$AP_1 = \frac{q}{z_1} \qquad AP_2 = \frac{q}{z_2} \qquad q = f(z_1, z_2)$$

e $z_1 = \ell$

$$MP_1 = \frac{\partial q}{\partial z_1} \qquad MP_2 = \frac{\partial p}{\partial z_2}$$

$$AP_\ell = \frac{q}{\ell}$$

PF. HAS CONSTANT RETURN TO SCALE

TOTAL PRODUCTIVITY DIVIDED BY ℓ WORKERS

EULERS THEOREM $\longrightarrow$ PF IF HOMOGENEOUS OF DEGREE α $\rightarrow 1$

$$\alpha\, q = \frac{\partial f}{\partial z_2} \cdot z_1 + \frac{\partial f}{\partial z_2} z_2$$

$$\boxed{q = \frac{\partial f}{\partial z_1} z_1 + \frac{\partial f}{\partial z_2} z_2}$$

HOW TO GET A P OF FIRST FACTOR? z_1

DIVIDE BOTH SIDE FOR z_1

$$\frac{q}{z_1} = \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2} \cdot \frac{z_2}{z_1}$$

$$AP_1 = MP_1 + MP_2 \cdot \frac{z_2}{z_1}$$

$$MP_1 = AP_1 - MP_2 \cdot \frac{z_2}{z_1} \qquad MP_2 = \frac{AP_1 - MP_1}{\frac{z_1}{z_2}}$$

$MP_{2} > 0$

IF $AP_1 > MP_1$

$MP_{2} < 0$

IF $AP_1 < MP_1$

4) $\pi(p, w) = p^2 w^{\alpha}$

a) PROVE PRODUCTION FUNCTION SATISFY HOMOG. OF DEGREE 1

TO FIND α IN BOTH PRICES HOMOGENEITY

MULTIPLY FOR A COMMON FACTOR $\lambda > 1$

$$\pi(\lambda p, \lambda w) = (\lambda p)^2 (\lambda w)^{\alpha} = \lambda^{2+\alpha} (p^2 w^{\alpha}) = \lambda^{2+\alpha} \pi(p, w)$$

λ?

WE NEED $2 + \alpha = 1$ SO $\alpha = -1$

TO HAVE 1 AS EXPONENT IN λ

SO HOMOGENEOUS OF DEGREE 1 $\Rightarrow$ WE HAVE CONSTANT RETURN TO SCALE

$$\pi(p, w) = p^2 w^{-1}$$

1) NON DECREASING IN p $\rightarrow$ CHECK 1° DERIVATIVE

$$\frac{\partial \pi}{\partial p} = \frac{2p}{w} \boxed{> 0} \Rightarrow$$ NON DECREASING IN P, ACTUALLY IS INCREASING IN P

PRICES ARE POSITIVE (IT'S A FACT)

2) NON-INCREASING IN w

$$\frac{\partial \pi}{\partial w} = -\frac{p^2}{w^2} < 0 \Rightarrow$$ DECREASING IN w

3) CONVEX IN P AND W

↳ CHECK HESSIAN (MATRIX OF 2° DERIVATIVES)

$$H = \begin{bmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} \end{bmatrix} \qquad \tilde{\Pi}_{11} = \frac{\partial^2 \tilde{\Pi}}{\partial p \partial p} \qquad \tilde{\Pi}_{12} = \frac{\partial^2 \tilde{\Pi}}{\partial p \partial w}$$

$$\tilde{\Pi}_{21} = \frac{\partial^2 \tilde{\Pi}}{\partial w p} \qquad \tilde{\Pi}_{22} = \frac{\partial^2 \tilde{\Pi}}{\partial w \partial w}$$

$$\begin{vmatrix} \frac{2}{w} & -\frac{2p}{w^2} \\ -\frac{2p}{w^2} & \frac{2p^2}{w^3} \end{vmatrix}$$

COMPUTE 2° DER HERE JUST AS EXERCISE

$$\delta^2 - p^2 \quad w^{-2} = -p^2 \, -2 w^{-3} = \frac{2p^2}{w^3}$$

CONVEX → POSITIVE SEMIDEFINITE

$$\frac{2}{w} \geq 0 \qquad \frac{2p^2}{w^3} \geq 0 \qquad |H| \geq 0$$

$$\frac{4p^2}{w^4} - \frac{4p^2}{w^4} = 0$$

CONVEX IN P AND W IN ALL ARGUMENTS

(5) $f(z_1, z_2) = K(z_1 \cdot z_2)^{1/4}$

K = CONSTANT

a) UNCONDITIONAL DEMAND FUNCTION z_1, z_2?

THIS IS A COBB - DOUGLAS

$1/2$ = CRS → FUNCTION IS CONCAVE

$$\max_{z_1, z_2} \tilde{\Pi} = \frac{p \cdot K(z_1 z_2)^{1/4}}{q} - w_1 z_1 - w_2 z_2$$

UNCONSTRAINT MAX PROBLEM

$q = f(z_1, z_2)$

MP₁ PRICE OF FACTOR 1

$$\text{FOCs} = \frac{\delta \tilde{\Pi}}{\delta z_1} = \boxed{pK \frac{1}{4} z_1^{-3/4} z_2^{1/4}} - w_1 = 0$$

PRICE 2

$$\frac{\delta \tilde{\Pi}}{\delta z_2} = \boxed{pK z_1^{1/4} \frac{1}{4} z_2^{-3/4}} - w_2 = 0$$

MP₂

ALSO WE HAVE NON NEGATIVITY CONSTRAINT

$z_1 \geq 0 \quad z_2 \geq 0$

BUT CORNER SOLUTION NOT POSSIBLE SINCE

IF $z_1 = 0 \quad z_1 \cdot z_2 = 0$

$$|MRTS| = \frac{z_2}{z_1} = \frac{w_1}{w_2}$$

SLOPE OF ISOQUANT

SLOPE OF ISOPROFIT LINE

$$z_2 = \frac{w_1}{w_2} z_1$$

LINE WITH SAME PROFIT

TO SOLVE SYSTEM TAKE EQUATIONS

1. EQN $\quad z_2 = \frac{w_1}{z_1 p_k}$

$$p_k \, z_1^{-3/4} \, \frac{1}{4} z_2^{1/4} = w_1$$

$$z_2^{1/4} = \frac{w_1}{p_k^{1/4} \, z_1^{-3/4}}^{4} \qquad z_2 = \frac{w_1^4}{(p_k)^4 \left(\frac{1}{4}\right)^4 z_1^{-3}}$$

TWO EXPRESSION FOR z_2 → SO EQUATE TWO SIDES

$$\boxed{z_2 =} \left(\frac{w_1}{p_k}\right)^4 z_1^3 \left(\frac{1}{4}\right)^{-4}$$

$$\boxed{z_2} = \left(\frac{w_1}{w_2}\right) z_1$$

$$\left(\frac{w_1}{p_k}\right)^4 z_1^3 \left(\frac{1}{4}\right)^{-4} = \left(\frac{w_1}{w_2}\right) z_1$$

$$\left(\frac{w_1}{w_2}\right) z_1 = \frac{w_2^4}{(p_k)^4} z_1^3 \left(\frac{1}{4}\right)^{-4}$$

$$\left(\frac{w_1}{w_2}\right) \cdot \left(\frac{1}{4}\right)^4 \cdot \frac{(pk)^4}{w_1^4} = z_1^2$$

$$w_1^{-3} w_2^{-1} \left(\tfrac{1}{4}\right)^4 (pk)^4 = z_1$$

$$z_1^* = w_1^{-3/2} w_2^{-1/2} \left(\tfrac{1}{4}\right)^2 (pk)^2$$

UNCONDITIONAL DEMAND FOR z_1

FOR z_2 JUST REPLACE z_1

$$z_2^* = \left(\frac{w_1}{w_2}\right) w_1^{-3/2} w_2^{-1/2} \left(\tfrac{1}{4}\right)^2 (pk)^2 =$$

$$= w_1^{-1/2} w_2^{-3/2} \left(\tfrac{1}{4}\right)^2 (pk^2)$$

UNCOND. DEMAND FOR z_2

UNCONSTRAINT WITH QUANTITY, I CAN CHOOSE WHAT QUANTITY I WANT

TO CHECK CONVEXITY HESSIAN!

PROVE IF FUNCTION CONCAVE, SINCE IS ALSO QUASI CONCAVE. WE KNOW CONCAVE FROM START SINCE COBB-DOUGLAS

FIND SUPPLY FUNCTION

PROD FUNCTION IN OPTIMAL UNCONDITIONAL DEMANDS

$$q^* = f(z_1^*, z_2^*) = k(z_1^* z_2^*)^{1/4} =$$

$$= k\Big[\underbrace{\big(w_1^{-3/2} w_2^{-1/2} \left(\tfrac{1}{4}\right)^2 pk^2\big)}_{z_1} \cdot \big(w_1^{-1/2} w_2^{-3/2} \tfrac{1}{4}^2 pk^2\big)\Big]^{1/4} =$$

$$= k \cdot \big(w_1^{-4/\cancel{2}} w_2^{-2} \tfrac{1}{4}^4 pk^4\big)^{1/4} = \qquad \frac{-3-1}{2}$$

$$= k\,(w_1^{-1/2} w_2^{-1/2})\,\tfrac{1}{4}\,pk = k \underbrace{\left(\frac{1}{w_1 w_2}\right)^{1/2} \tfrac{1}{4}\,pk}_{A}$$

$q(w_1, w_2, p)$

how quantity depends on prices

$$\frac{\partial q}{\partial p} = Ak > 0 \quad \longrightarrow \text{ supply is increasing in } p$$

EX. 2

$$f(z) = 2^{3/4} \, z_1^{1/4} \, z_2^{1/4}$$

CONDITIONAL FACTOR DEMAND

SOLVING MINIMISATION COST PROBLEM

CMP

$$\min_{z_1, z_2} \; w_1 z_1 + w_2 z_2 \qquad \text{s.t.} \quad q = 2^{3/4} z_1^{1/4} z_2^{1/4} \geq q$$

$$\Downarrow$$

$$= q$$

$$L = w_1 z_1 + w_2 z_2 + \lambda \left(q - \underbrace{2^{3/4} z_1^{1/4} z_2^{1/4}}_{\text{PRODUCTION FUNCTION}} \right)$$

FOCs

$$\frac{\partial L}{\partial z_1} = w_1 + \lambda \left(-2^{3/4} \tfrac{1}{4} z_1^{-3/4} z_2^{1/4} \right) = 0$$

$$\frac{\partial L}{\partial z_2} = w_2 + \lambda \left(-2^{3/4} z_1^{1/4} \cdot \tfrac{1}{4} z_2^{-3/4} \right) = 0$$

RATIO OF FOCS

$$w_1 + \lambda \left(\cancel{-2^{3/4}} \cancel{\tfrac{1}{4}} z_1^{-3/4} z_2^{1/4} \right) = w_2 + \lambda \left(\cancel{-2^{3/4}} z_1^{1/4} \cdot \cancel{\tfrac{1}{4}} z_2^{-3/4} \right)$$

$$\frac{w_1}{w_2} = \frac{z_2}{z_1} \qquad \qquad z_1 = \frac{w_1}{w_2} \cdot z_2$$

WE DON'T HAVE q HERE BUT ENTER ON THE CONSTRAINT

$$\frac{\partial L}{\partial \lambda} = q - 2^{3/4} z_1^{1/4} z_2^{1/4}$$

$$z_2^{1/4} = q\, z_1^{-1/4}\, 2^{-3/4}$$

$$z_2 = q^4 z_1^{-1}\, 2^{-3}$$

$$z_2 = q^4 z_1^{-1}\, 2^{-3}$$

(*)

↓ REPLACE

$$z_1 = \frac{w_1}{w_2} \cdot q^4 z_1^{-1}\, 2^{-3} \rightarrow z_1^2 = \frac{w_1}{w_2}\, q^4 \frac{1}{2^3}$$

$$z_1^* = q^2 \sqrt{\frac{w_1}{w_2}}\, \sqrt{\frac{1}{8}}$$

CONDITIONAL DEMAND FOR z_1

(LINKE z_1 WITH q)

FOR z_2 WE CAN DO IT IN MANY WAYS

EASIER IS REPLACE IN $z_1 = \frac{w_1}{w_2} z_2$

$$z_2^* = \frac{w_2}{w_1}\, q^2 \sqrt{\frac{w_1}{w_2}} \cdot \sqrt{\frac{1}{8}} =$$

$$= w_2^{1/2}\, w_1^{-1/2}\, q^2 \sqrt{\frac{1}{8}} = q^2 \sqrt{\frac{w_2}{w_1}} \cdot \sqrt{\frac{1}{8}}$$

CONDITIONAL FACTOR DEMAND FOR z_2

ANOTHER THING : <u>COST FUNCTION</u>

$$c(w_1, w_2, q) = w_1 z_1^* + w_2 z_2^* =$$

$$= w_1 \left(q^2 \sqrt{\frac{w_1}{w_2}} \sqrt{\frac{1}{8}} \right) + w_2 \left(q^2 \sqrt{\frac{w_2}{w_1}} \sqrt{\frac{1}{8}} \right)$$

$$q^2 \sqrt{\frac{1}{8}} \left[w_2 \, w_1^{1/2} w_2^{-1/2} + w_2 \cdot w_2^{1/2} w_1^{-1/2} \right]$$

$$= q^2 \sqrt{\frac{1}{8}} \left[\underbrace{w_1^{3/2} \cdot w_2^{-1/2} + w_2^{3/2} \, w_1^{-1/2}}_{A} \right]$$

A ALL POSITIVE SINCE SQUARE ROOTS

IF $q \uparrow$, $C \uparrow$ SO COST FUNCTION INCREASING IN q SO

<u>MARG COST IS POSITIVE</u>

Advanced Microeconomic Theory

Chapter 4: Alternative solution of PMP, aggregate supply

Alternative Representation of PMP

FOC of the alternative version of the PMP implies P=MC(q). So the firm's supply curve is the locus in which price is equal to the marginal cost, i.e. in practice the marginal cost curve, but only for the portion in which P>AC (long run) or P>AVC (short run), where AVC are average variable costs (in the short run it makes sense to distinguish between variable and fixed costs because some factors are fixed, in the long run it doesn't, since all factors are variable).

Alternative Representation of PMP

- Using the cost function $c(w,q)$, we write the PMP as follows

$$\max_{q\geq 0} \quad pq - c(w,q)$$

(NB. Now the choice variable is q, i.e. the quantity of output and not the input quantities (i.e. z).

This is useful if we have information about the cost function, but we don't know the production function $q = f(z)$.

After solving the cost minimisation problem we can find the cost function that is the value of function problem. So with this value function we can set the PMP (profit maximisation problem) in a different way. The profit is the difference between revenue and total cost.

It indicates the minimum case of quantity q.

The profit is only a function of one variable q so exogenous.

The quantity in such way to maximise profit. We know quantity cannot be negative. In this way is useful when we don’t know the production function (the function linking input to outputs).

Now to solve this problem we compute the FOC: the derivative of the profit function with respect to q.

Alternative Representation of PMP

- Let us now solve this alternative PMP
$$\max_{q \geq 0} \quad pq - c(w,q)$$
- FOCs for q^* to be profit maximizing are
$$p - \frac{\partial c(w,q^*)}{\partial q} \leq 0$$
and in interior solutions
$$p - \frac{\partial c(w,q^*)}{\partial q} = 0$$
- That is, at the interior optimum q^*, price equals marginal cost, $\frac{\partial c(w,q^*)}{\partial q}$.

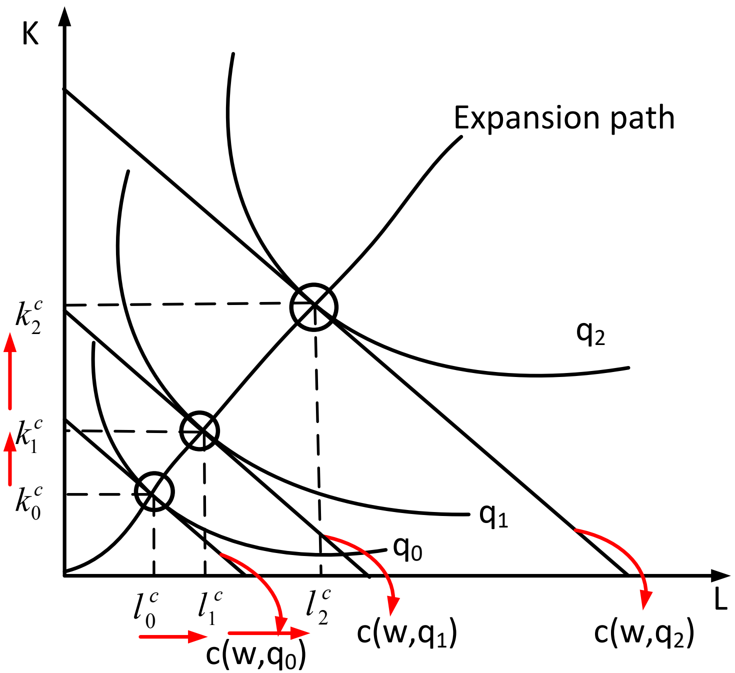
Now to solve this problem we compute the FOC: the derivative of the profit function with respect to q. The derivative of the total revenue it price (which is price).
This derivative should be 0 so optimum price should be equal to the marginal cost.

As maximum we should also consider the Second order condition (SOC).

Firm's Expansion Path

- The ***expansion path*** is the locus of cost-minimizing tangencies. (Analogous to the *wealth expansion path* in consumer theory)
- The curve shows how inputs increase as output increases.
- Expansion path is positively sloped if both k and l are *normal* inputs, i.e.,

$$\frac{\partial k^c(w,q)}{\partial q} \geq 0, \frac{\partial l^c(w,q)}{\partial q} \geq 0$$



In the following slide we introduce the concept of the firm expansion path: By solving the CMP varying quantity (that are constraint in this CMP problem) we can find the optimal combination of factor minimising cost to produce different levels of quantity corresponding to the constrain to the CMP. After we have found this tangency points (the FOC for CMP is the tangency between the isoquant and the isocost line). After findings this points we can link them and reobtain a curve that is the firm expansion path. If increasing (this lines have positive slope) then both inputs are normal.
To increase the q produced the firm must an increase quantity of both factors. In this case factors are capital and labour.

Firm’s Expansion Path

- If the firm’s expansion path is a ***straight line***:
  - All inputs must increase at a **constant proportion** as firm increases its output.
  - The firm’s production function exhibits constant returns to scale and it is, hence, homothetic.
  - If the expansion path is straight and coincides with the **45-degree line**, then the firm increases all inputs by the **same proportion** as output increases.
- The expansion path does not have to be a straight line.
  - The use of some inputs may increase faster than others as output expands
    - Depends on the shape of the isoquants.

However the firm expansion path doesn’t have to be a straight line.
In case it is the inputs must increase at a constant proportions as firm increases its output.
This means that also the curve (firm expansion path) is homothetic.
If expansion path is 45° line this mean that not only inputs increase in constant proportion but also increase in exactly the same proportion.

We will see that there are cases in which expansion path is not a straight line.

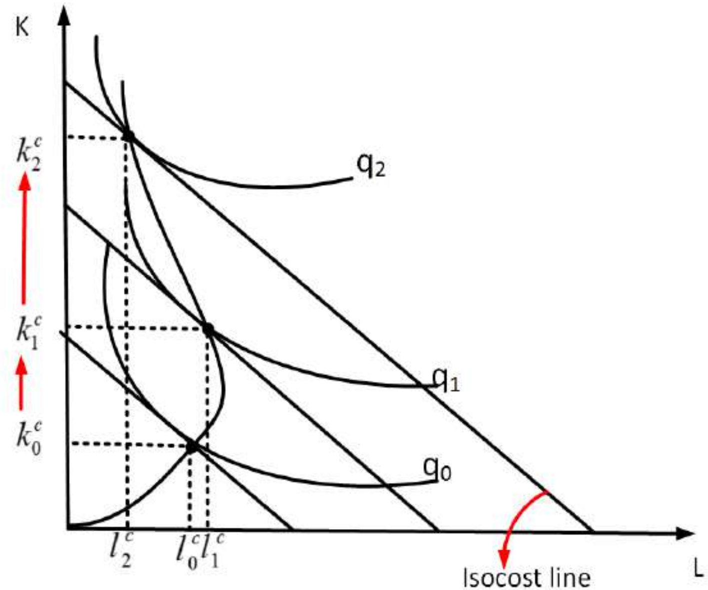
Firm's Expansion Path

- The expansion path does not have to be upward sloping.
  - If the use of an input falls as output expands, that input is an *inferior* input.
- k is *normal*

$$\frac{\partial k^c(w,q)}{\partial q} \geq 0$$

but l is inferior (at higher levels of output)

$$\frac{\partial l^c(w,q)}{\partial q} < 0$$



Let’s take this case in which we have:

- two factor K and L
- isocost line
- Different levels of q.

We obtain a line that first increases and then decreases. As for capital, to increase the quantity you have to use more capital. But actual labour is normal input up to a certain level of quantity but then for go to q0 to q1 you have to decrease the quantity of labour (L). After reaching the level q1, L became an inferior input.

Cost and Supply: Single Output

- Let us assume a **given vector of input prices** $\bar{w} \gg 0$ (i..e input prices are given). Then, $c(\bar{w}, q)$ can be reduced to $C(q)$. Then, average and marginal costs are

$$AC(q) = \frac{C(q)}{q} \text{ and } MC = C'(q) = \frac{\partial C(q)}{\partial q}$$

- Hence, the FOCs of the PMP can be expressed as

$p \leq C'(q)$, and in interior solutions $p = C'(q)$

i.e., **all output combinations such that $p = C'(q)$ are the (optimal) supply correspondence of the firm $q(p)$.**

Introducing two aspects: ***Average cost and Marginal cost***

As i said if we consider the cost function then we consider input prices as given. In fact, **we have over line on w.**

Cost are all function of quantity then we can apply definition of average cost which is total cost divided by quantity.

Marginal cost is the derivative of total cost with respect to quantity.

In the alternative setting of MP the FOC we can see prices <= marginal cost.

For Interior solution (at optimal level of quantity) prices = marginal cost.

The equation p = MC is the firm supply curve.

Cost and Supply: Single Output

- We showed that the cost function $c(w, q)$ is homogenous of degree 1 in input prices, w.
  - Can we extend this property to the AC and MC? Yes!
  - For the average cost function,

$$AC(tw, q) = \frac{C(tw, q)}{q} = \frac{t \cdot C(w, q)}{q}$$
$$= t \cdot AC(w, q)$$

In some previous lecture we saw that cost function is homogeneous of degree 1 in the input prices.
So if you increases all prices by the proportional alpha also the total cost increases by the proportional alpha.
Let’s check whether this property also extend to the average cost and Marginal cost.

We have to apply the definitions.
So average cost is total cost/q. We are multiplying all by t(constant), and increases all prices by t the proportional cost will increase by t.
So this can be rewritten as t which multiply the average cost.

So we prove the AC is homogeneous in input prices if C function is homogeneous of degree one in input prices.

Cost and Supply: Single Output

– For the marginal cost function,

$$MC(tw,q) = \frac{\partial C(tw,q)}{\partial q} = \frac{\partial \boxed{[tC(w,q)]}}{\partial q} = \frac{t \cdot \partial C(w,q)}{\partial q}$$

$$= t \cdot MC(w,q)$$

(Isn't this result violating Euler's theorem? No!

- The above result states that $c(w,q)$ is homog(1) in inputs prices, and that $MC(w,q) = \frac{\partial C(w,q)}{\partial q}$ is also homog(1) in input prices.
- Euler's theorem would say that: If $c(w,q)$ is homog(1) in inputs prices, then its derivate with respect to input prices, $\frac{\partial C(w,q)}{\partial w}$, must be homog(0).

We can do similar check for the marginal cost. We have to compute total derivative with respect to q if the cost function is homogeneous of degree one we can bring outside the cost function and then we have to compiute the derivative with respect to q which is t which multiply the derivative of the cost function with respect to q. Which is t that multiply the MC.

So we have proved that if cost function is homogeneous of degree one also MC is homogeneous of degree 1.

Graphical Analysis of Total Cost

- With constant returns to scale, total costs are proportional to output.

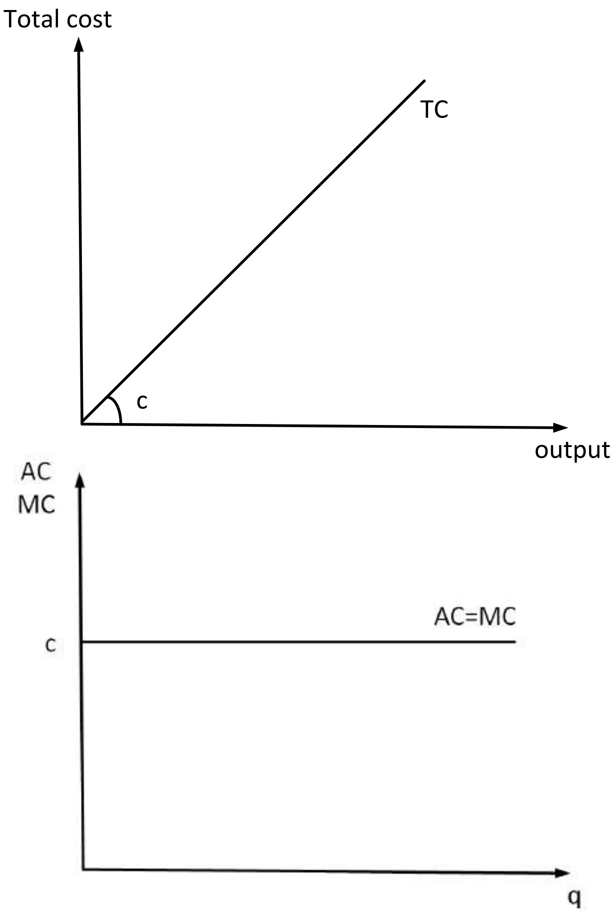
$$TC(q) = c \cdot q$$

- Hence,

$$AC(q) = \frac{TC(q)}{q} = c$$

$$MC(q) = \frac{\partial TC(q)}{\partial q} = c$$

$$\Longrightarrow AC(q) = MC(q)$$



Constant return to scale technology with Total cost

Total costs are proportional to outputs so in this case the total cost is a straight line starting form origin in.
TC can be see as c multiply by quantity.
If you want to double quantity you have to use twice the original input quantity so given input prices the total cost will doubled.
Now we can compute the average cost bu dividing the total cost to q and we get c.
MC is the derivative of TC with respect to q which is against c.
TC proportional to output is the case of constant return to scale.
The average cost and MC are the same and constant.
So graphical representation is an orizonthal line in which TC is straight line starting from origin who slope is c that is a cost.

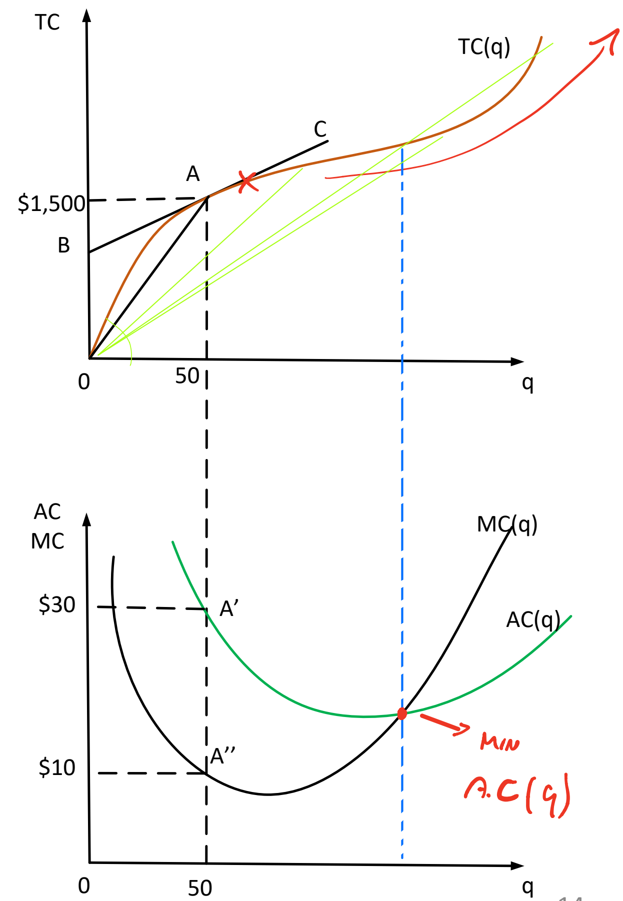
Cost and Supply: Single Output

- Suppose that TC starts out as concave and then becomes convex as output increases.
  - TC no longer exhibits constant returns to scale.
  - One possible explanation for this is that there is a third factor of production that is fixed as capital and labor usage expands (e.g., entrepreneurial skills).
  - TC begins rising rapidly after *diminishing returns* set in.

there are also cases in which the cost function is a little bit different so it’s more complex.
TC in which function starts as a concave function at to a point in which actually the function changes concavity so in this case it became convex.
From a certain point(certain level of quantity)
Imaging we have TC function and we want to draw MC and AC. The average cost is the slope of the TC function and if you compute the slope in the portion of the total cost function which is concave the slope will be decreasing so this means that MC will be decreasing up to the point in which function changes concavity.

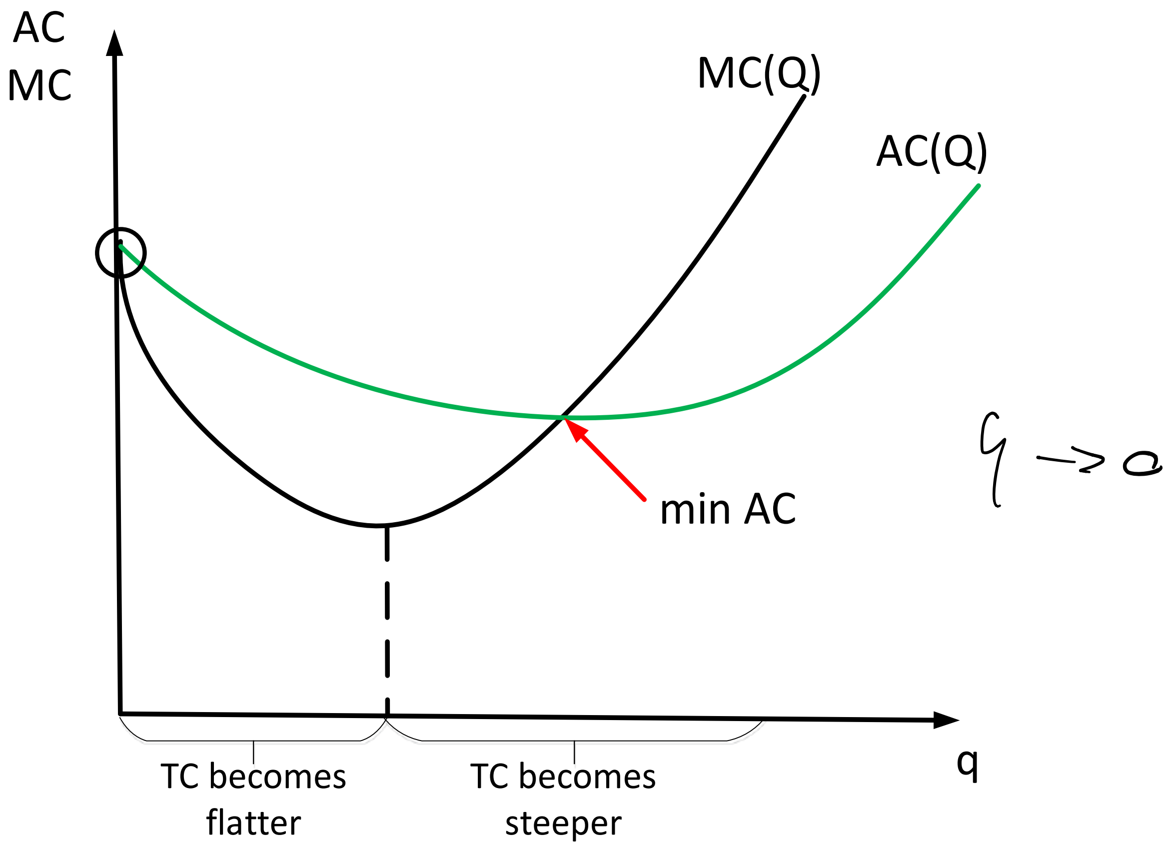
Cost and Supply: Single Output

- TC initially grows very rapidly, then becomes relatively flat, and for high production levels increases rapidly again.
- MC is the slope of the TC curve.



From A to C the function became convex and this mean that slope of TC is increasing. To compute the slope, we have to take a point and compute the tangent of the point and it’s easy to see that the slope of first one is increasing.
What about the AC? To compute the AC we have to take a point in TC and connect this to the origin and the AC in this this point will be the slope of this line connecting the point of TC to the origin. In this case you can check that up to a point the AC will be decreasing up to the point in which the AC is tangent to the TC.
A peculiarity of MC and AC, the MC cuts the AC in the minimum of the AC 7function. So, the point is the minim in the AC function.

Cost and Supply: Single Output



This is actually the zoom of the previous picture. The fact that marginal cost crossed the AC in itsminimum this can be proved analytically.
We want to prove that MC and AC start from the same point, to prove this we have to check the limit of the cost MC and AC when q tend to 0.

Cost and Supply: Single Output

- ***Remark 1***: AC=MC at $q = 0$.
  - Note that we cannot compute
$$AC(0) = \frac{TC(0)}{0} = \frac{0}{0}$$
  - We can still apply l'Hopital's rule
$$\lim_{q\to 0} AC(q) = \lim_{q\to 0} \frac{TC(q)}{q} = \lim_{q\to 0} \frac{\frac{\partial TC(q)}{\partial q}}{\boxed{\frac{\partial q}{\partial q}}} = \lim_{q\to 0} MC(q)$$
  - Hence, AC=MC at $q = 0$, i.e., AC(0)=MC(0).

What about the AC? Is the TC divided by q and we have to compute the AC in 0 (when quantity is 0). The TC if q is 0 will be 0, and the denominator will be 0 too. The limit is undefined, and we have to apply the De l'Hopital's rule I which to compute the AC (which depend to q –> 0) we can compute the derivative of the numerator and den with respect to q. The der of numerator is the MC and the derivative of the denominator is 1.
So we obtain by using de l'Hopital that the limit of the AC with q tends to 0 equal to the limit of MC with q tends to 0.
So this is what we wanted to prove.
The two curves tend to the same point when q tend to 0.

Cost and Supply: Single Output

- ***Remark 2***: When MC<AC, the AC curve decreases, and when MC>AC, the AC curve increases.
  - *Intuition*: using example of grades
  - If the new exam score raises your average grade, it must be that such new grade is better than your average grade thus far.
  - If, in contrast, the new exam score lowers your average grade, it must be that such new grade is than your average grade thus far.

Another interest property of the AC that is when MC < AC then AC curve decreases and when MC>AC then the AC curve increases.
If you take the example of grades: to increase the GPA must be that the last grade that you got is larger that previous GPA before that exam.

Cost and Supply: Single Output

- ***Remark 3***: AC and MC curves cross (AC=MC) at exactly the minimum of the AC curve.
  - Let us first find the minimum of the AC curve

FOC:

$$\frac{\partial AC(q)}{\partial q} = \frac{\partial\left(\frac{TC(q)}{q}\right)}{\partial q} = \frac{q\boxed{\frac{\partial TC(q)}{\partial q}} - TC(q)\cdot 1}{q^2}$$

$$= \frac{q\cdot \boxed{MC(q)} - TC(q)}{q^2} = 0$$

  - The output that minimizes AC must satisfy

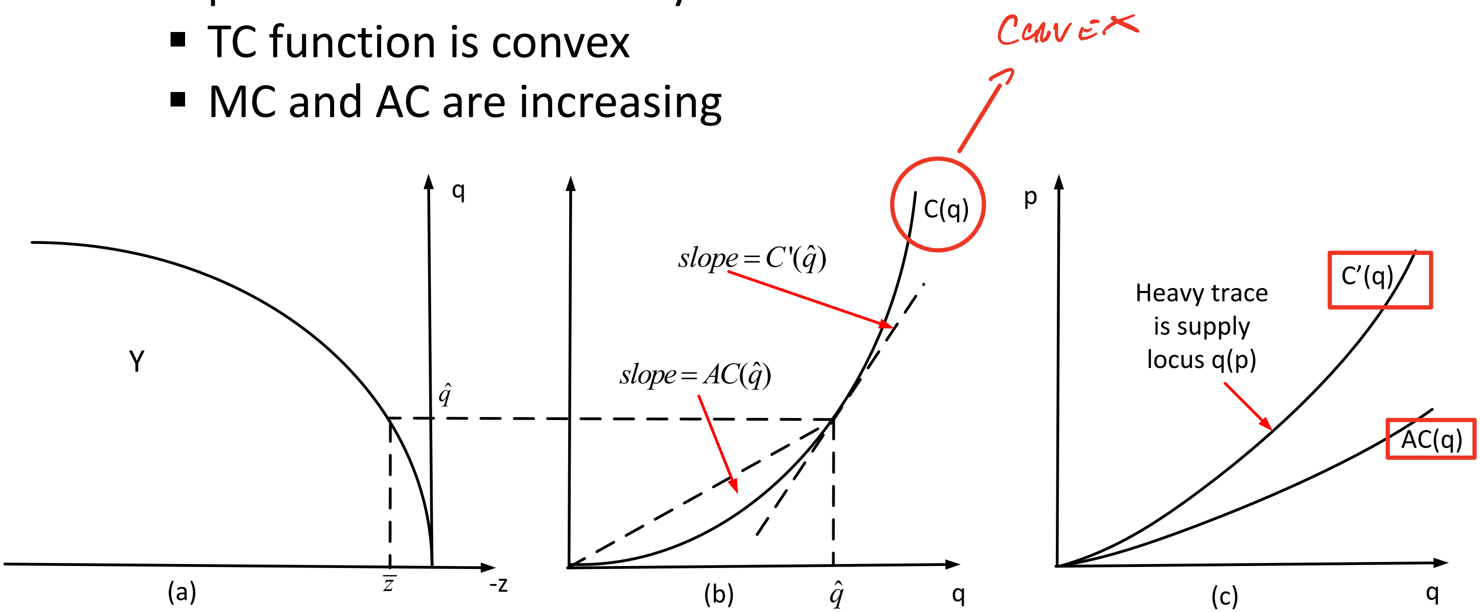
$$\frac{1}{q}(MC(q) - \frac{TC(q)}{q}) = 0 \implies MC(q) = \frac{TC(q)}{q} \leftarrow AC(q)$$

  - Hence, $MC = AC$ at the minimum of AC.

Now we are proving what he anticipated before: the MC crosses the AC in the minimum of AC.
To prove this we have to find the min of the AC.
To do it we check the FOC that is: der of AC with respect to q. After computing this we apply the rule of derivative of the ratio. So der TC with respect to q is MC. At the end all of that must be equal to 0 (for the FOC). Then we collect 1/q and we get in the parentheses the MC – TC/q. Since q cannot be equal to 0 (in general) then for the product to be equal to 0 it must be the case in which the term in parentheses must be equal to 0. This happens when MC is equal to AC.
So MC = AC.
We have seen that the FOC for the Min of the AC implies that the MC must be equal to AC in its minimum. So MC must crosses AC in the minimum of the AC.

Cost and Supply: Single Output

- ***Decreasing returns to scale***:
  - an increase in the use of inputs produces a less-than-proportional increase in output.
    - production set is strictly convex
    - TC function is convex
    - MC and AC are increasing

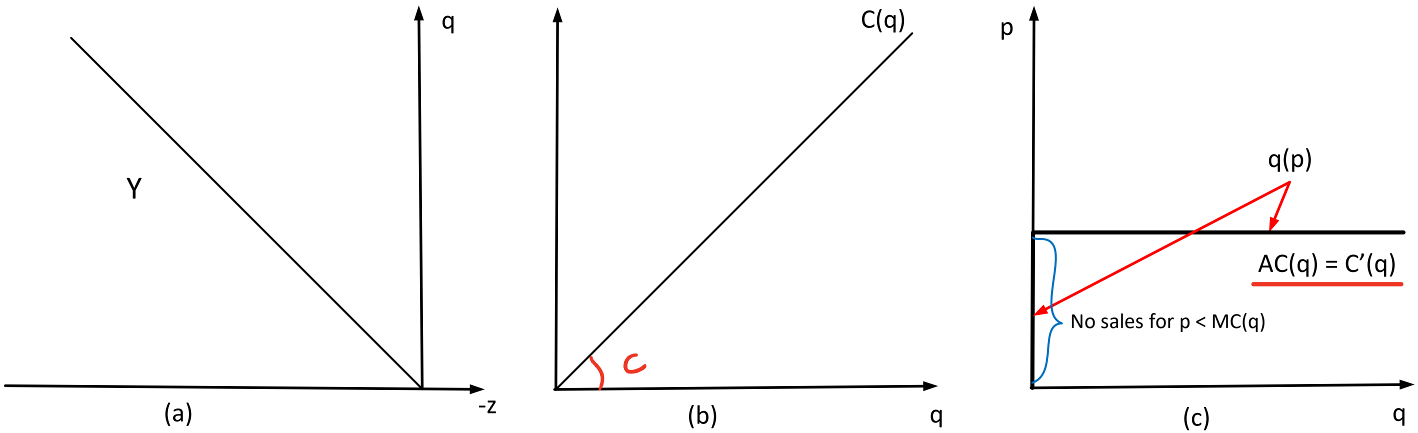


Now we check other cases and examples of Cost functions: this is an example of a cost function that is convex. In particular, this correspond to the case of a production that has decreasing return to scale. So the cost function corresponding to a production function that has decreasing RTC is convex. Then, given the TC we can also draw the MC and AC. MC is the slope of the convex cost function (TC) so the MC will be increasing since the cost function is convex.
We can also compute the AC: it is the slope of the segment connecting each point in the cost unction to the original and in these points the MC is larger than the AC. This happened for all points of the convex cost function: this implies that AC will always lie below the MC function.
If AC increasing the MC must be always larger than AC

Cost and Supply: Single Output

- ***Constant returns to scale***:
  - an increase in input usage produces a proportional increase in output.
    - production set is weakly convex
    - linear TC function
    - constant AC and MC functions

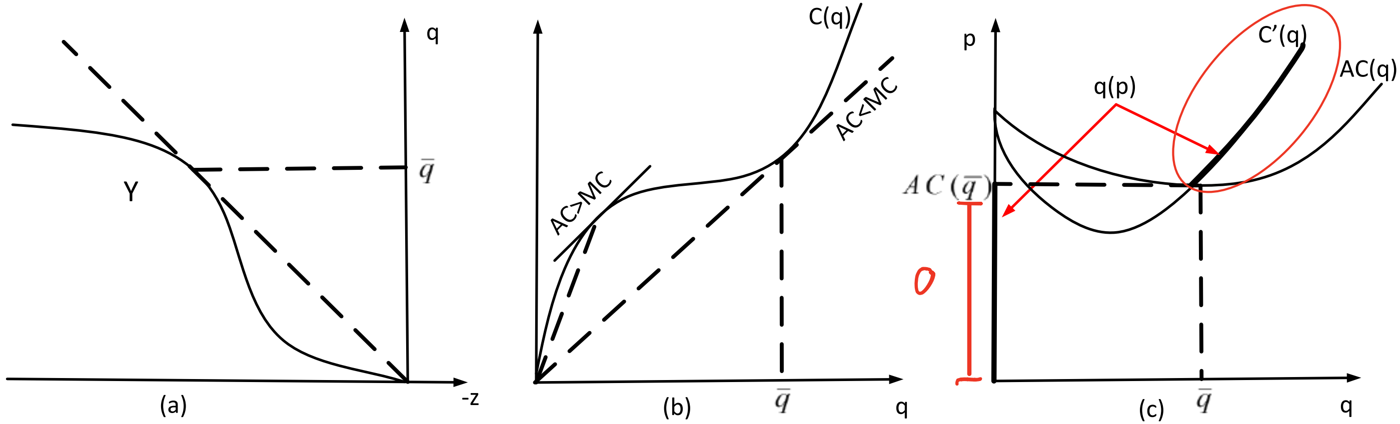
P=MC



This is a case of CRT production function and in this case if you increase all input by the same proportion also output increase by the proportion which implies that the cost function is linear and is a straight line from the origin. In this case the AC and MC are constant and correspond to the slope of the TC.
So AC = MC and they are constant.
Also, is important to notice that if the price is below to the MC we know that the FOC is the price = MC. If prices below to the MC actually the firm supply is 0 quantity.
If prices is below MC the firm do not produce

Cost and Supply: Single Output

- ***Increasing returns to scale***:
  - an increase in input usage can lead to a more-than-proportional increase in output.
    - production set is non-convex
    - TC curve first increases, then becomes almost flat, and then increases rapidly again as output is increased further. P=MC



This is actually the case that we already see in which the TC has complex shape so before the cost function is concave and then became convex. So shape of the MC and AC is the ones that we already seen before.
The only thing to remember is the following: as in the case that we have just seen the supply function is given by the quantity between MC and the price. The only relevant part of the MC curve (which represent the firm supply) for the firm is the portion of the MC which is above (sopra) AC. So this means that for the prices above the MC the firm will not produce. This means that supply curve has spike corresponding to 0.
When price is above the AC, then the relevant portion of the MC function (which represent the firm supply) is 1.
Firm supply curve is the MC curve lieing above the AC curve.

Cost and Supply: Single Output

- Let us analyze the presence of *non-convexities* in the production set Y arising from:
  – Fixed set-up costs, K (is not capital here, are fixed costs…), that are non-sunk

$$C(q) = K + C_v(q)$$

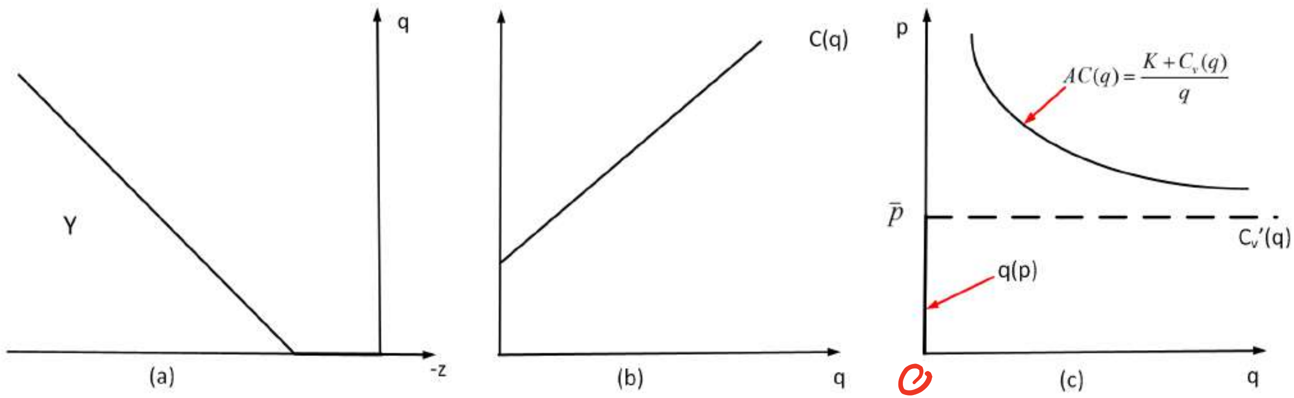
  where $C_v(q)$ denotes variable costs
    - with strictly convex variable costs
    - with linear variable costs
  – Fixed set-up costs that are sunk
    - Cost function is convex, and hence FOCs are sufficient

We may also have cases in which the firm have a cost that are fixed: cost that does not depend on quantity produced. So this is an example of TC in which we have two term (K and Cv(q)) in which one does not depend on quantity (i.e. K) and the second is the variable cost (with v index) which depend on q.
Also, fixed cost can be sunk or not sunk.
Not sunk cost are cost you can recover even when you are not producing. Imaging you buy a licence to produce a given good in a case you produce a 0 quantity which is the case in which you closed the activity, you can sell back your license and recover the value of the licence and in this case are not sunk fixed cost. Imaging the government deleted the mandatory needs for that license, in this case the cost can be sunk.
Sunk cost (costi non recuperabili)

Cost and Supply: Single Output

- ***CRS technology and fixed (non-sunk) costs***:
- Example: $C(q) = K + cq$
- If $q = 0$, then $C(q) = 0$, i.e., firm can recover K if it shuts down its operation.
  - MC is constant: $MC = C'(q) = C_v'(q) = c$
  - AC lies above MC: $AC(q) = \frac{C(q)}{q} = \frac{K}{q} + \frac{C_v(q)}{q} = \frac{K}{q} + c$

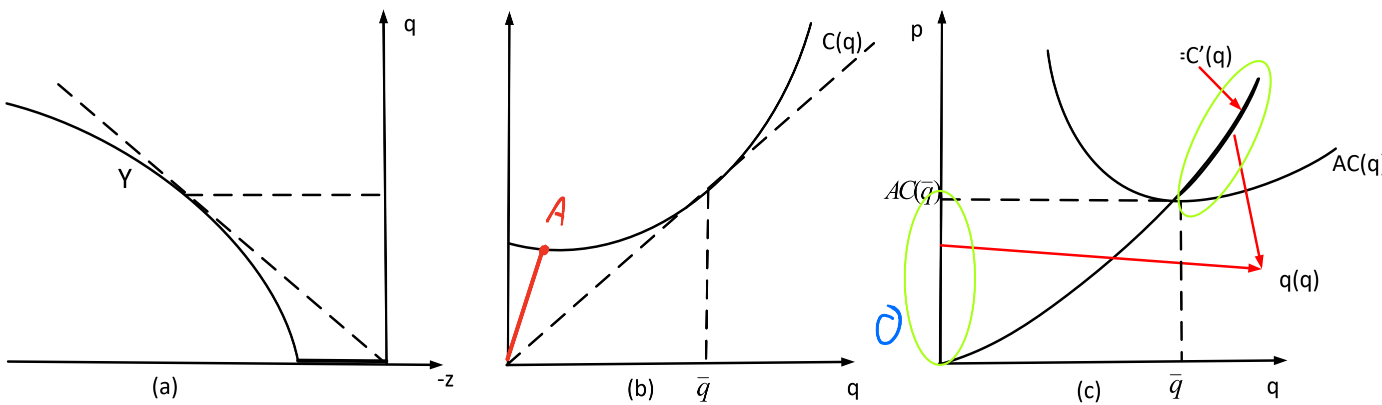
P=MC



So, let's start from the example of a C function including fixed non-sunk cost (K). This is an example in which the fixed cost is K and the variable cost is linear in quantity. So, in this case since the costs are not sunk if the firm produce 0 ($q = 0$) then the cost is 0. While, the MC is constant that is the derivative of TC with respect to q (that is c) but also the der of the variable cost that is the der of cq that is c. der of variable cost are both equal to c because variable cost is linear in q.
To compute AC we have to divide the TC by q which is $k/q + Cv/q$ which is equal to $k/q + c$.
When k goes to infinity (when firm produce a large quantity) the first term to 0 and when q became very large the AC will tend to the MC that is c.
AC is decreasing in q and when q tends to infinity this term became very close to 0 so K / q became very close to 0 and AC tends to c (which is equal to MC).
If the firm supply given by p and MC since AC is always above the MC the firm will never produce so the firm supply will have a spike in correspond to 0.
The idea is that when the price is lower than the AC is not convenient for the firm to produce because for each unit that the firm produce will bring to have: loss = prices – AC.

Cost and Supply: Single Output

- ***DRS technology and fixed (non-sunk) costs***:
  - MC is positive and increasing in q, and hence the slope of the TC curve increases in q.
  - in the decreasing portion of the AC curve, FC is spread over larger q.
  - in the increasing portion of the AC curve, larger average VC offsets the lower average FC and, hence, total average cost increases.



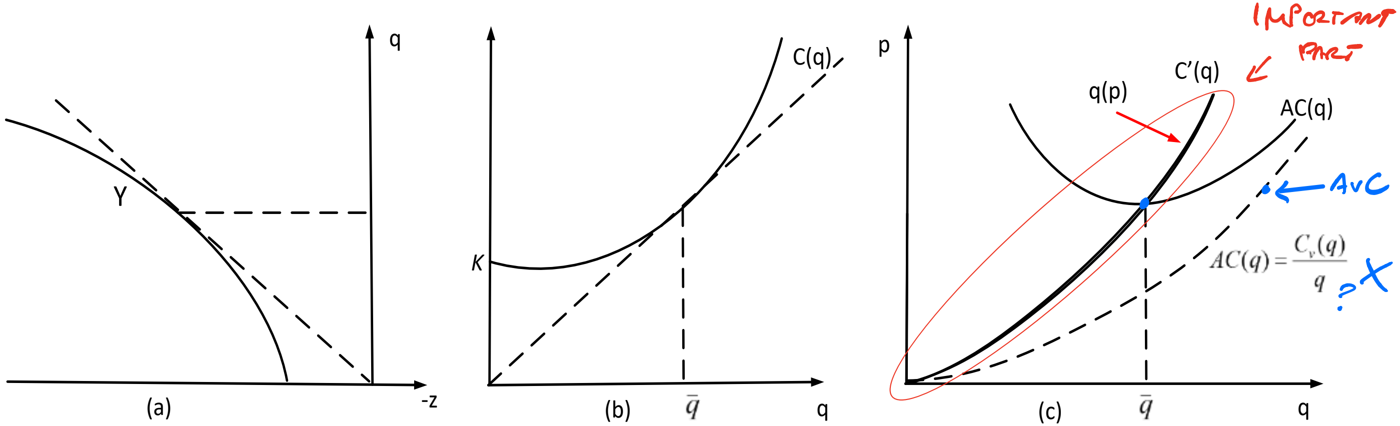
This is another example in which again we have DRS with fixed non-sunk costs. So now the cost function is not linear but is convex which is the case of DRT technology and again we have non sunk cost.
We can apply the same kind of reasoning and we know that if Cost function is convex the MC will be always increasing. However, the MC will cut the AC in its minimum (that is the point in which - - - line crossed).
Again, from supply the relevant bit of the firm supply are the increasing bit for prices larger that minimum of AC and in this bit that is the spike that tends to 0 when price is below the minimum of the AC curve.
[Second graph] You can find the shape of a larger cost just comparing for each point the slope of the AC that is the slope of the segment connecting the point in the TC with the origin and the slope of the TC function in that point that is the MC. In this point the MC is lower than the AC. That is for instance all points to the left of this quantity.

Cost and Supply: Single Output

$AvC = \frac{cq^2}{q} = cq$

$MC = 2cq$

- ***DRS technology and $\boxed{\text{sunk}}$ costs***:
- For instance: $C(q) = K + cq^2$
  - TC curve originates at K, given that the firm must incur fixed sunk cost K even if it chooses $q = 0$.
  - supply locus considers the entire MC curve and not only q for which MC>AC. $P = MC \longrightarrow$ ONLY FOR PORTION WHERE $\underline{MC > AvC}$



Another example in which we have DRS and sunk cost.
We have a convex cost function, but we have fixed cost that are sunk.
The things change a little bit in this case since we have a different firm supply.
The MC will be always increasing because TC is convex and will crosses in the minimum of AC. Now what happens is that if you compute the Average variable cost
AvC = c q^2 / q = c q
MC = 2 cq
You see that the MC is always above the AvC [that is the - - - curve in the 3° graph]
All bit of supply curve are all portions of the MC function.
So, to sums up:

- The firm supply curve is: P = MC —> only for portion in which the MC lies above to the AvC (i.e MC > AvC).

If you go back to the previous example you can check that this condition always holds.

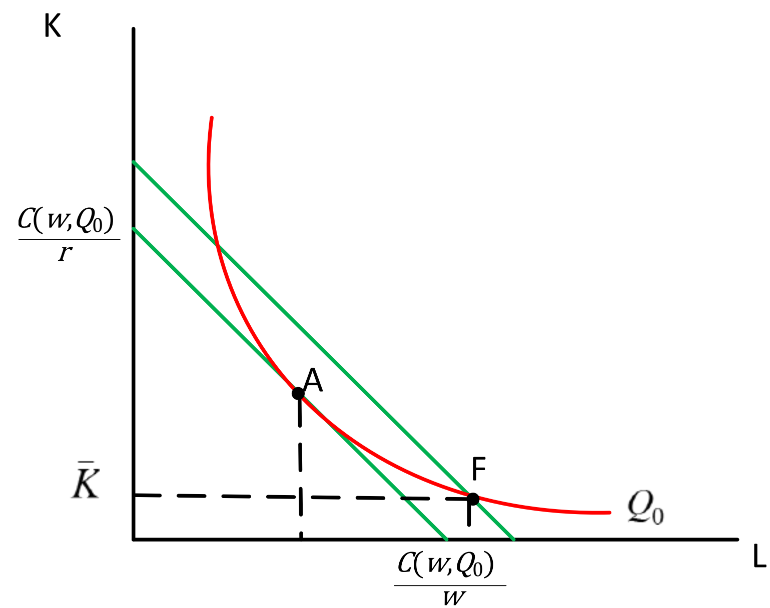
Short-Run Total Cost

- In the short run, the firm generally incurs higher costs than in the long run. In the short run some factors are fixed.
  - The firm does not have the flexibility of choosing all inputs (there are fixed inputs).
  - **To vary its output in the short-run, the firm must use non-optimal input combinations**
  - **The $MRTS$ will not be equal to the ratio of input prices.**

When some inputs fixed this depend on the time horizon we are considering. So, if you consider short time horizon the firm are not able to change the quantity of some inputs and we define this time horizon as short run.
While, the long run as the time horizon in which the firm can change the quantity of all factors and then this means that all factors are variable.

Short-Run vs Long-Run Total Cost

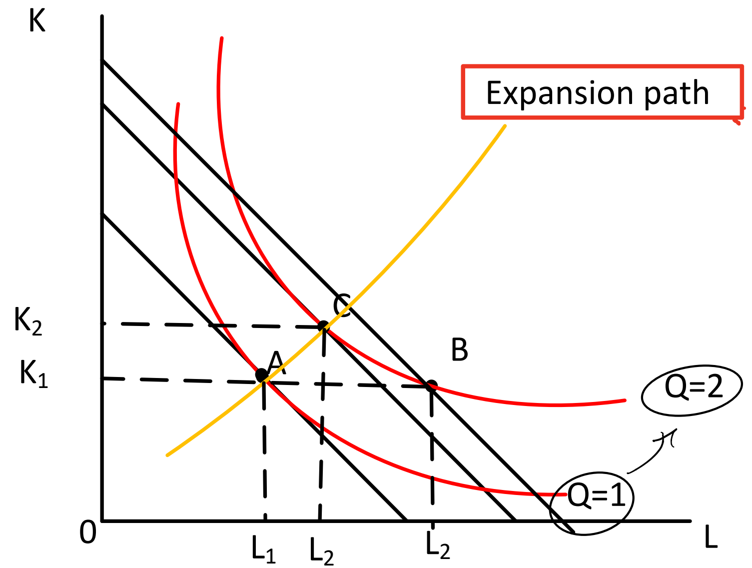
- In the short-run
  - capital is fixed at $\bar{K}$
  - the firm cannot equate $MRTS$ with the ratio of input prices.
- In the long-run
  - Firm can choose input vector A, which is a cost-minimizing input combination.



So a thing to keep in mind is that the long run TC, that Is the cost in which the firm has the freedom to choose the Optimal quantity of all factors is always equal or lower than the short run TC.

Short-Run vs Long-Run Total Cost

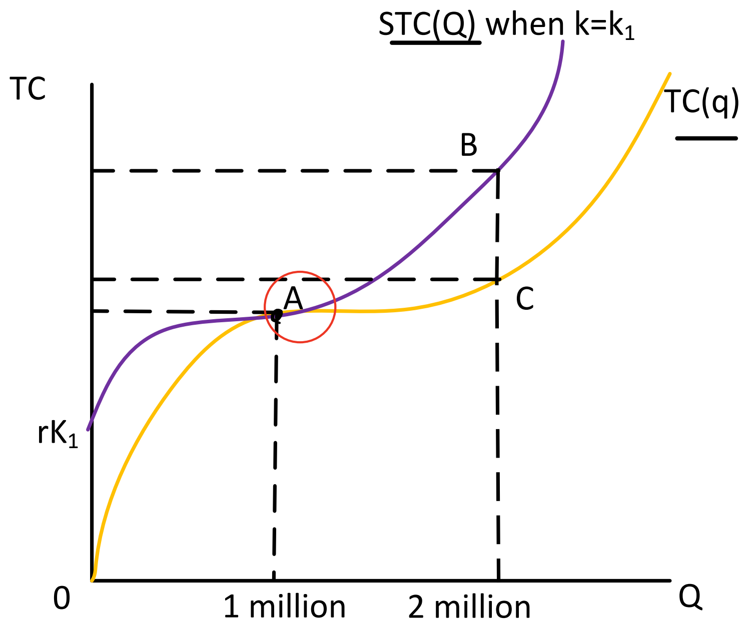
- $q = 1$ million units
  - Firm chooses (k_1, l_1) both in the long run and in the short run when $k = k_1$.
- $q = 2$ million units
  - Short-run (point B):
    - $k = k_1$ does not allow the firm to minimize costs.
  - Long-run (point C):
    - firm can choose cost-minimizing input combination.



Consider the following situation: the firm has to solve the CMP so we have different level of quantities, the optimal condition in the case of inferior solution is the tangency point between the isoquant and isocost. So A and C will be the optimal solution in the long run and the lie in the expansion path. Imaging that now the short run cannot change the Optimal quantity of capital, but at the same time the firm want to increase the production from Q = 1 to Q = 2. In this case the firm cannot choose the Optimal combination C as to choose the Optimal short run combination that is in the point B. The combination in point B lies on an isocost that is above the isocost where C is located.
So this means that when the firm can choose the Optimal quantity of all factors, cost will be generally lower in respect in which the firm is constraint in some factors(this happen in the short run that can be 1 year for example). Labour is a variable cost in the short run (you can hire new workers) but if you have to set up a new building this will takes sometimes.

Short-Run vs Long-Run Total Cost

- The difference between long-run, $TC(q)$, and short-run, $STC(q)$, total costs when capital is fixed at $k = k_1$.
- (the two curves are tangent when k_1 is the optimal demand for capital to produce Q=1 million)

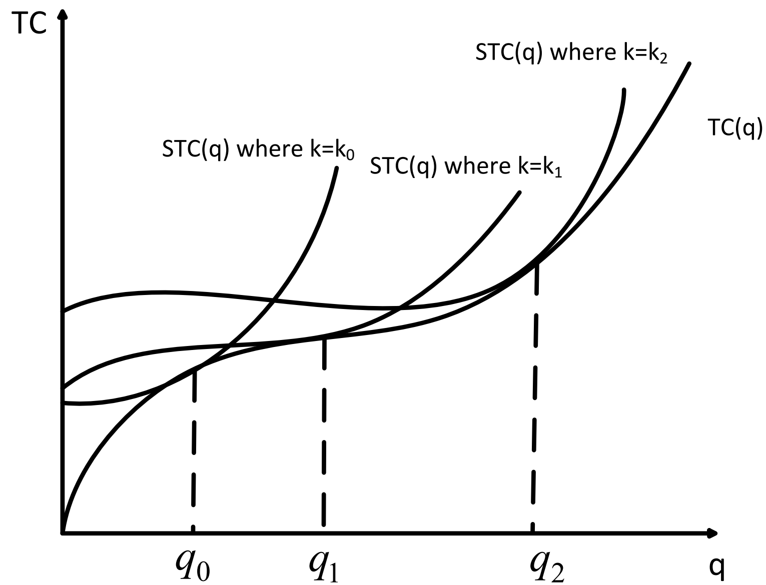


This can be also represented in a graphic in which we have short run total cost (STC) and long run total cost (TC).
When one of the factors is constraint (like K is constrain to the quantity).
STC lies above the long-run TC and they touch in one point (point A) and there is the point in which the optimal input quantity to produce one million is k1. It happens that the amount of k in which you are constraint is the Optimal amount that is required to produce one million also in the long run. In point A long-run TC and STC are the same.

Short-Run vs Long-Run Total Cost

- The long-run total cost curve $TC(q)$ can be derived by varying the level of k.

- Short-run total cost curves $STC(q)$ lies above long-run total cost $TC(q)$.



The fact that STC is always above the long run and they touch in one point.

Short-Run vs Long-Run Total Cost

- Summary:
  - In the long run, the firm can modify the values of all inputs.
  - In the short run, in contrast, the firm can only modify some inputs (e.g., labor, but not capital), i.e. the **variable inputs**.

Short-Run vs Long-Run Total Cost

$\bar{z}_2$ = FIXED QUANTITY

- ***Example***: Short- and long-run curves
  - In the long run,
  $$C(q) = \bar{w}_1 z_1 + \bar{w}_2 z_2$$
  where both input 1 and 2 are variable.
  - In the short run, input 2 is fixed at $\bar{z}_2$, and thus
  $$C(q|\underline{\bar{z}_2}) = \bar{w}_1 z_1 + \bar{w}_2 \bar{z}_2$$
  FIXED COST
    - This implies that the only input that the firm can modify is input 1.
    - The firm chooses z_1 such that production reaches output level q, i.e., $f(z_1, \bar{z}_2) = q$.

The long run cost function is the lower envelope of the STC.
If we want to define the STC we can start from the long run in which inputs are variable while w1 and w2 are the cost of the two inputs that are given (w barrate).
In the short run we are constrain in a given quantity of the two factors (so one is fixed) like z2. We define a cost function conditional on Z2 = _Z2 (_ è barrato) so w2 _z2 became a fixed cost since doesn’t not vary on the quantity. Only choice variable is z1 for this problem

Short-Run vs Long-Run Total Cost

- ***Example*** (continued):
  - When the demand for input 2 is at its *long-run value*, i.e., $z_2(w,q)$, then
$$\boxed{C(q) = C(q|z_2(w,q))} \text{ for every } q$$
and also
$$\boxed{C'(q) = C'(q|z_2(w,q))} \text{ for every } q \text{ (*)}$$
i.e., values and slopes of long- and short-run cost functions coincide.
  - Long- and short-run curves are tangent (*) at $z_2(w,q)$.

If the value of z2 to which our constraint is the long run optimal value to produce the quantity q then we will have this equality which mean that the long rung cost function and short cost function are equal.
We can compute derivative with respect to q and the two things must holds. So, slope of the long-run CT is equal to the slope of the short-run CT. Long and short run cost curve are tangent at the quantity z2 when z2 is the optimal long run input demand for producing the quantity q.

Short-Run vs Long-Run Total Cost

- ***Example*** (continued):
  – Since

$$C(q) \le C(q|z_2) \text{ for any given } z_2,$$

then the long-run cost curve $C(q)$ **is the *lower envelope*** of the short-run cost curves, $C(q|z_2)$.

By gathering these two conditions that short cost run is higher than the long run expect for being equal to the latter in one point. We define the long run cost as the lower envelope of the short run curve.

Aggregation in Production

The aggregation in production and 1° and 2° fundamental theorem of welfare economics.
As to aggregating production he will explain in a different way to the book.
Imagine we have J firm in the economy. From 1 to J.

J FIRMS $\quad j = 1 \dots J$

$$\text{PMP} \quad \max_{q_j} \Pi_j = p \cdot q_j - c(q_j)$$

For each of this firm we have seen that PMP imply maximise profit with respect to the quantity produced by the firm. Profit change are equal to Prices multiply by qj minus the cost function that depend on the quantity qj. -- c(qj).

FOC the derivative of the profit with respect to qj = p – der C /qj = 0 so P = MC

$$\text{FOC} \quad \frac{\partial \Pi_j}{\partial q_j} = p - \underbrace{\boxed{\frac{\partial c(q_j)}{\partial q_j}}}_{MC} = 0 \Rightarrow p = MC$$

Social plan: can be the government that want to maximise total profits that is the overall profit in the economy. This mean the profit function became the summation all firm profits and can be written as p q1 – c(q1) + pq2 – c(q2) … pqj – c(qj)

$$\text{SOCIAL PLANNER} \quad \max_{q_1, q_2 \dots q_j} \sum_j \Pi_j = p q_1 - c(q_1) + p q_2 - c(q_2) \dots p q_j - c(q_j)$$

If we solve the problem of social planner, is easily to check if we have j FOC of the following form:
P – der C(q1)/ der q1 = 0 … p – der C(qj) / der qj = 0

We will see the very same FOC for the individual firm profit maximisation problem. Is like you can decompose the social planner problem in a single maximization problem of the individual firm.

$$p - \frac{\partial c(q_1)}{\partial q_1} = 0 \quad \dots \quad p - \frac{\partial c(q_S)}{\partial q_S} = 0$$

Since this condition are the same, this mean that the optimal quantity that max profit will be the same when the individual firm decides and when the social planner decides.

This is an interesting result: it implies that if firm decides qj to maximise their profits. So if we let firm to the decides to max their profits will also imply maximisation of total profit in the economy.

This is so called ***decentralisation result***

IF FIRMS DECIDE $q_S \rightarrow$ MAX Π_S

$\Downarrow$

WILL IMPLY MAXIMISATION OF TOTAL PROFITS IN THE ECONOMY

$\Downarrow$ CALLED

DECENTRALIZATION RESULT

Firm decides independently to max the profit we will obtain the same result of social planner that decided what firm produce to maximise not individual firm profit but total profits.

To obtain the aggregate supply Y we will have to sum the individual firm supply that depends on the input and output prices

$$Y(p) = \sum_{s} y_s(p)$$

Important notice is that aggregate supply depends on prices.
We have seen that if p increases the firm supply increases and this will also imply that if p increase also the aggregate supply increase. Obvious because if each of this firm supply increases then also the summation of the all individual supply will increases.
So Law of supply holds also for aggregate supply: If prices increases the supply increase.

$p \uparrow \quad y_s(p) \uparrow$

$p \uparrow \quad Y(p) \uparrow$

LAW OF SUPPLY HOLDS ALSO FOR AGGREGATE SUPPLY

Aggregation in Production

- Let us analyze under which conditions the "law of supply" holds at the aggregate level (i.e. if p increases firm supply increases)
- **An aggregate production function maps aggregate inputs into aggregate outputs**
  - In other words, it describes the **maximum level of output that can be obtained if the inputs are efficiently used in the production process.**

Aggregation in Production

- Consider J firms, with production sets $Y_1, Y_2, \dots, Y_J$.
- Each Y_j is non-empty, closed, and satisfies the free disposal property.
- Assume also that every supply correspondence $y_j(p)$ is single valued, and differentiable in prices, $p \gg 0$.
- Define the **aggregate supply correspondence as the sum of the individual supply correspondences**

$$y(p) = \sum_{j=1}^{J} y_j(p) \; = \; \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j(p) \right\}$$

where $y_j \in y_j(p)$ for $j = 1,2, \dots, J$.

Aggregation in Production

- The ***law of supply*** is satisfied at the aggregate level.
- Two ways to check it:
  1) Using the derivative of every firm's supply correspondence with respect to prices, $D_p y_j(p)$.
     – $D_p y_j(p)$ is a symmetric positive semidefinite matrix, for every firm j.
     – Since this property is preserved under addition, then $D_p y(p)$ must also define a symmetric positive semidefinite matrix.

Aggregation in Production

2) Using a revealed preference argument.
   - For every firm j,
     $$[p - p'] \cdot \left[y_j(p) - y_j(p')\right] \geq 0$$
   - Adding over j,
     $$[p - p'] \cdot [y(p) - y(p')] \geq 0$$
   - This implies that market prices and aggregate supply move in the same direction
     - the law of supply holds at the aggregate level!

Aggregation in Production

- Is there a "***representative producer***"?
  - Let Y be the aggregate production set,
  $$Y = Y_1 + Y_2 + \ldots + Y_j = \left\{ y \in \mathbb{R}^L \colon\ y = \sum_{j=1}^{J} y_j \right\}$$
  for some $y_j \in Y_j$ and $j = 1, 2, \ldots, J$.
  - Note that $y = \sum_{j=1}^{J} y_j$, where every y_j is just a feasible production plan of firm j, but not necessarily firm j's supply correspondence $y_j(p)$.
  - Let $\pi^*(p)$ be the profit function for the aggregate production set Y.
  - Let $y^*(p)$ be the supply correspondence for the aggregate production set Y.

Aggregation in Production

- Is there a "***representative producer***"?
  - Then, there exists a representative producer:
    - Producing an aggregate supply $y^*(p)$ that exactly coincides with the sum $\sum_{j=1}^{J} y_j\,(p)$; and
    - Obtaining aggregate profits $\pi^*(p)$ that exactly coincide with the sum $\sum_{j=1}^{J} \pi_j\,(p)$.
  - *Intuition*: The aggregate profit obtained by each firm maximizing its profits separately (taking prices as given) is the same as that which would be obtained if all firms were to coordinate their actions (i.e., y_j's) in a joint PMP (decentralization result)

Aggregation in Production

- Is there a "***representative producer***"?
  - It is a "decentralization" result: to find the solution of the joint PMP for given prices p, it is enough to "let each individual firm maximize its own profits" and add the solutions of their individual PMPs.
  - *Key*: price taking assumption
    - **This result does not hold if firms have market power**.
    - Example: oligopoly markets where firms compete in quantities (a la Cournot).

First FTWE

- **First Fundamental Theorem of Welfare Economics:**

  If a production plan $y \in Y$ is profit maximising for a price vector $p \gg 0$, then y must be efficient.

A production plan y is efficient when there is no other feasible production plan y' producing more output with the same amount of inputs (or producing the same output with less inputs)

The other important things are: First fundamental theorem of welfare economics (1° FTWE) and second fundamental theorem of welfare economics (2° FTWE).
The FTWE: if a production plan y that belongs to the production set is profit maximising for a price vector p, then y must be efficient.
We have to define first what is a efficient production plan:
we have seen that the production plan in efficient if there is not other feasible production plan y' which allows the firm to produce more output with the same amount of inputs.

First FTWE

- Proof (by contradiction). Suppose $y \in Y$ is profic maximizing, i.e. $py \geq py'$ for any other $y' \in Y$, different from y, **but that it is not efficient**. Then there must be a production plan $y' \in Y$ such that $y' \geq y$ (i.e. it allows to produce more). However, multiplying both sides by p, we get $py' \geq py$. Then y cannot be profit maximising. We reached a contradiction. So y must be efficient.
- Hint: Remember $py = p_q q - \mathrm{w}_1 \mathrm{z}_1 - \mathrm{w}_2 \mathrm{z}_2 - \cdots$
- Where q is the output

This can be proved by contradiction.
Suppose we have production plan y that is profit maximise (product between p and y must be greater or equal to (any othe production plan y') p*y'. p * y are profits: p is the vector including the prices of the goods but also the negative inputs prices. Not only p*q but also include w-1, w-2 up to the price of the last input. If you take the product between this two vector we will obtain the profit. We assume that this vector is not efficient. If not efficient there must be another production plan y' such this production plan allows to produce more of the same of y.
We can multiply by vector p in both side and we got py' >= py. So we obtain a contradiction with respect to the initial condition that stated that py >= py'. Since we reached the contradiction this must be the case of production y is efficient.

Second FTWE

- Second Fundamental Theorem of Welfare Economics:

  If a production set Y *is convex, then every efficient production plan plan* $y \in Y$ *is a profit-maximising production plan, for some non-zero price vector* $p \geq 0$.

(Proof in the book, not necessary)

Another theorem FTWE we introduce the 2° FTWE: if the production set y is convex then every efficient production plan y is a profit-maximising production plan for some non zero price vector(p >=0). In this case price vector can also contains some zero, but the important thing is that not all prices can be equal to 0.

NO PROOF for this 😊

UNIVERSITÀ DEGLI STUDI DI MILANO

Advanced Microeconomic Theory

Chapter 6: Partial Equilibrium

To note:

1) Firm supply in the short run: Marginal cost curve above the Average Variable Costs (AVC) curve (the reason is that as long as $P>AVC$, by producing you will lower the firm losses even if $P<ATC$, indeed $profits=(P-ATC)q$, but $ATC=AVC+AFC$, and $AFC*q=FC$, where AFC are average fixed costs, so $profits=(P-AVC)q-FC$ and if $P>AVC$ then $(P-AVC)q>0$. Thus by producing $q>0$ you will cut a bit losses that are maximum when $q=0$ since $profits=-FC$)

2) Firm supply in the long run: In the long run, we have no FC, so $AVC=ATC$ and we can only consider ATC. While in the short run there might be positive profits, in the long run firm entry will shift aggregate supply to the right up to the point in which the equilibrium price becomes equal to the minimum of the ATC, so $P^*=Min(ATC)$. The long-run firm supply is infinitely elastic (i.e. horizontal) at the price level $P=Min(ATC)$. Note that in Perfect Competition there are no barriers to firm entry.

3) Demand for the individual firm is infinitely elastic, i.e. horizontal at the market equilibrium price. Intuition, the firm can sell any amount at the market equilibrium price.

Outline

- Partial Equilibrium Analysis
- Comparative Statics
- Welfare Analysis

Partial Equilibrium Analysis

- In a competitive equilibrium (CE), all agents must select an optimal allocation given their resources:
  - Firms choose profit-maximizing production plans given their technology;
  - Consumers choose utility-maximizing bundles given their budget constraint.
- A competitive equilibrium allocation will emerge at a price that makes consumers' purchasing plans coincide with the firms' production decision.

First of all we have to define what partial equilibrium is:
In a competitive equilibrium (CE) since we are considering perfect competition all agent must select an optimum allocation given their results.
We have seen firms solving PMP so selection profit maximise the production plan with constrain given by available techlogies while consumer choose utilty maximising bundle of goods given their budget constrain.
In this situation CE is an allocation of goods amount consumers and producers making consumer purchasing plains coinciding with firms production decision.
So this mean that the amount produce buy firms must be exactly equal to the amount consumer intend to buy.

Start with Firms problem

Partial Equilibrium Analysis

- ***Firm***:
  - Given the price vector p^*, firm j's equilibrium output level q_j^* must solve

$$\max_{q_j \geq 0} \quad p^* q_j - c_j(q_j)$$

    which yields the FOC:

$$p^* \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$

  - That is, every firm j produces until the point in which its marginal cost, $c_j'(q_j^*)$, coincides with the current market price.

Start with Firms problem Given price vector p*, which is the competitive equilibrium. Firms have to maximise their profit so we write profit function as usual: is given by total revenue - total cost.
Since we are considering the individual firm, all quantity are indexed by j and also the Cost indexed by j since firms may have different cost functions.
Quantity cannot be negative.
At the optimum the price must be smaller or equal that the marginal cost. This inequality holds with equal sign if the solution is an interior solution (so quantity is positive).
At the opt firms produce at the point price is equal to the marginal cost.
Price we are considering is the price of the equilibrium.

Partial Equilibrium Analysis

- ***Consumers***:
  - Consider a quasilinear utility function

$$u_i(m_i, x_i) = m_i + \phi_i(x_i)$$

    where m_i denotes the numeraire (i.e. all income spent in other goods, except x_i), and $\phi_i'(x_i) > 0$, $\phi_i''(x_i) < 0$ for all $x_i > 0$ (ϕ_i increasing and concave)

  - Normalizing, $\phi_i(0) = 0$. Recall that with quasilinear utility functions, the wealth effects for all non-numeraire commodities are zero.

Now we consider a consumer optimisation maximisation problem and we take as example the quasi linear function.
Utility of individual i depends on the mi (amount of income spent on all the other goods expect for good x) and also utility depends on good x.

Function is quasi linear because is linear with one of the two goods but in this case is linear in the m good that are also define as the numeraire because are define in term of income and not unit of goods that are consumed by the individual excluding the good xi.
We also assume for convenience that the function FI is increasing (1° der positive) and concave (2° der is negative) for all values of Xi. Also we assume that when xi = 0 the utilty that comes from Xi is equal to 0.

Numeraire: in mathematical economics it is a tradable economic entity in terms of whose price the relative prices of all other tradables are expressed

Partial Equilibrium Analysis

– Consumer i's UMP is

$$\max_{m_i \in \mathbb{R}_+,\, x_i \in \mathbb{R}_+} \quad m_i + \phi_i(x_i)$$

$$\text{s.t.} \quad \underbrace{m_i + p^* x_i}_{\text{Total expend.}} \leq \underbrace{w_i + \textstyle\sum_{j=1}^{J} \theta_{ij} (\underbrace{p^* q_j^* - c_j(q_j^*)}_{\text{Profits}})}_{\text{Total resources (endowment+profits)}}$$

– θ_{ij} share of firm *j* owned by consumer *i*. $\sum_i^I \theta_{ij} = 1$.

– The budget constraint must hold with equality (by Walras' law). Hence,

$$m_i = -p^* x_i + \left[w_i + \sum_{j=1}^{J} \theta_{ij} \left(p^* q_j^* - c_j(q_j^*) \right) \right]$$

NB. I skipped the labor supply model, but we are assuming that the individual devotes all her time endowment (normalized to one) to working in the market (so as labor income is $w_i * 1$).

Given this utility function we can set up the consumer utility maximisation problem so the consumer has to choice the optimal value of the inputs mi and xi and maximising the value of utility subjected to the budget constrain.
So by considering the Partial equilibrium analysis the Budget constraint is considered in another way: On the left hand side we have the total expenditure that is equal to mi: that is income spent for all the other good expect xi + total expenditure that the consumer will be able to by the goods xi (price of xi * quantity).
On the right hand side we have income defined by wi + profits. We know this are profit since we have the difference between total revenue and Total costs for firm j. While thetaij can be interpret as the share of firm j posses by the consumer i. We assume that workers get income from too many resources from labour but also from firms. So we assume that they own some fraction of the firms (is like holding some stock of the firms).
If we sum all share across individual: share of firm j possesses by all I consumer they have to sum to 1.
By walras law we know that at the optimal the budget constrain will hold with equality and we can rewrite budged constrain with equal sign and if we do that we can isolate mi in the left hand side.
Mi = total revenue + income of the consumer
So after having done this we can replace mi in the utility function.
Mi in the utility function we get problem in only one variable that is xi.

Partial Equilibrium Analysis

– Substituting the budget constraint into the objective function,

$$\max_{x_i \in \mathbb{R}_+} \quad \phi_i(x_i) - p^* x_i + \left[w_i + \textstyle\sum_{j=1}^{J} \theta_{ij}\left(p^* q_j^* - c_j(q_j^*)\right)\right]$$

– FOCs wrt x_i yields

$$\phi_i'(x_i^*) \leq p^*, \text{ with equality if } x_i^* > 0$$

– That is, consumer increases the amount she buys of good x until the point in which the marginal utility she obtains exactly coincides with the market price she has to pay for it.

So this is the new utilty function after replacing mi, so the Problem of the consumer became to maximise this utilty function with respect to the xi. If we compute of FOC we have to compute derivative of this new utilty function with respect to xi so we obtain FI * XI <= p* and must hold with equality in type case of interior solution (X*i >0).

In this case if we have an interior solution we have the price equal to the marginal utility.

Partial Equilibrium Analysis

– Hence, an allocation $(x_1^*, x_2^*, \dots, x_I^*, q_1^*, q_2^*, \dots, q_J^*)$ and a price vector $p^* \in \mathbb{R}^L$ constitute a CE if:

$$p^* \leq c_j'(q_j^*), \text{ with equality if } q_j^* > 0$$

$$\phi_i'(x_i^*) \leq p^*, \text{ with equality if } x_i^* > 0$$

$$\textstyle\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^* \text{ (market clearing)}$$

– Note that the these conditions do not depend upon the consumer's initial endowments w_i (implication of quasi-linear utility).

Given the solution of the firm maximisation problem and consumer utilty maximisation problem we find an allocation of goods which is a vector whose components are consumption of good x for the I consumer and quantity produce by the J firms and also an allocation not only contain quantity but also price vector for all goods.
This allocation consists of **competitive equilibrium** if hold:

- the FOC for the consumer
- the FOC for the firms
- and the ***Market clearing condition hold.***

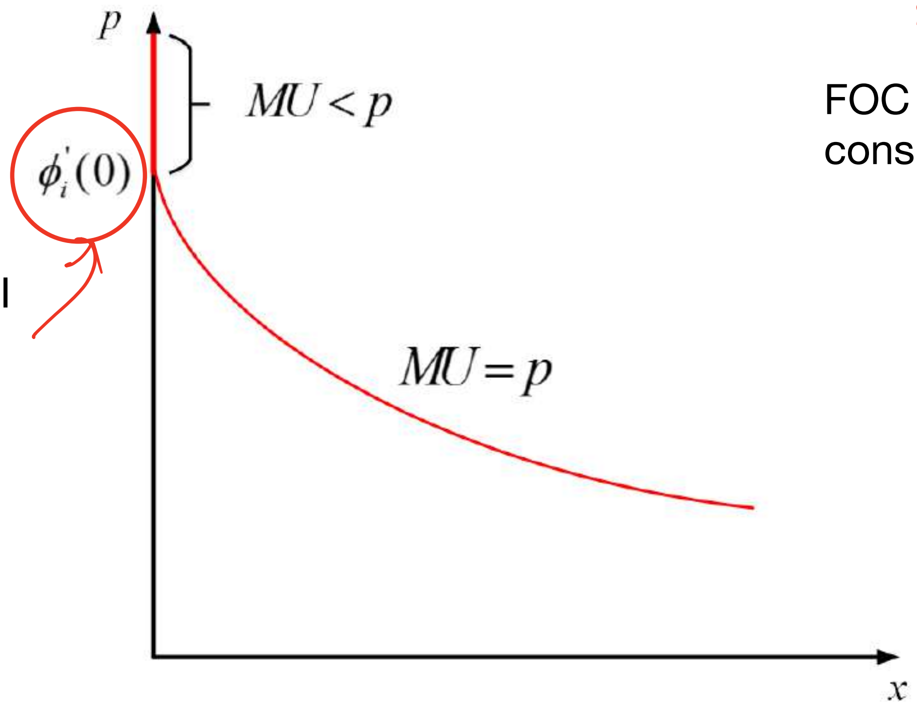
This condition states that the Total quantity demanded by the consumer must be equal to the total quantity produced by firms (in equilibrium)

An implication of the utility function that we have choice (quasi linear) the consumer initial endowments wi doesn't enter in this condition.

Partial Equilibrium Analysis

- The individual demand curve, where $\boxed{\phi_i'(x_i^*) \leq p^*}$

FOC for the consumer



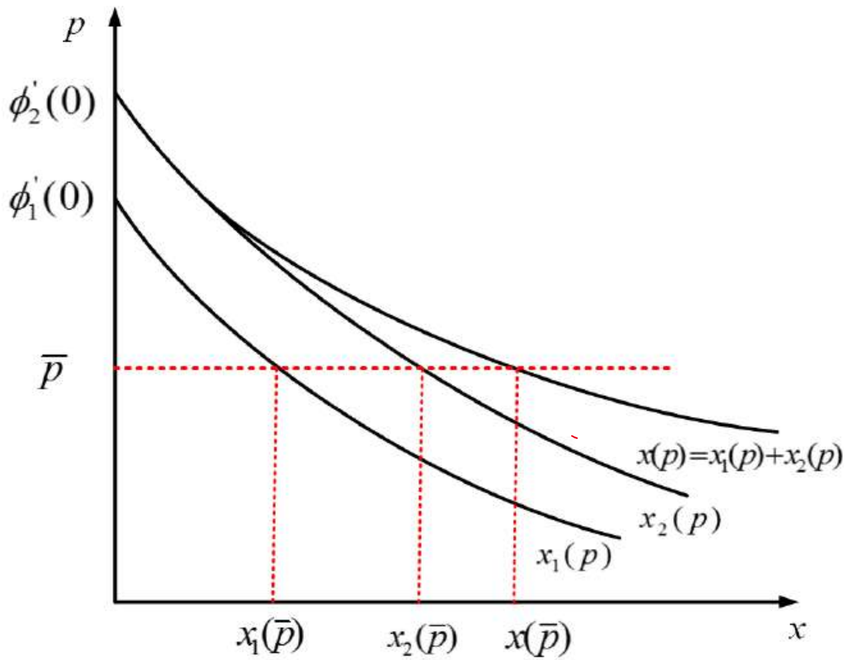
Marginal level of zero unit

So the next step is to draw the individual demand curve: this is defined by the FOC for the consumer.
If the price is larger Than the Mu given by the 0 unit of the goods this means that even for unit the Marginal utilty is Lower Than the price and this mean that the consumer is not consuming any unit of good xi.
Below the level defined by the marginal level of zero unit, then the consumer will start to buy unit of good xi and in particular since margin utility is decreasing and price increases the guy will consumer a Lower quantity of the good.
This individual demand is decreasing.

Partial Equilibrium Analysis

- **Horizontally summing** individual demand curves yields the aggregate demand curve.

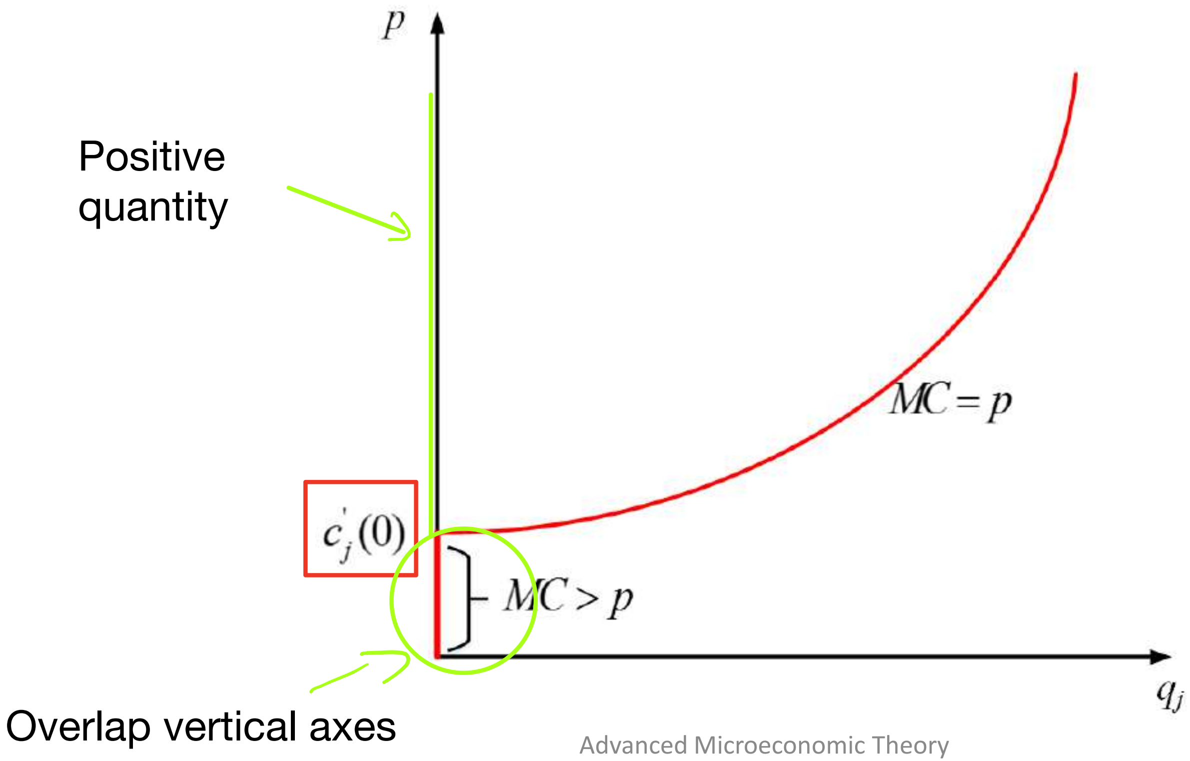
Interval



Given the demand of different consumer we can also compute and dream the total demand curve in the market: done with horizontally summed demand curve of the individual consumers.
In this case we have different demand curve.
Horizontally sum the demand curve:
If the price is between the two FI values, you see that only the individual with high demand will demand good xi. So this mean that for prices in this interval the total demand will be only the demand of the high demand consumer.
For lower prices, we have to horizontally sum the two: so the demand will be the sum of the demand of the first consumer + the demand of the second consumer.
So total demand in the market for price p = x1(p) + x2 (p).
So we are summing optimal demand of the consumers.

Partial Equilibrium Analysis

- The firm individual supply curve, where $\boxed{p^* \leq c_j'(q_j^*)}$

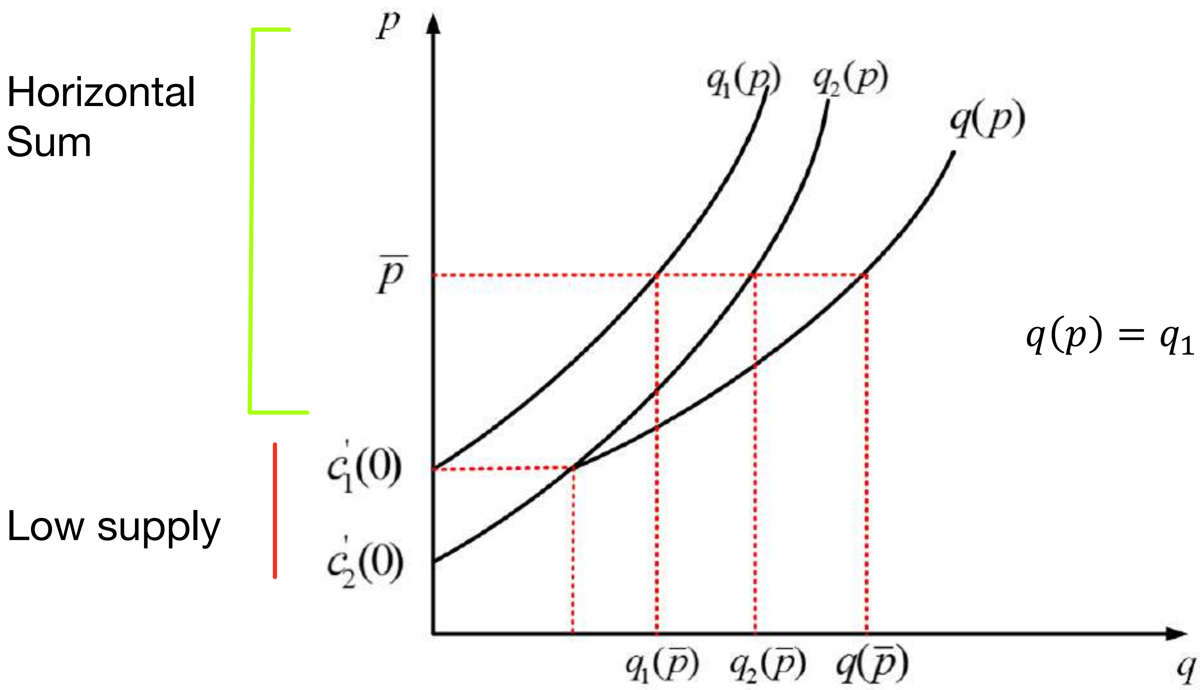


You can do something similar also for firms. In this case the FOC states that the price must be smaller or equal to the marginal costs. O you have to draw the individual this supply curve.
If the price is below the MC of the zero units of the good, then the firm will not produce any unit of the good: this mean that individual supply curve overlap the vertical axes. For prices in this interval the firm will supply null quantity.

If the price increases (more than zero) the firm will supply positive quantity and FOC is defined by inequality by the price and the MC.
Since the MC is increasing this also implies that the supply curve is increasing.

Partial Equilibrium Analysis

- **Horizontally summing** individual supply curves yields the aggregate supply curve.



$$q(p) = q_1(p) + q_2(p)$$

We can also apply the same to supply of the things we did for demand.
So we can obtain the firm supply simply by horizontally summing the supply of individuals firms.
Also in this case we have two curves and we have high and low supply.
Prices in the region, supply will be the low supply, for price above the region we have to horizontally sums the supply of the two firms.
By horizontally summing individual demand curves we can obtain aggregate market demand curve and we can do the same for the individual supply curve and by horizontally summing we can obtain the total market supply.

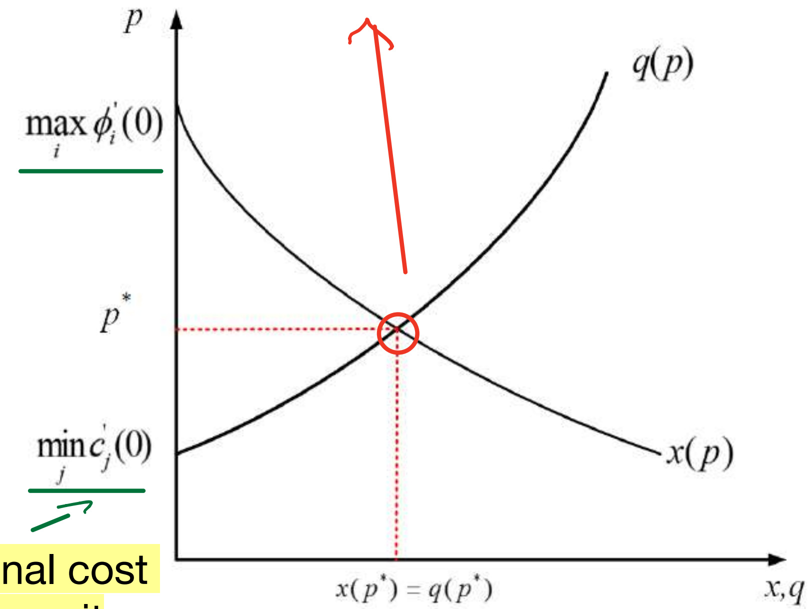
Partial Equilibrium Analysis

- Superimposing aggregate demand and aggregate supply curves, we obtain the CE allocation of good x.

- To guarantee that a CE exists, the equilibrium price p^* must satisfy

$$\boxed{\max_i \ \phi_i'(0) \geq p^* \geq \min_j \ c_j'(0)}$$

Equilibrium



Marginal cost at zero unit

i.e. the aggregate supply starts below the aggregate demand

After having obtained the two aggregate market supply and demand curve we can draw it in a graph. Vertical axes we have price and in the horizontal we have the quantities of the good.
We have supply curve positive sloped and consumer demand curve negatively sloped.
The equilibrium is given by the crossing point of the two curves.
As to the intercept of the aggregate supply curve this is given by the minimum marginal cost of the firm supplying that good: minimal marginal cost is computed at zero units.
The same for the aggregate demand curve: vertical intercept is given by the maximum of the marginal utility of the individual consumer when they are compute in 0.

$\phi_2'(0)$

So this intercept are in practise the FI intercept of the consumer.

$\phi_1'(0)$

The same for the aggregate supply in which is the minim marginal cost compute at zero unit.

In order to have a competitive equilibrium it must be the case that the intercept of the supply curve is below the intercept of the demand curve. This condition is stating this: **the max marginal utility consumer computed in 0 must be greater or equal to the minimum of the marginal cost computed in 0 for the firm j.**
In this case the crossing point will be between vertical intercept of demand and vertical intercept of aggregate supply.

If this condition is not met we have a something similar to the next graph

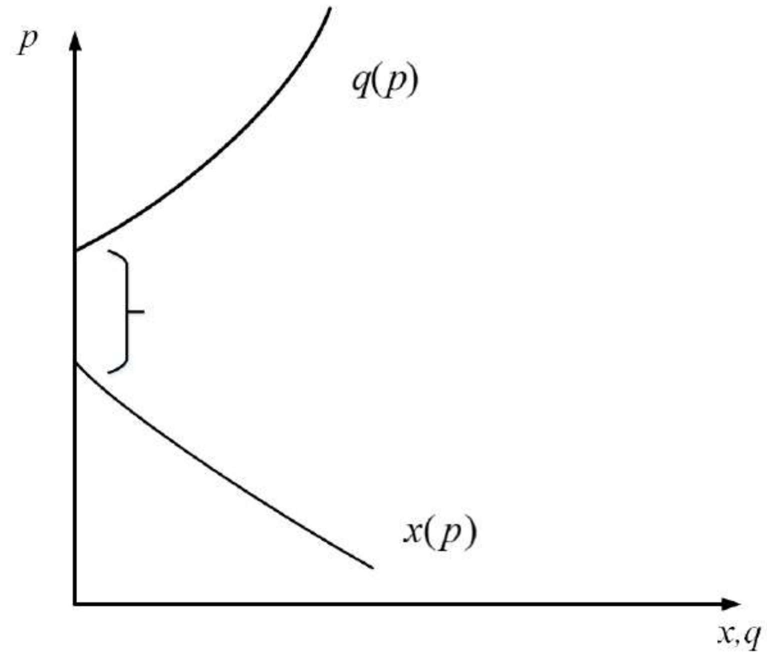
Partial Equilibrium Analysis

- If we have

$$\max_i \phi_i'(0) < \min_j c_j'(0),$$

Then aggregate supply starts above aggregate demand

and there is *no* positive production or consumption of good x representing a CE.



supply curve starts above the demand curve so there is no crossing point between the two curve and this mean there is not competitive equilibrium or CE does not exist.

Partial Equilibrium Analysis

- Also, since $\phi_i'(x_i)$ is downward sloping in x_i, and $c_j'(q_i)$ is upward sloping in q_i, then aggregate demand and supply cross at a unique point.
  - Hence, the **CE allocation is unique**.

Also is important to notice that since the supply curve is positively sloped and demand curve is negatively sloped : which is saying that FI' is downward sloping in xi and c' is upward sloping in qi, the aggregate demand and supply cross at a unique point:

CE allocation is unique

Partial Equilibrium Analysis

- ***Example 6.1***:
  - Assume a perfectly competitive industry consisting of two types of firms: 100 firms of type *A* and 30 firms of type *B.*
  - Short-run supply curve of type *A* firm is
$$s_A(p) = 2p$$
  - Short-run supply curve of type *B* firm is
$$s_B(p) = 10p$$
  - The Walrasian market demand curve is
$$x(p) = 5000 - 500p$$

We make an example on how we can find the competitive equilibrium in case of Perfect competition (since we have competitive equilibrium we assume we are in perfect competition).
Percent competition is a form of marketing in which some several condition needs to be meet like:

- firms are small enough that they cannot affect the price at they which they sell their good
- All firms produce an homogenous good, so goods are not different one from the other —> consumer is indifferent from which firms or supply to buy (He only care of the price!)
- All consumer and firms are perfectly informed: consumer are informed about all the prices applied by all firms —> in the market will emerge one price at which all the firms will sell their good. This depends on the fact that if a firm apply a larger price then all consumer will go to buy to the firms that apply Lower prices.

In this example there are two type of firms: A and B.
In the market we have 100 of A and 30 of B.
Then we have short run supply A given by quantity.
Supply of A depends on prices ecc.
We also have aggregate demand which depend on price
X(p) = 5000 - 500p
We have intercept (5000) and the slope (- 500 p).
From that we can obtain aggregate supply of the market

Partial Equilibrium Analysis

- ***Example 6.1*** (continued):
  - Summing the individual supply curves of the 100 type-*A* firms and the 30 type-*B* firms,
    $S(p) = 100 \cdot 2p + 30 \cdot 10p = 500p$
  - The short-run equilibrium occurs at the price at which quantity demanded equals quantity supplied,
    $$5000 - 500p = 500p, \text{ or } p = 5$$
  - Each type-*A* firm supplies: $s_A(p) = 2 \cdot 5 = 10$
  - Each type-*B* firm supplies: $s_B(p) = 10 \cdot 5 = 50$

Aggregate supply of the market is given by the sum of the supply of firm A and B: 500p

Having the total supply we have just to equate total demand and total supply. We obtain an equation in which we have only one variable which is the price. So we obtain the price and the optimal price is 5 (competitive price). Now we can replace this price in the supply of the two firms A and B and we get the optimal supply of individual of type A and B.

For firm A = 2 * 5 = 10
For firm B =10 * 5= 50

Comparative Statics

Comparative Statics

- **Goal:** How the CE changes in the presence of taxes?
- **Taxes:** price received by firms and price paid by consumers differ
- Notation:
  - $\hat{p}_i(p,t)$ is the effective price paid by the consumer
  - $\hat{p}_j(p,t)$ is the effective price received by the firm

Consumer taxes:

- **Per unit tax**: $\hat{p}_i(p,t) = p + t$.
  - Example: $t = \$2$, regardless of the price p
- **Ad valorem tax:** $\hat{p}_i(p,t) = p + pt = p(1+t)$
  - Example: $t = 0.1$ (10%).

Comparative statics helps to answer a question like:
How CE changes in the presence of taxes?
So when the government imposed some taxes on consumer or producer.
before taking some example is useful to stress that in presence of taxes there is a difference between the price payed by consumer and receive by firms.
We refers to price of consumer as pi and pj receive by the firm.
In this case we said that the firm reiceves different price because what happen in reality is that firm receive prices and then they give the taxe revenue to the government.
Two different case of consumer taxes:

- Per unit tax: each unit of good that consumer buy price increase buy the amount p
- Ad valorem tax: tax is a percentage payed in the amount of the price. In this case the price is p(1+t) where t is the tax rate.

Comparative Statics

- ***Implicit Function Theorem (IFT)***:
  – You have a condition (equation) under the form
  $F(x,p)=0$ (1)
  This can a first-order condition for the consumer UMP, for instance, where x is the demand for a good and p its price. We ask: how the optimal demand changes if p increases?
  We can give an answer by totally differentiating (1), i.e.
  $$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial P}dp=0\,;\frac{dx}{dp}=-\frac{\frac{\partial F}{\partial p}}{\frac{\partial F}{\partial x}}$$
  The same can be obtained by recognizing that in the optimum x depends on p, $F(x(p),p)=0$ and computing the derivative wrt p, i.e. $\frac{\partial F}{\partial x}x'(p)$+$\frac{\partial F}{\partial p}=0$; $x'(p)=-\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial x}}$

The IFT:
Image we have a condition in which an expression is equal to 0 and these expression depend on x and p. There arguments can be both variable or one variable and one parameter.
We are asking the following question: (imaging FOC consumer) how optimal demand change on price changes?
Optimal demand increase or decrease when p increases?
So to answer this question we can totally deferenciate the expression in the left hand side
Der F with respect to x + partial der F with respect to P. We can isolate on the lef hand side dx/dp.
What we get is saying that der of x with respect to p(how demand change with respect to price?)

This can also obtained in another way in which der of F with respect to x which multiply der of x' with respect to p + der F with respect to p and this must be equal to 0.
From this expression we can isolate x'(p) and get the same result.

Comparative Statics

- ***Implicit Function Theorem***: an example
- Example, $u(x)$, budget constraint $m = px$, where m is income.
- FOC: $u'(x) = p$ or $F(x, p) \equiv u'(x) - p = 0$
- How x changes when p increases?
- By applying the IFT:
- $\frac{dx}{dp} = -\frac{\frac{\partial F}{\partial p}}{\frac{\partial F}{\partial x}} = -\frac{-1}{u''} = \frac{1}{u''} < 0$ if $u'' < 0$ (i.e. utility is strictly concave)

Example by applying the IFT. We have the consumer utilty maximisation problem and we get generic utilty function that depends on x and constrain is equal to the total income that is equal to total expenditure x * price of x.
By FOC we know the optimal the margina utitly must be equal to price.
We can write this FOC as the IFT, so this is done by simply moving p on the left, and we get marginal utility - p = 0.
Now if we are looking for what happen to the opt demand of x with respect to p increase. We can just apply IFT.
Der x with respect to p is equal to - d F/ dp / dF/dx
Computing this derivative we will get 1/u'' where u'' is second derivative of u.
If u'' < 0 (strictly concave) then if p increase the opt demand decrease. Dx /Dp <0.
This is an example of using IFT.

Comparative Statics

- ***Per unit (consumer) tax*** (Example 6.2):
  - The expression of the aggregate demand is now $x(p+t)$, because the effective price that the consumer pays is actually $p+t$.
  - In equilibrium, the market price after imposing the tax, $p^*(t)$, must hence satisfy
$$x(p^*(t)+t) = q(p^*(t))$$
  - if the per unit tax is marginally increased, and functions are differentiable at $p = p^*(t)$,
$$x'(p^*(t)+t) \cdot (p^{*\prime}(t)+1) = q'(p^*(t)) \cdot p^{*\prime}(t)$$

Another example: we are wondering how competitive price changes when government introduce a per unit consumer taxes: so each consumer has to pay an amount p for each amount of unit x he buys.
How the opt demand change when taxes is introduces? We can answer this question whit market clearing condition.

Market clearing condition (MCC) states that after introducing the taxes, the consumer demand must be equal to the firm supply.
It's important to notice that the price payed by the consumer is not p* but the sum of p* + t where t is the tax.

Also, with this notation we know that opt price p* will also be a function of tax rate t.
After MCC we can get the derivative of it with respect to t.
By collecting p' we can rewrite it in the following form [next slide]

Comparative Statics

– Rearranging, we obtain

$$\underline{p^{*\prime}(t)\cdot\left[x'(p^*(t)+t)-q'\big(p^*(t)\big)\right]}$$
$$=\boxed{-x'(p^*(t)+t)}$$

p ? WHEN t ↑

– Hence,

Final result —> $$\boxed{p^{*\prime}(t)=-\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\big(p^*(t)\big)}}$$

– Since $x(p)$ is decreasing in prices, $x'(p^*(t)+t)<0$, and $q(p)$ is increasing in prices, $q'\big(p^*(t)\big)>0$,

$$p^{*\prime}(t)=-\frac{x'(p^*(t)+t)}{\underbrace{x'(p^*(t)+t)}_{-}-\underbrace{q'\big(p^*(t)\big)}_{+}}=-\frac{(-)}{(-)}=- \quad (*)$$

Rearreging: we can now get p’
This derivative is telling us how the optimal price changes when t increases.

Check the same of this expression:
Demand is decreasing in prices and num has a negative sign.
Supply is increasing in price so q’ is larger to 0 and taken with - sign is less than zero and what we get is a negative number.
So if per unit tax is introduce, the equilibrium price will decrease.

Comparative Statics

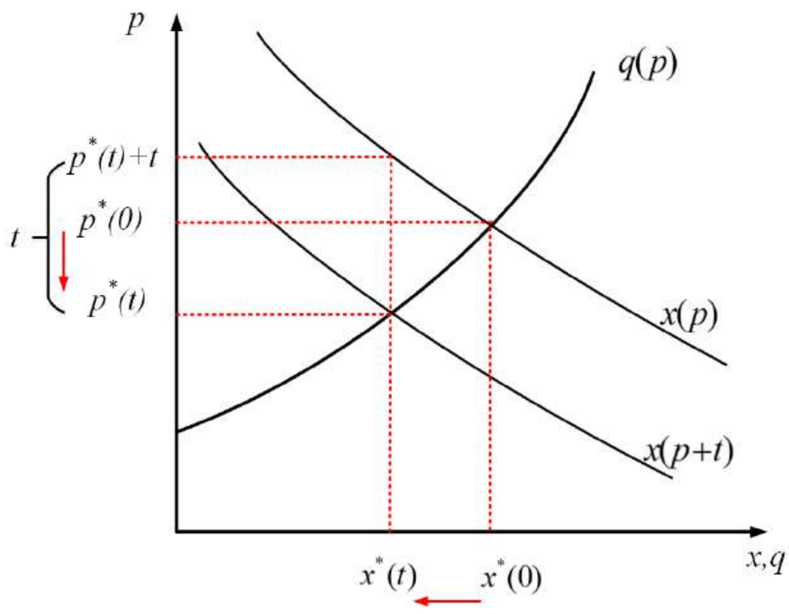
- Hence, $p^{*\prime}(t) < 0$ (the equilibrium price decreases with the tax).
- Moreover, $p^{*\prime}(t) \in (-1,0]$ as the denominator of (*) is larger in absolute value than the numerator.
- Therefore, $p^{*}(t)$ decreases in t.
  - That is, the **price received by producers** falls in the tax, but less than proportionally.
- Additionally, since $p^{*}(t) + t$ is the price paid by consumers, then $p^{*\prime}(t) + 1$ is the marginal increase in the price paid by consumers when the tax marginally increases.
  - Since $0 > p^{*\prime}(t) > -1$, then $p^{*\prime}(t) + 1 < 1$, and the price paid by consumers raises less than proportionally (i.e. the tax is not totally borne by consumers)

If the tax rate increase the equilibrium price decrease.
From the previous expression we can also get the magnitude. Since we know that for sure the denominator is all negative and also numerator. The denominator will be larger in absolute value than the nominator. So the derivative is also < 1. So it's included in the interval (-1,0].
This implies that the price recieved by producer falls in the tax but less than proportionally because p is less than one.

What about the payed by consumer is the price received by producer + the tax so the evaluation will be the derivative of this expression so p' + 1 but we know that p' < 1 so this implies that the price payed by the consumer is less than 1.
We are summing 1 to 1 and taking a negative quantity so will be less than 1.

Comparative Statics

- *No tax:*
  - CE occurs at $p^*(0)$ and $x^*(0)$
- *Tax:*
  - x^* decreases from $x^*(t=0)$ to $x^*(t)$
  - Consumers now pay $p^*(t)+t$
  - Producers now receive $p^*(t)$ for the $x^*(t)$ units they sell.



Now we show graphically the comparitive statics.
We have two consumer demand:

- aggregate consumer demand (decreasing) corresponding the case with the tax
- aggregate consumer demand (decreasing) corresponding the case with per unit tax
- Aggregate supply (increasing)

When a tax is introduced in the market, the aggregate demand shift downward on the left —> there will be a reduction in the opt demand but also in the price receive by firms.
At the same time there will be an increase of the price for the consumer.
Indeed, in the new equilibrium firm receive price p* that is the function of t while consumer payed p*+ t rate.
There is a difference between price payed by consumer and receive by firms.

Comparative Statics

- ***Per unit Tax*** (Extreme Cases):
  - *a) The supply is very responsive to price changes (i.e. very elastic, close to be horizontal), i.e., $q'\big(p^*(t)\big)$ is large.*

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\big(p^*(t)\big)} \to 0$$

If $q' \to +\infty$

$$\frac{x'}{x'+\infty}$$

$p \to 0$

  - – Therefore, $p^{*\prime}(t) \to 0$, and the price received by producers does not fall.
  - – However, consumers still have to pay $p^*(t) + t$.
    - ▪ A marginal increase in taxes therefore provides an increase in the consumer's price of

$$p^{*\prime}(t) + 1 = 0 + 1 = 1$$

    - ▪ **The tax is solely borne by consumers.**

Depending on the elasticity on the firm supply there may be some extreme cases:

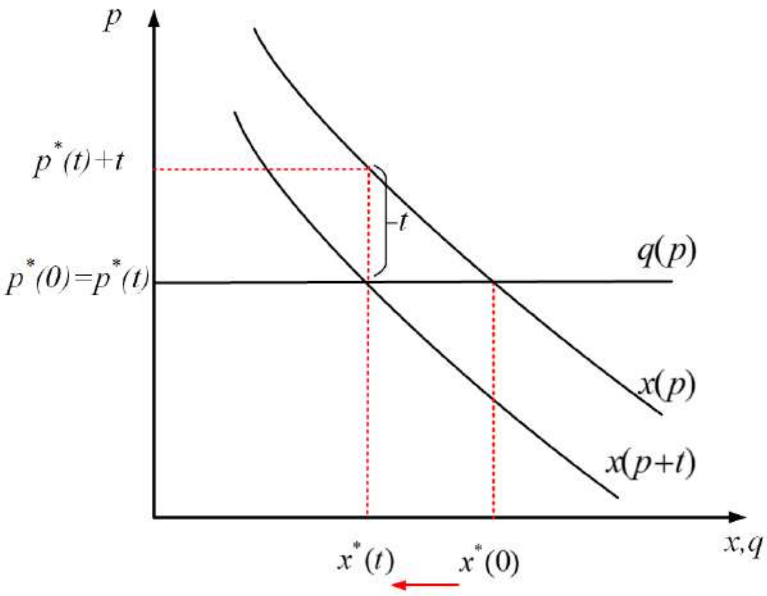
- firm supply is very elastic

when elasticity tends to infinity the firm supply is horizontal.
Let’s start from the previous condition which is the derivative of p*’ with respect to t.
Now we have to check this quantity on the right when the supply is very elastic so q’ tends to infinity. If q’ —> infinity the denominator will be very large and p’ will tend to 0.
So price received by firm when supply is very elastic doesn’t change.

What about the price payed by consumer?
In this case the derivative p’ + 1 = 0 +1, so consumer will pay 1. So if the supply if very elastic the all tax is borne (sostenuta) by consumers.
So: **Price will increase the same amount of the tax**

Comparative Statics

- A very elastic supply curve
  - The price received by producers almost does not fall.
  - But, the price paid by consumers increases by exactly the amount of the tax.



Supply is horizontal so infinity elastic

So if the demand curve shift downward the opt quantity in equilibrium decrease but price receive by firm doesn’t change. So all tax borne by consumer. So price payed by consumer will be previous price p* + t (total amount of the tax).

Comparative Statics

b) *The supply is not responsive to price changes, i.e., $q'\left(p^*(t)\right)$ is close to zero (i.e. vertical supply).*

$$p^{*\prime}(t) = -\frac{x'(p^*(t)+t)}{x'(p^*(t)+t)-q'\left(p^*(t)\right)} = -1$$

- Therefore, $p^{*\prime}(t) = -1$, and the price received by producers falls by $1 for every extra dollar in taxes.
  - **Producers bear all the tax burden**
- In contrast, consumers pay $p^*(t) + t$
  - A marginal increase in taxes produces an increase in the consumer's price of

$$p^{*\prime}(t) + 1 = -1 + 1 = 0$$

  - **Consumers do not bear tax burden at all.**

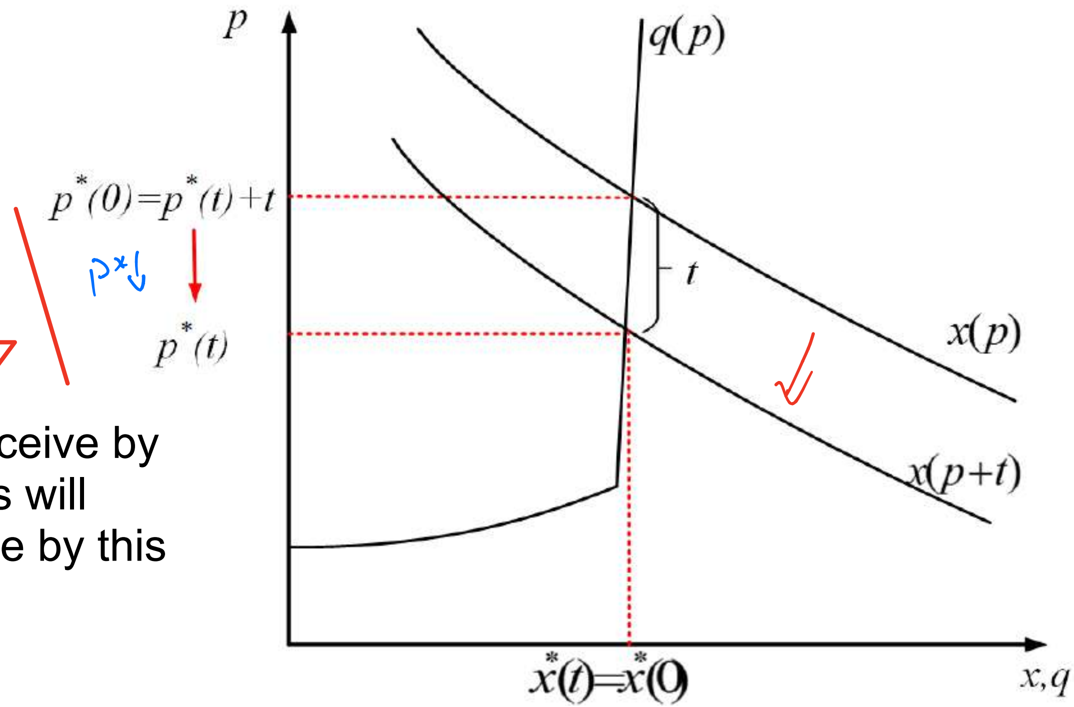
- Firm supply very inelastic: elasticity tend to 0

We will have vertical supply and in this case what we get that q’ tends to 0 and the ratio will tend to 1 and we the minus signi is -1.
When firm supply is vertical: the derivative of p’ with respect to tax will be -1.
This is telling us that price recevive by producer falls by same amount of the tax. So **Producers bear all the tax burden**. (Carico della tassa).

What about consumer? He pays the price propose by the firm + the tax rate so p’ +1 = -1 +1 = 0.
So change in the price payed by consumer will be 0.
So consumer does not bear the tax at all.

Comparative Statics

- Inelastic supply curve



Price receive by the firms will decrease by this amount

Firm supply is very inelastic (almost vertical). This imply that downward shift of the demand will only case a change in price but not in quantity. In particular the all tax rate is payed by firm. Consumer will pay the same price as the case before the introduction of the tax, while the price receive by the firm will decrease by this amount and is the different between the old and new prices.

Comparative Statics

Demand

- ***Example 6.3***:
  - Consider a competitive market in which the government will be imposing a tax t per unit.
  - Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve is $q(p) = ap^{\gamma}$, where $a > 0$ and $\gamma > 0$.
  - What are ε and γ? (The elasticities of Demand and Supply)
  - Let us evaluate **how the equilibrium price is affected by a marginal increase in the tax**.

Supply

Comparative Statics

$x(p^*) = q(p^*)$

- ***Example 6.3*** (continued):
  - The change in the price received by producers at $t = 0$ is (using the previous expression):

$\partial x \quad \partial q$

$$p^{*\prime}(0) = -\frac{x'(p^*)}{x'(p^*) - q'(p^*)} = -\frac{A\varepsilon p^{*\varepsilon \cancel{-1}}}{A\varepsilon p^{*\varepsilon \cancel{-1}} - a\gamma p^{*\gamma \cancel{-1}}} \cdot p$$

$$= -\frac{A\varepsilon p^{*\varepsilon}}{A\varepsilon p^{*\varepsilon} - a\gamma p^{*\gamma}} = -\frac{\varepsilon x\cancel{(p^*)}}{\varepsilon x\cancel{(p^*)} - \gamma q\cancel{(p^*)}} = -\frac{\varepsilon}{\varepsilon - \gamma}$$

Since $x = q$ because equilibrium

Where we use the fact that in equilibrium $x(p^*) = q(p^*)$ (market clearing condition)

  - The change in the price paid by consumers at $t = 0$ is

$$p^{*\prime}(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}$$

We want to compute the change in the price after the introduction of the tax rate. We know from the last part of the lecture that is equal by x’ / x’ - q’.

$$\overline{A\varepsilon p^{*\varepsilon-1} - a\gamma p^{*\gamma-1}}$$

Derivative of demand - derivative of supply

We can multiply both numerator and denomitator by p this means that -1 in the exponent goes away.

$x(p^*) = q(p^*)$

A * p epsilon is x and a*y is q.

Since equilibrium must the demand must be equal to supply and we can cancel all this term. Left esp / eps - y

This is the changing price production by introduction of the tax (or marginal change of the tax)

The change In the price paid by consumer is p(t) +t.
If we compute the derivative we get p’ +1.
If we replace with the condition before
p’ +1 = - eps /eps -y +1 and if you do the compute the expression we get
p’(0) = - y/ eps -y

So we get that the change in the price by the consumer.

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\gamma = 0$ (i.e., supply is perfectly inelastic), the price paid by consumers (change= $-\frac{\gamma^{0}}{\varepsilon-\gamma_{0}}$) in unchanged, and the price received by producers (change=$-\frac{\varepsilon}{\varepsilon-\gamma_{0}}$)decreases be the amount of the tax.
    - That is, producers bear the full effect of the tax.
  - When $\varepsilon = 0$ (i.e., demand is perfectly inelastic), the price received by producers (change= $-\frac{\varepsilon^{0}}{{}_{0}\,\varepsilon-\gamma}$) is unchanged and the price paid by consumers (change=$-\frac{\gamma}{\varepsilon-\gamma}$) increases by the amount of the tax.
    - That is, consumers bear the full burden of the tax.

Now what happen as consequence of the introduction of the tax when gamma (y) = 0. Which mean supply is perfect inelastic.
Price of the consumer is unchanged since supply is perfectly inelastic.
If we replace 0 to gamma we test -1, so producer will bear all the effect of the tax.

If eps = 0 price receive by producer is unchanged and the consumer will bear full burden of the tax since equal to 1.

Comparative Statics

- ***Example 6.3*** (continued):
  - When $\varepsilon \to -\infty$ (i.e., demand is perfectly elastic), the price paid by consumers is unchanged (change= $-\frac{\gamma}{\varepsilon-\gamma}$), and the price received by producers (change= $-\frac{\varepsilon}{\varepsilon-\gamma}$) decreases by the amount of the tax. $\to 1$
  - When $\gamma \to +\infty$(i.e., supply is perfectly elastic), the price received by producers $\left(\text{change} = -\frac{\varepsilon}{\varepsilon-\gamma}\right)$ is unchanged and

    the price paid by consumers (change= $-\frac{\gamma}{\varepsilon-\gamma}$) 1

    increases by the amount of the tax.

Two other cases: perfectly elastic demand and supply!

- perfectly elastic demand:

eps —> - infinity and the change in the price by consumer tend to 0. So price paid by consumer is unchanged.

Price received by producer is -1.

- perfect elastic supply:

y —> + infinity and the change price of producer is unchanged and price paid by consumer increase by the amount of the tax.

Welfare Analysis

Welfare Analysis

- Let us now measure the changes in the aggregate social welfare due to a change in the competitive equilibrium allocation.
- Consider the **aggregate surplus (=area between supply and demand)**

$$\boxed{S = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)}$$

- Take a differential change in the quantity of good k that individuals consume and that firms produce such that $\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j$.
- The change in the aggregate surplus is

$$dS = \sum_{i=1}^{I} \phi_i'(x_i)dx_i - \sum_{j=1}^{J} c_j'(q_j)dq_j$$

How do we measure welfare?
We can use EV or CV.
Usually is customise in PF to use aggregate surplus: can be measure by the sum of all the consumer of the total utility given by consuming the good xi - the summation over all firm of the marginal cost of producing the quantity qj.
This is called **aggregate surplus** since the aggregate surplus because is the difference between the total utility given by the consumption - total cost that firm bear to produce that quantity that is consumed in the market.

We want to check what happen if there is a change in the quantity produced and consumed. <u>The change in this quantity affect aggregate surplus.</u>

To be in equilibrium we must be the case that total variation in the consumer demand must be equal to the total variation in the supply.
Then, what happen when this quantities change?
Then we can rewrite when quantity are going to change.

$$\underbrace{dS}_{\text{CHANGE IN SURPLUS}} = \sum_{i=1}^{I} \underbrace{\phi_i'(x_i)}_{\text{MU DEMAND}} \underbrace{\boxed{dx_i}}_{\text{DEMAND}} - \sum_{j=1}^{J} \underbrace{c_j'(q_j)}_{\text{MARGINAL COST OF SUPPLY}} \underbrace{\boxed{dq_j}}_{\text{SUPPLY}}$$

We take the total differential of this expression (S) to get dS.

Welfare Analysis

- Since

  Marginal utility

  - $\phi_i'(x_i) = P(x)$ for all consumers; and
    - That is, every individual consumes until MU=p.
  - $c_j'(q_j) = C'(q)$ for all firms
    - That is, every firm's MC coincides with aggregate MC

then the change in surplus can be rewritten as

$$dS = \sum_{i=1}^{I} P(x)dx_i - \sum_{j=1}^{J} C'(q)dq_j$$

$$= P(x)\sum_{i=1}^{I} dx_i - C'(q)\sum_{j=1}^{J} dq_j$$

change in aggregate demand

change in aggregate supply

Now if we are in equilibrium we know from the FOC that Mu of all consumer must be equal to the price
Also marginal cost of the firm must be equal to the total aggregate marginal cost.

So we can replace this expression in the variation of the surplus.
Then, since x and q doesn’t on an index we can bring them outside the summation.
So: P(x) * variation of all quantity consumed - C’ * variation in all quantities produced by firm.
In practise one is change in aggregate demand and the other is the change in aggregate supply

Welfare Analysis

dq

- But since $\boxed{\sum_{i=1}^{I} dx_i = \sum_{j=1}^{J} dq_j} = dx$ (change in aggregate demand or supply), and $x = q$ by market feasibility, then

$$\boxed{dS = [P(x) - C'(q)]dx}$$

$\phi'(x)$

- *Intuition*:
  – The change in surplus of a marginal increase in consumption (and production) reflects the **difference between the consumers' additional utility and firms' additional cost of production.**

The change in aggregate demand (since we are in equilibrium) must be equal to the change in the aggregate supply.
So dx = dq

Variation in total surplus: (Price - Marginal cost) * variation in quantity.

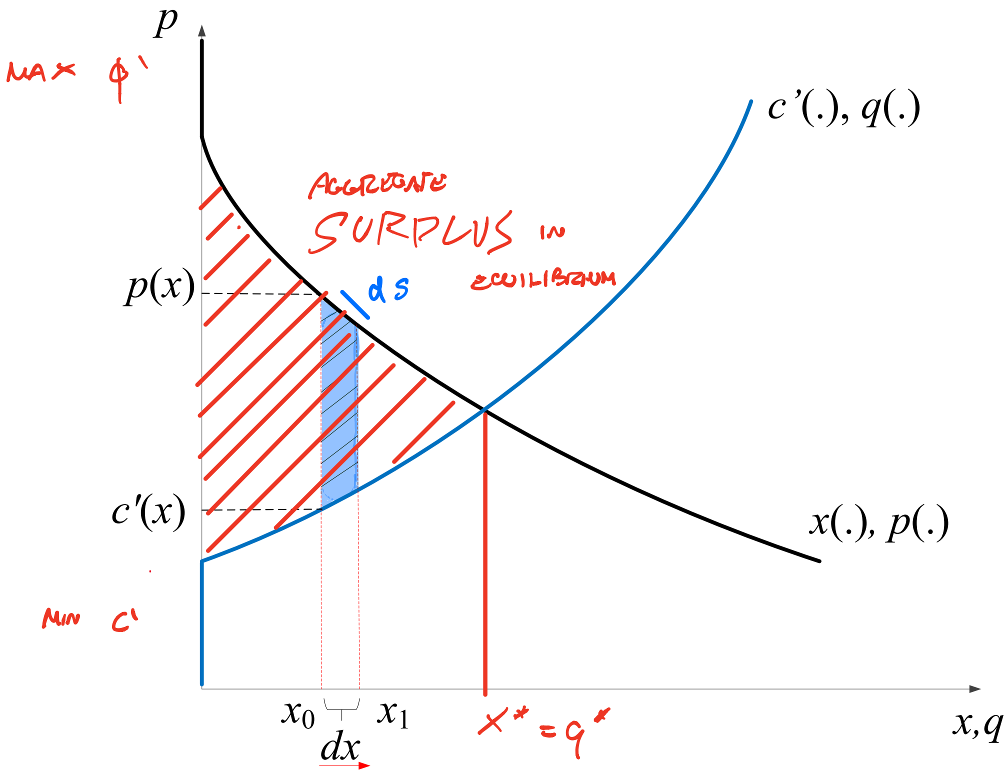
The main intuition is that the change in surplus of a marginal increase in consumption reflect the difference between consumer additional utitly and firms' additional cost of production.

Welfare Analysis

- Differential change in surplus

Max of marginal utility

Min of marginal cost



Change in surplus can also be draw in a graph:
We have aggregate demand and supply.
Aggregate demand —> curve of marginal utility
Curve of supply —> curve of marginal cost

We assuming Mc increasing while Mu decreasing. The the intercept in this point was the max in the Mu, and this is the minimum in the Mcost

We know that for price above max demand is 0
For price below min c' supply is 0.

What happen if quantity increases from x0 to x1.
We find the aggregate surplus in this area included between the Mu curve and Mc curve.
From x0 to x1 the change in aggregate surplus will be the highlight area.

ds

Main difference between graph and expression before is that we were using discrete changes so xi and we used summation symbol.
We can also consider the continuos changes.

Welfare Analysis

- We can also integrate the above expression, eliminating the differentials, in order to obtain the total surplus for an aggregate consumption level of x:

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)]ds$$

where $S_0 = S(0)$ is the constant of integration, and represents the aggregate surplus when aggregate consumption is zero, $x = 0$.
  - $S_0 = 0$ if the intercept of the marginal cost function satisfies $c_j'(0) = 0$ and the intercept of the marginal utility function satisfies $u'(0) = 0$.

We can also consider the continuos changes in quantities like from 1 to 2 to 3 ecc.
If change continuously we can use integrals instead of summation.
We can rewrite the surplus in the following way.

$$S(x) = S_0 + \int_0^x [P(s) - C'(s)] ds$$

$x = 0$

Surplus when quantity is equal to 0

IS EQUIVALENT TO $ds = (P_x - C')dx$

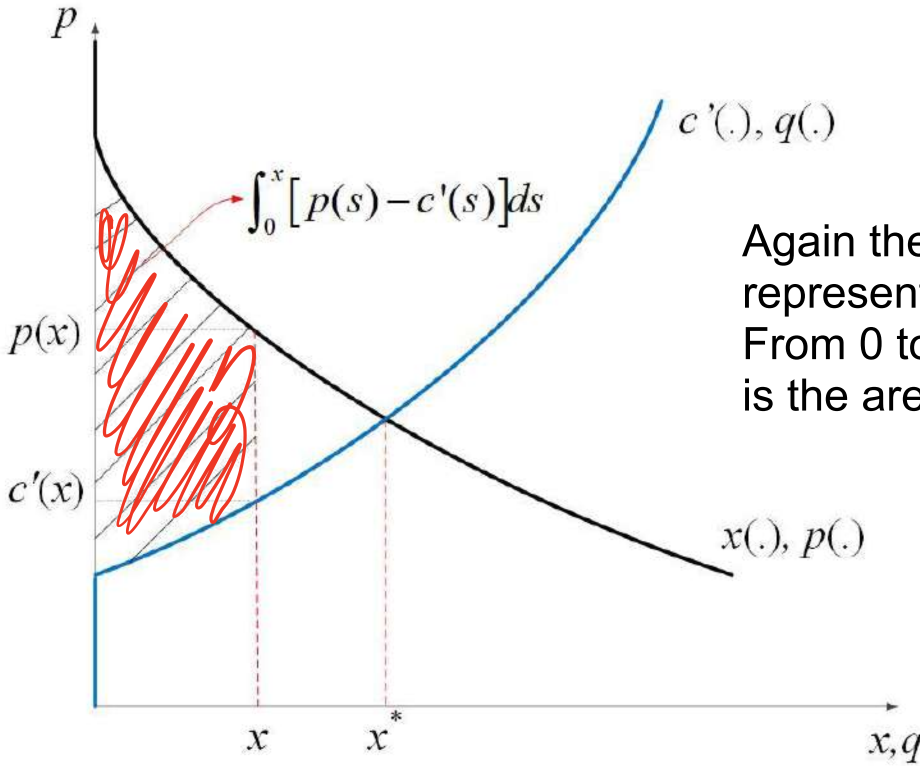
BUT THIS IS FOR DISCRETE

We also have the constant of integration that is S0. That is the aggregate surplus when quantity is equal to 0 and S0 is given by difference in Mc when q = 0 and Mu when q = 0.

If this difference is equal to 0 then also the constant integration will be equal to 0. This is the same if the Mc of 0 = 0 and Mu = 0 also S0 = 0.

Welfare Analysis

- Surplus at aggregate consumption x



Again the graphical represention of surplus. From 0 to the quantity x so is the area drawn here.

Welfare Analysis

- For which consumption level is aggregate surplus $S(x)$ maximized?
  - Differentiating $S(x)$ with respect to x, FOC for a maximum:

$$S'(x) = P(x^*) - C'(x^*) \leq 0$$

INTERIOR OPT

$$\text{or,} \quad P(x^*) \leq C'(x^*)$$

=0

  - The second order (sufficient) condition is

$$S''(x) = \underbrace{P'(x^*)}_{-} - \underbrace{C''(x^*)}_{+} < 0$$

    i.e. , $S(x^*)$ is concave.

    - **Then, when $x^* > 0$, aggregate surplus $S(x)$ is maximized at $P(x^*) = C'(x^*)$.**

for which consumption level (quantity) aggregate surplus maximise?
Imagine we have social planner that has to decide the quantity in such way aggregate surplus is maximise.
We have to compute the FOC with respect to x. S0 doesn't depend on X and the only one is the integral.
So we remain with the thing inside the integral

$$S(x) = \cancel{S_0} + \int_0^x [P(s) - C'(s)]ds$$

SOC—> aggregate surplus must be concave function so the derivatives must be < 0.

When X* > 0 (interior solution and holds with equality), what implies? Implies that price in equilibrium must be equal to the marginal cost in equilibrium.

Aggregate surplus is maximise in the point in which the price is equal to the marginal cost —> FOC of the competitive equilibrium.

CE is the allocation of good in which maximise the aggregate surplus

Welfare Analysis

- **Therefore, the CE allocation maximizes aggregate surplus.**

EXERCISE BY yourself

Welfare Analysis

- ***Example 6.4***:
  - Consider an aggregate demand $x(p) = a - bp$ and aggregate supply $y(p) = J \cdot \frac{p}{2}$, where J is the number of firms in the industry.
  - The CE price solves

$$a - bp = J \cdot \frac{p}{2} \quad \text{or} \quad p = \frac{2a}{2b+J}$$

  - Intuitively, as demand increases (number of firms) increases (decreases) the equilibrium price increases (decreases, respectively).

Welfare Analysis

- ***Example 6.4*** (continued):
  - Therefore, equilibrium output is

$$x^* = a - b\frac{2a}{2b+J} = \frac{aJ}{2b+J}$$

  - Surplus is

$$S(x^*) = \int_0^{x^*} p(x) - C'(x)dx$$

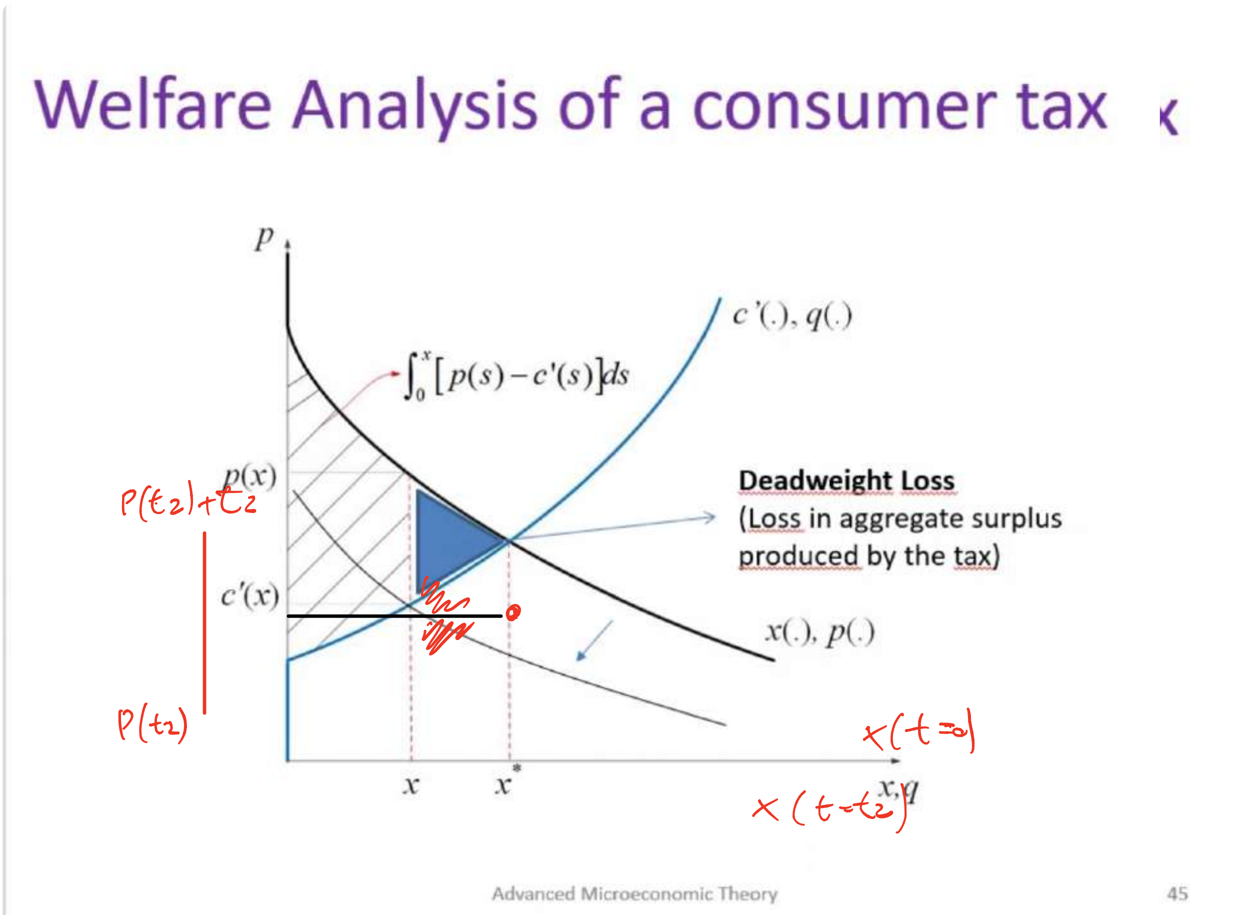
where $p(x) = \frac{a-x}{b}$ and $C'(x) = \frac{2x}{J}$ (as P=MU and P=MC, in the demand and supply, respectively).

  - Thus,

$$S(x^*) = \int_0^{x^*} \left(\frac{a-x}{b} - \frac{2x}{J}\right) dx = \frac{a^2 J}{4b^2 + 2bJ} = \frac{a^2}{2b\left(1+\frac{2b}{J}\right)}$$

which is increasing in the number of firms J.

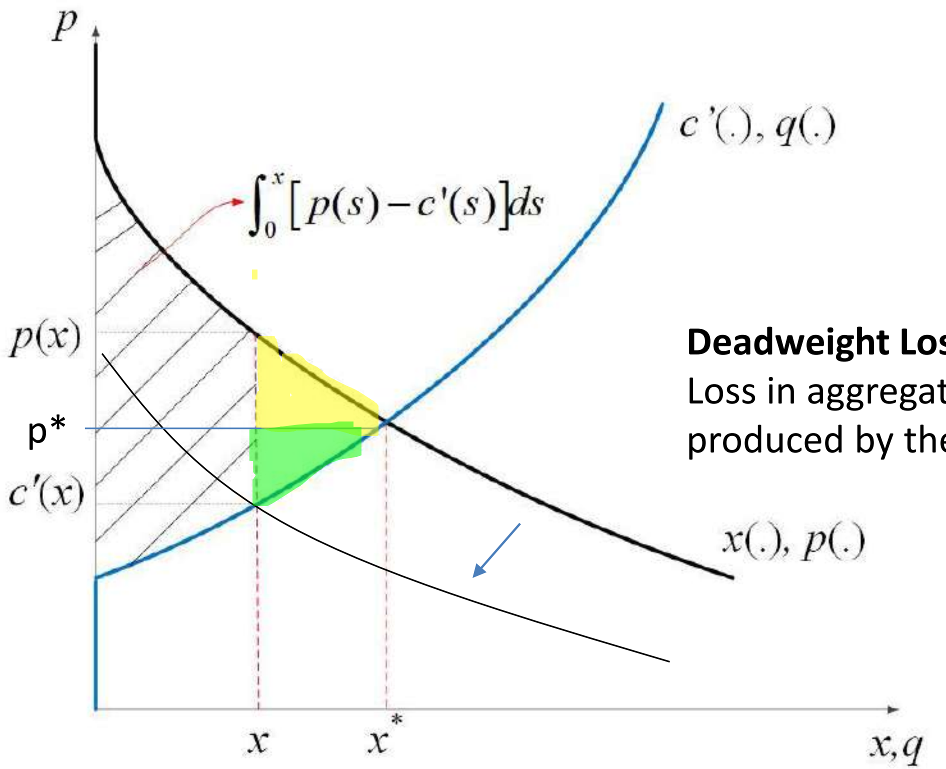
Important thing is to reach this point



Represent graphically the change in the aggregate total surplus in the case of introduction of a tax.
A per unit consumer tax is introduced in the market and the effect will be shift downward of the demand. This will be the demand when x(t = 0) and the downward will be x (t = t2).
We will see that the equilibrium price falls but also quantity of demand walls and what i will obtain is a difference in price payed by consumer which is p(t2) + t2.

Which is the Chang of total surplus? Is the change in the Mu and the Mc curve and what is lost is the quasi-triangle. What happened is that you can split the area in two triangles
Above triangle is change in consume surplus
Below triangle is the change in the produce surplus.
Consumer surplus is define as the area above the price and below the demand while the producer surplus is the area below the price and above supply.

Welfare Analysis of a consumer tax



Deadweight Loss =
Loss in aggregate surplus produced by the tax

Area btw demand and equilibrium price p*= Consumer Surplus (CS) / Area btw supply and p* = Producer Surplus (PS = profits)

Exercise 1 - production

Set. 3

$$q = z_1^{1/5} z_2^{1/3}$$

$\boxed{a}$ CONDITIONAL FACTOR DEMAND → MINIMIZE COST FUNCTION

$$\min_{z_1, z_2 \geq 0} \; w_1 z_1 + w_2 z_2 \quad \text{s.t. } z_1^{1/5} z_2^{1/5} \geq q$$

$$z_1^{1/5} z_2^{1/5} = q \qquad z_1 = \frac{q^5}{z_2}$$

$$w_1 \cdot \frac{q^5}{z_2} + w_2 z_2 \qquad \text{FOC} \quad \frac{\partial TC}{\partial z_2} = -\frac{w_1 q^5}{z_2^2} + w_2 = 0$$

$$\frac{z_2^2}{w_1 q^5} = \frac{1}{w_2} \qquad z_2 = \left(\frac{w_1 q^5}{w_2}\right)^{1/2}$$

NOW REPLACE IN z_1

$$z_2^* = q^{5/2} \frac{w_1^{1/2}}{w_2^{1/2}}$$

$$z_1^* = \frac{q^5}{z_2} \qquad z_1 = \frac{q^5}{q^{5/2}\left(\frac{w_1}{w_2}\right)^{1/2}} = q^{5-5/2} \cdot \left(\frac{w_2}{w_1}\right)^{1/2} = q^{5/2} \cdot \left(\frac{w_2}{w_1}\right)^{1/2}$$

NOW REPLACE z_1^*, z_2^* IN COST FUNCTION

$$C(w_1, w_2, q) = w_1 z_1^* + w_2 z_2^* = w_1 q^{5/2} \cdot \left(\frac{w_2}{w_1}\right)^{1/2} + w_2 q^{5/2} \left(\frac{w_1}{w_2}\right)^{1/2} =$$

$$= q^{5/2} w_1 \cdot w_2^{1/2} w_1^{-1/2} + q^{5/2} w_2 \cdot w_1^{1/2} w_2^{-1/2} = q^{5/2} w_1^{1/2} w_2^{1/2} + q^{5/2} w_1^{1/2} w_2^{1/2} =$$

$$= 2 q^{5/2} (w_1 w_2)^{1/2}$$

$\boxed{b}$ SUPPLY AND PROFIT FUNCTION

$$q(w, p) \qquad \pi(w, p)$$

$$C(w_1, w_2, q) = 2\, q^{5/2} (w_1\, w_2)^{1/2}$$

PROFIT MAXIMIZATION PROBLEM OF THE FIRM

$$\pi = p \cdot q - 2\, q^{5/2} (w_1, w_2)^{1/2}$$

↳ DIFFERENCE BETWEEN TOTAL REVENUE AND TOTAL COST

FOC WITH RESPECT TO QUANTITY: $\dfrac{\partial \pi}{\partial q} = 0$

$$\frac{\partial \pi}{\partial q} = p - \cancel{2}\, \frac{5}{\cancel{2}}\, q^{3/2} (w_1, w_2)^{1/2} = 0$$

$$q^{3/2} (w_1, w_2)^{1/2} = \frac{p}{5} \qquad q = \frac{p^{2/3}}{5^{2/3} (w_1 w_2)^{1/3}}$$

FIRM SUPPLY FUNCTION

REPLACE IN PROFIT TO FIND PROFIT FUNCTION

$$\Pi = p \cdot q - 2 q^{5/2} (w_1, w_2)^{1/2} =$$

$$= p \cdot \frac{p^{2/3}}{5^{2/3}(w_1 w_2)^{1/3}} - 2\left(\frac{p^{2/3}}{5^{2/3}(w_1 w_2)^{1/3}}\right)^{5/2} (w_1 w_2)^{1/2}$$

$$= \frac{p^{5/3}}{5^{2/3}(w_1 w_2)^{1/3}} - 2 \frac{p^{5/3}}{5^{5/3}(w_1, w_2)^{5/6}} (w_1, w_2)^{1/2} =$$

$(w_1, w_2)^{-1/2}$

$$= \frac{p^{5/3}}{5^{2/3}(w_1 w_2)^{1/3}} - 2 \frac{p^{5/3}}{5^{5/3}(w_1, w_2)^{1/3}} =$$

$$= \frac{p^{5/3}}{5^{2/3}(w_1, w_2)^{1/3}} \left(1 - \frac{2}{3}\right) = \frac{p^{5/3}}{5^{2/3}(w_1 w_2)^{1/3}} \left(\frac{3}{5}\right)$$

PROFIT FUNCTION!

$\boxed{C}$ TO DO! w_1, w_2 AND COMPUTE OPTIMAL QUANTITY

Exercise 2

SET 2

$$c(w, q) = \frac{1}{2} q^2 \sqrt{2w_1 w_2}$$

$\boxed{a}$ FIND PROFIT FUNCTION

$\pi(p, w)$

$$\max_{q \geq 0} \; p \cdot q - c(\underbrace{w}_{\text{EXOGENOUS}}, q) = p \cdot q - c(q) = p \cdot q - \frac{1}{2} q^2 \sqrt{2w_1 w_2}$$

$$\frac{\partial \pi}{\partial q} \rightarrow p - q\sqrt{2w_1 w_2} = 0 \qquad \underline{q^*} = \frac{p}{\sqrt{2w_1 w_2}}$$

REPLACE IN PROFIT!

$$\pi = p \cdot q^* - \frac{1}{2} q^{*2} \sqrt{2w_1 w_2} =$$

$$= p \cdot \frac{p}{\sqrt{2w_1 w_2}} - \frac{1}{2} \frac{p^2}{2w_1 w_2} \sqrt{2w_1 w_2} =$$

$$= \frac{p^2}{\sqrt{2w_1 w_2}} - \frac{1}{2} \frac{p^2}{\sqrt{2w_1 w_2}} = \frac{1}{2} \frac{p^2}{\sqrt{2w_1 w_2}} \left(1 - \frac{1}{2}\right) =$$

$$= \frac{1}{2} \frac{p^2}{\sqrt{2w_1 w_2}}$$

EXERCISE 2 (SKETCH)

INTEREST RATE OF CAPITAL

$$\pi(w, r, p) = \frac{p^2}{4w} + \frac{p^2}{4r}$$

ASSUME w, r, p EXOGENOUS

$\boxed{a}$ FIND COST FUNCTION

$$c(w, r, q) = w \cdot z_e + z_k r$$

PRICE FOR LABOUR

CONDITIONAL DEMAN FOR LABOUR

HOW DO WE FIND z_k?

UNCONDITIONAL FACTOR DEMAND → APPLY HOTELLING LEMMA

$$\ell(p, w, r) = \frac{\partial \pi(p, w, r)}{\partial w} = \frac{p^2}{4w^2}$$

$$K(p, w, r) = -\frac{\partial \pi(p, w, r)}{\partial r} = \frac{p^2}{4r^2}$$

WE CAN OBTAIN SUPPLY NOW $q = F(p)$

LINK BETWEEN p AND q

$$q(p, w, r) = \frac{\partial \pi(p, w, r)}{\partial p} = \frac{2p}{4w} + \frac{2p}{4r} = \frac{p(w+r)}{2wr}$$

ISOLATE p → $p = \frac{2wrq}{w+r}$

INVERSE SUPPLY

NOW REPLACE P IN ZL e ZK

$$\ell(p,w,r) = \frac{p^2}{4w^2} = \frac{\left(\frac{2wrq}{w+r}\right)^2}{4w^2} = \frac{r^2}{(w+r)^2} q^2$$

$$K(p,w,r) = \frac{p^2}{4r^2} = \frac{\left(\frac{2wrq}{w+r}\right)^2}{4r^2} = \frac{w^2}{(w+r)^2} q^2$$

NOW WE CAN WRITE THE COST FUNCTION

$$c(w,r,q) = w \frac{r^2}{(w+r)^2} q^2 + r \frac{w^2}{(w+r)^2} q^2 = \frac{wr}{w+r} q^2$$

Exercise 3 - set 3

PRODUCTION FUNCTION $q = f(z_1, z_2) = \dfrac{\theta}{1 + z_1^{-\delta} z_2^{-\varepsilon}}$

☐ FIND PRODUCT ELASTICITY AND TYPE OF RETURN TO SCALE THAT FUNCTION REPRESENTS

PRODUCTION $f(z) = q$

FIRST FACTOR

$$\zeta_{q,z_i} = \frac{\delta f(z)}{\partial z_i} \frac{z_i}{f(z)} = \frac{\frac{\delta f(x)}{f(z_1)}}{\frac{\delta z_1}{z_1}}$$

$$\zeta_{q,z_1} = -\theta \left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-2} (-\delta) \left(z_1^{-\delta-1} z_2^{-\varepsilon}\right) \frac{z_1}{\theta (1 + z_1^{-\delta} z_2^{-\varepsilon})^{-1}} =$$

$$= \delta \left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-1} z_1^{-\delta} z_2^{-\varepsilon}$$

APPLY FOR 2° FACTOR

$$\zeta_{q,z_2} = \frac{\delta f(z)}{\partial z_2} \frac{z_2}{f(z)} =$$

$$= -\theta \left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-2} (-\varepsilon) \left(z_1^{-\delta-1} z_2^{-\varepsilon}\right) \frac{z_2}{\theta (1 + z_1^{-\delta} z_2^{-\varepsilon})^{-1}} =$$

$$= \varepsilon \left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-1} z_1^{-\delta} z_2^{-\varepsilon}$$

SCALE OF ELASTICITY IS SUM OF OUTPUT ELASTICITIES OF TWO INPUT

$$\zeta_{q,t} = \sum_{i=1}^{m} \zeta_{q,z_i} = \zeta_{q,z_1} + \zeta_{q,z_2} =$$

$$= \delta \left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-1} z_1^{-\delta} z_2^{-\varepsilon} + \varepsilon \left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-1} z_1^{-\delta} z_2^{-\varepsilon} =$$

$$= (\delta + \varepsilon)\left(1 + z_1^{-\delta} z_2^{-\varepsilon}\right)^{-1} z_1^{-\delta} z_2^{-\varepsilon}$$

SCALE OF ELASTICITY WILL GIVE US RETURN TO SCALE

SCALE ELASTICITY:

- > 0 → INCREASING RETURN TO SCALE
- $= 0$ → CONSTANT RETURN TO SCALE
- < 0 → DECREASING RETURN TO SCALE

IN THIS CASE DEPENDS ON z_1, z_2 INPUTS SO CAN HAVE ANY OF THIS RETURN TO SCALE

Exercise 5 set 2

$$TC = 4q^3 - 12q^2 + 20q + 16$$

$\boxed{q}$
- AVERAGE TOTAL COST
- FIX COST
- VARIABLE COST
- AVERAGE VARIABLE COST
- AVERAGE FIX COST
- MARGINAL COST

$$ATC = \frac{TC}{q} = \overbrace{4q^2 - 12q + 20}^{AVC} + \overbrace{\frac{16}{q}}^{AFC}$$

$FC = 16 \rightarrow$ PART OF TC THAT DOESN'T DEPENDS ON q

$VC = 4q^3 - 12q^2 + 20q \rightarrow$ PART OF TC THAT DEPENDS ON q

$$\overset{\frac{VC}{q}}{AVC} = 4q^2 - 12q + 20$$

$$\overset{\frac{FC}{q}}{AFC} = \frac{16}{q}$$

$$MC = \frac{\delta TC}{\delta q} = 12q^2 - 24q + 20$$

$\boxed{b}$ FIND Q SUCH THAT MC IS MINIMUM

TAKE MC AND MINIMISE IT ⟶ FOC

$$MC = 12Q^2 - 24Q + 20$$

$$\text{FOC} \quad \frac{\delta MC}{\delta Q} = 0 \rightarrow 24Q - 24 = 0$$

$$Q = 1$$ QUANTITY THAT MINIMIZE MC

$\boxed{C}$ LEVEL OF OUTPUT MC = AVC ?

Q SUCH THAT MC = AVC

$$MC = \frac{\delta TC}{\delta Q} = 12Q^2 - 24Q + 20$$

$$AVC = 4Q^2 - 12Q + 20$$

$$MC = AVC$$

$$12Q^2 - 24Q + \cancel{20} = 4Q^2 - 12Q + \cancel{20}$$

$$8a^2 - 12a = 0 \qquad 2a^2 - 3a = 0$$

$$2a^2 = 3a \qquad 2a = 3 \qquad a = \frac{3}{2} = 1.5$$

$\boxed{d}$ ATC REACH MINIMUM?

RUFFINI RULE

Summary equilibrium in perfect competition

We saw choice of consumer and choice of firm.

CONSUMER

$\max U(x)$

INCOME

S.T. $m = \underline{p \cdot x}$

TOTAL EXPENDITURE

$U'(x) = p$ CONSUMER DEMAND

MARGINAL UTILITY

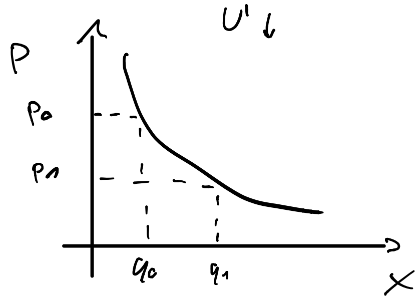
FIRM

TOTAL REVENUE − TOTAL COST

$\max \Pi = p \cdot q - C(q)$

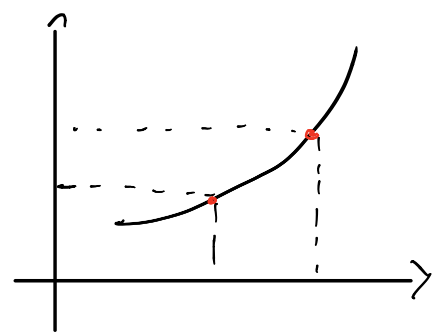
PMP

$c'(q) = p$ FIRM SUPPLY



INDIVIDUAL CONSUMER DEMAND FOR GOOD X

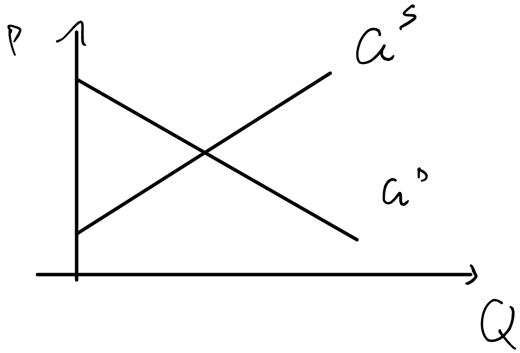
$P \downarrow$ CONSUMER INCREASE CONSUME



SUPPLY IS INCREASING SINCE WE ASSUME MARGINAL COST IS INCREASING

IF WE WANT FIRM TO PRODUCE MORE FIRM MUST APPLY HIGHER PRICE (TO COVER MARGINAL COST)

We have seen that we can aggregate he individual supply and demand function to find the market demand and supply.

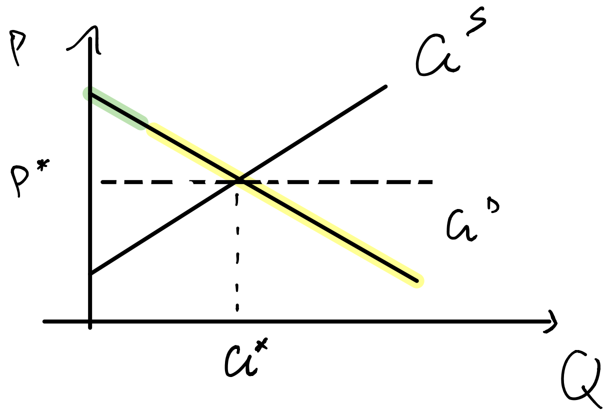


MARKETS

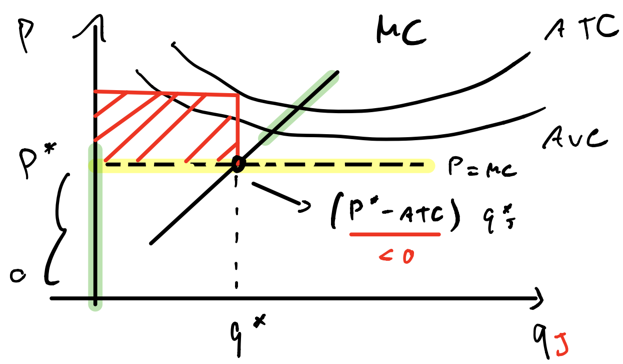
The aggregate supply can be found by horizontally summing the individual firm supply, while aggregate demand can be obtain summing horizontally the individual consumer demand.

Now he will show how is possible from equilibrium to find the individual quantity produced by individual firm.

SHORT TERM → SOME FACTORS ARE FIXED



MARKET



INDIVIDUAL FIRM

We are in the short term and some factor are fixed (usually the capital K is fixed).

We have market on the left and individual firm on the right.
On the vertical axes we price and on the horizontal we have total quantity in market and on right we have quantity produce by firm j (qj)

On the right we have Average total cost, average variable cost and marginal cost. MC cuts ATC and AVC in the minimum and we want to see the amount produce by firm if market equilibrium is defined by aggregate production Q* and equilibrium price p*.
Consider this case. The optimal price is p* in the market and we have to see how much firm j is producing.
For individual firm equilibrium will be defined by crossing point between equilibrium price and marginal cost firm.
Is important that AVC e ATC are above the p*.
How much this firm is going to produce? Is immediately possible to see in the red square that if the price < AVC the firm will income the loss (negative profits) and negative profit will be the shaded red area. How can we say that this are the negative profit incorred by the firm? We can see that the area in the box is (p* - ATC) qj *

In the equilibrium the firm does not produce anything.

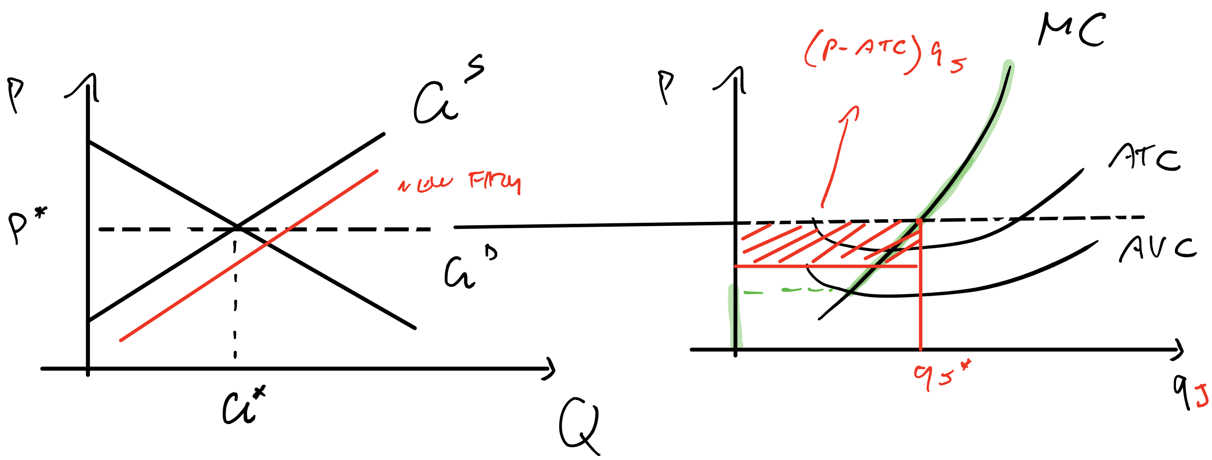
So profit are negative! And we can rewrite it in this way:

$$\Pi = p \cdot q - c(q) = \left(p - \frac{c(q)}{q}\right) q =$$

$$= (p - ATC)\, q$$

$$p < ATC \rightarrow \Pi < 0$$

Imagine the situation like this



The price is above both AVC and ATC and now the amount of profit made by the firm is in the red area.

In the short run the firm makes positive profit.
In the long run?
If profits are positive is convenient to a firm to enter the market. What is the effect of new firms entering the market? Produce shift toward the right of the supply function.
This imply a shift down of the equilibrium price and will end when equilibrium price will be equal to the minimum of the ATC.

Firm enter as long a the price is large than ATC —> because profits are positive

P > ATC —> $\pi > 0$

P = ATC —> long run total cost
There are fixed cost (AVC = ATC)

$\pi = (P - ATC)\, q$

$\pi = 0$

NO CONVENIENCE IN ENTERING MARKET

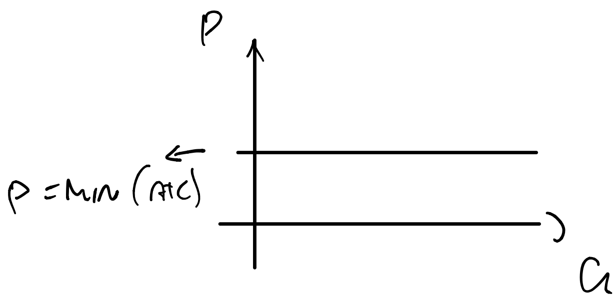
This i called the so called long run equilibrium

In the long run price is equal to the minimum of ATC

P = MIN (ATC) ⟶ EFFICIENT EQUILIBRIUM

Efficient because:

- there are no profit
- Consumer payers the lower possible price



Firm supply is infinitevely elastic and horizonatal of level of minimum of ATC.

Advanced Microeconomic Theory

Chapter 7: Monopoly

Outline

- Barriers to Entry
- Profit Maximization under Monopoly
- Welfare Loss of Monopoly
- Multiplant Monopolist

Barriers to Entry

Perfect competition: high number of firms, firms are price takers (cannot decide the price to charge the consumer)

In Monopoly there is only one producer:

- can be barriers to entry:
    - **Legal barrier**, if you invent a new product (new technology) you can protect your self by registering a patent and this gives you the right to produce it. (Common in farmaceuca industry)

    - **Structural barrier**: some firms may have advantage in lower cost or demand. This may depend on a superior techlogy that get a patent or loyal group of costumer.

    - **Strategic**: monopolist fight newcomers driving them out of market (price war)

Barriers to Entry

- ***Entry barriers***: elements that make the entry of potential competitors either impossible or very costly.

- Three main categories:
  1) ***Legal***: the incumbent firm in an industry has the legal right to charge monopoly prices during the life of the patent
     – *Example*: newly discovered drugs

Barriers to Entry

2) ***Structural***: the incumbent firm has a cost or demand advantage relative to potential entrants.
   - superior technology
   - a loyal group of customers
     - positive network externalities (Facebook, eBay)

3) ***Strategic***: the incumbent monopolist has a reputation of fighting off newcomers, ultimately driving them off the market.
   - price wars

Profit Maximization under Monopoly

Profit Maximization

- Consider a demand function $\boxed{x(p),}$ which is continuous and strictly decreasing in p, i.e., $x'(p) < 0$.
- We assume that there is price $\bar{p} < \infty$ such that $x(p) = 0$ for all $p > \bar{p}$.
- Also, consider a general cost function $c(q)$, which is increasing and convex in q.

Problem of monopolist

Profit maximisation problem for monopolist.

Since in the market there is only one producer, this mean that the demand for the single firm is the same as the market demand (since in the market there is only one firm). Demand for monopolist is negatively sloped.
So derivative of the demand with respect to price is negative.

Assume we assume that for a price $p' <$ infinite such that $x(p) = 0$. All prices $p > p'$ have demand equal to 0.

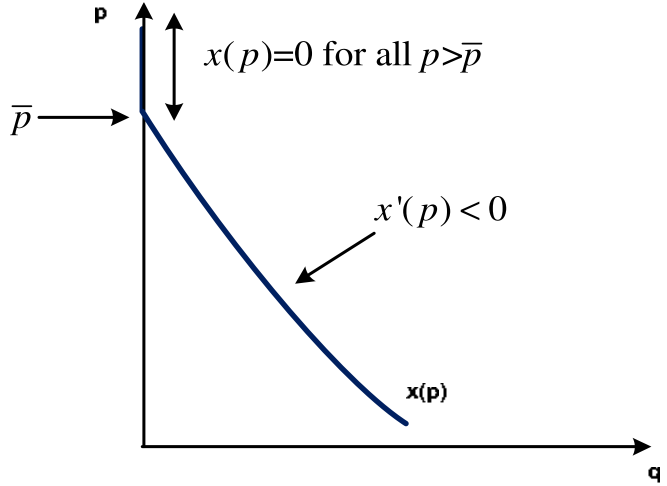
We also consider a cost function increasing in the quantity and is convex.
$C'(q) > 0$ and $c''(q) > 0$.
Now we draw demand for monopolist.

Demand for monopolist

Profit Maximization

- $\bar{p}$ is a "choke price"
- No consumers buy positive amounts of the good for $p > \bar{p}$.

Price below chock price have demand 0



Profit Maximization

AGGREGATE DEMAND

$x = x(p)$

- Monopolist's decision problem is

$$\max_{p} \underbrace{px(p)}_{TR} - \underbrace{c(x(p))}_{TC}$$

- Alternatively, using $x(p) = q$, and taking the inverse demand function $p(q) = x^{-1}(p)$, we can rewrite the monopolist's problem as

$$\boxed{\max_{q \geq 0} \; p(q)q \; - c(q)}$$

$p = p(q^*)$

Now we set the Maximisation problem.

Since monopolist has market problem can decide the price!
Monopolist maximisation problem both in prices and quantities.
The decision of the monopolist is to maximising profit.
Profit = Total revenue - total cost

We see immediately that choice variable is price. So set price in a way you can maximise profit.
Demand is a function of prices (quantity demanded and prices given by aggregate demand)
we can set up the problem in a similar way switching quantity to price as a choice variable.
The the problem is to maximise with respect to quantity.

We compute the derivative of profit with respect to quantities as usual.

Profit Maximization

FOC FOR MAX

- Differentiating with respect to q,

$$\frac{\partial \pi}{\partial q} = \underbrace{p(q^m) + p'(q^m)q^m}_{MR} - \underbrace{c'(q^m)}_{MC} \leq 0$$

- Rearranging,

$$\underbrace{p(q^m) + p'(q^m)q^m}_{MR = \frac{d[p(q)q]}{dq}} \leq \underbrace{c'(q^m)}_{MC}$$

with equality if $\underline{q^m > 0}$. EQUALITY $\longrightarrow$ MR = MC

- Recall that total revenue is $TR(q) = p(q)q$

Profit Maximization

- In addition, we assume that $p(0) \geq c'(0)$.
  - That is, the inverse demand curve originates above the marginal cost curve.
  - Hence, the consumer with the highest willingness to pay for the good is willing to pay more than the variable costs of producing the first unit.
- Then, we must be at an interior solution $q^m > 0$, implying

$$\boxed{\underbrace{p(q^m) + p'(q^m)q^m}_{MR} = \underbrace{c'(q^m)}_{MC}}$$

We assume to allow for the existence of and equilibrium that price must be >= marginal cost when quantity is 0.

Profit Maximization

FOC $q^m > 0$

- Note that

$$\overbrace{p(q^m) + \underbrace{p'(q^m)q^m}_{-}}^{MR} = c'(q^m)$$

Demand negatively sloped so derivative is < 0

- Then, $p(q^m) > c'(q^m)$, i.e.,

monopoly $\boxed{\text{price}} > MC$

- Moreover, we know that in competitive equilibrium $\boxed{p(q^*) = c'(q^*).}$

$P > MR = MC$

In the perfect competition was p = MC in the optimal case

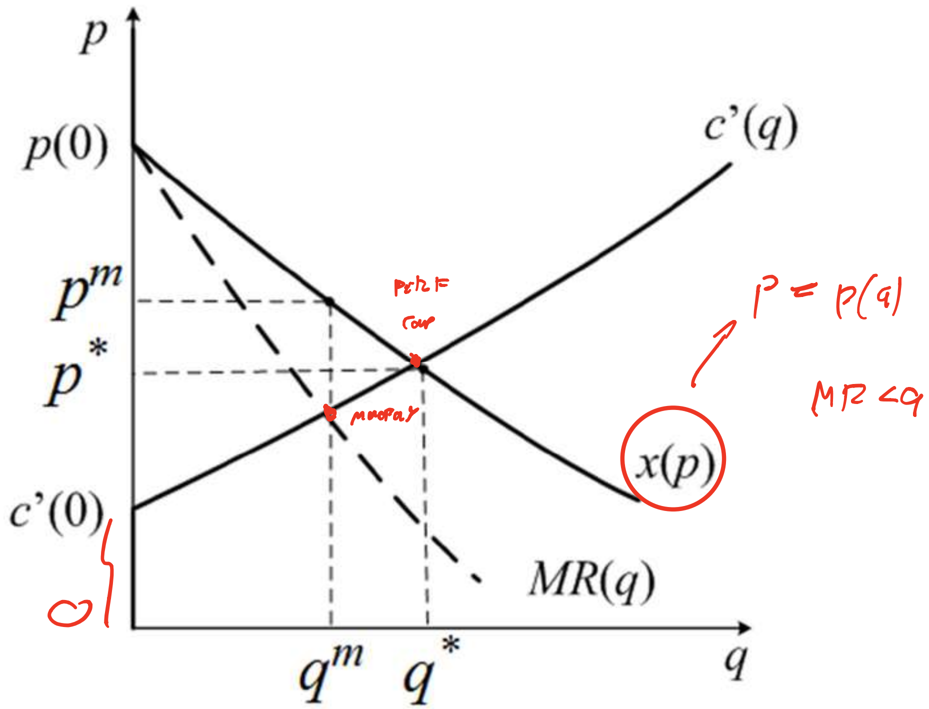
- Then, $p^m > p^*$ and $q^m < q^*$.

P monopoly > p equil

In equilibrium production is lower (than monopoly) and price higher in monopoly

Inverse demand since p is in the vertical axes

Profit Maximization



Equilibrium in the monopoly is the equality between Marginal revenue and marginal cost

$MR = \frac{\partial TR}{\partial q^m}$

Profit Maximization

- Marginal revenue in monopoly

$$MR = p(q^m) + p'(q^m)q^m \longrightarrow p'(q^m) < 0$$

MR describes two effects:

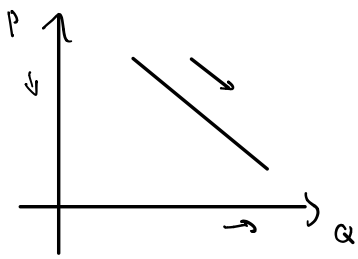
– **A *direct* (positive) effect**: an additional unit can be sold at $p(q^m)$, thus increasing revenue by $p(q^m)$.

– **An *indirect* (negative) effect**: selling an additional unit can only be done by reducing the market price of all units (the new and all previous units), ultimately reducing revenue by $p'(q^m)q^m$.

  - *Inframarginal units* – initial units before the marginal increase in output.

Marginal revenue is given by price and the second term p' (that is less than 0).
Marginal revenue is the derivative of total revenue with respect to quantitity produced.
This expression is saying that when we increase production we have two effects:

- a direct (positive) effect: if produce more and can selle this new unit we gain the price of that units. Means TR increase by total price.
- A indirect(negative) effect: to produce more we have to move along the demand curve, so if we move along the demand curve if we want to increase quantity price must falls.



We have indirect effect when increase production because we have to charge a small price not only in the last unit we sell but also all previous unit called **inframarginal unit**: all unit was selling before increasing quantity and reducing prices.

Profit Maximization

FOC

- Is the above FOC also sufficient?
  - Let's take the FOC $p(q^m) + p'(q^m)q^m - c'(q^m)$, ≤ 0
    and differentiate it wrt q,

$$\boxed{\underbrace{p'(q) + p'(q) + p''(q)q}_{\frac{dMR}{dq}} - \underbrace{c''(q)}_{\frac{dMC}{dq}} \leq 0}$$

$\to$ SOC

< 0

> 0

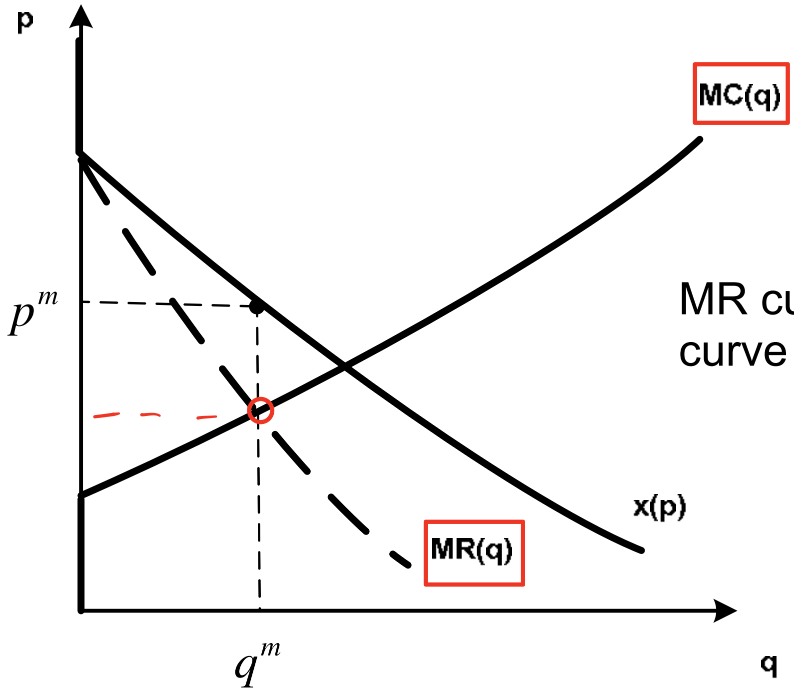
  - That is, $\frac{dMR}{dq} \leq \frac{dMC}{dq}$.
  - Since MR curve is decreasing and MC curve is weakly increasing, the second-order condition is satisfied for all q.

Second order condition.
We already seen the FOC in which marginal revenue - marginal cost must be <= 0 to zero. Then, we see the SOC.
We have to derive again to check the SOC, with respect to quantity.
So for the SOC to be satisfy we need the derivative <= 0. Given the assumption that Mrevenue is decreasing (means first term $p'(q) + p'(q) + p''(q)q < 0$) and also we assume cost function is convex ($c''(q)$ positive).
Negative term summed we got negative term.
This means that the FOC is also sufficient for a maximum.
In case of Interior solution the marginal revenue = marginal cost is the point in which profit maximise.

When we referred to the monopolist equilibrium is the point in which marginal cost cross marginals revenue.

Profit Maximization

Price in demand curve



MR curve crosses MC curve from above

Quantity in demand curve

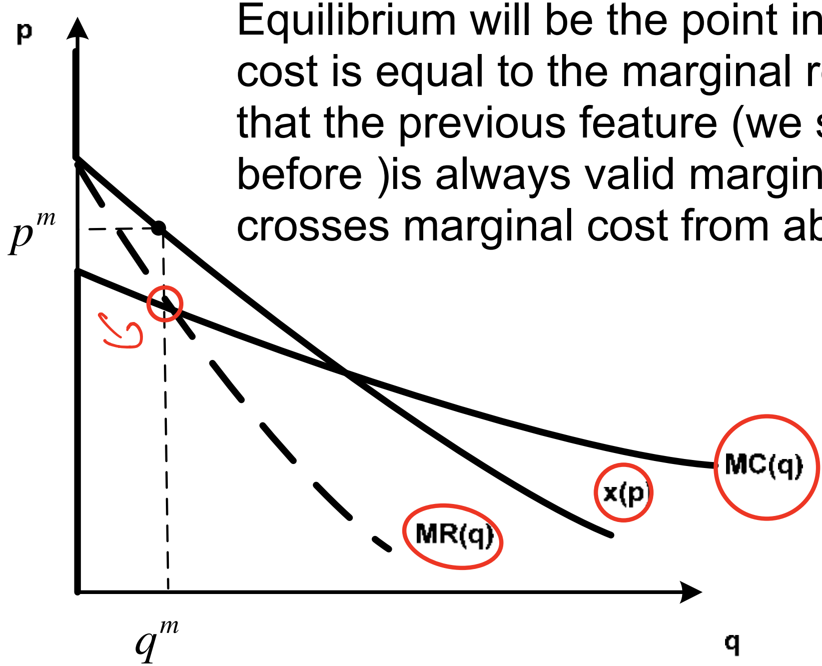
Profit Maximization

- What would happen if MC curve was decreasing in q (e.g., concave technology given the presence of increasing returns to scale)?
  - Then, the slopes of MR and MC curves are both decreasing.
  - At the optimum, MR curve must be steeper MC curve.

In case marginal cost decreasing in q implies the case of concave technology which has increasing return to scale. In this case will be decreasing and q and we will have the following situation [next graph]

Marginal cost decreasing. We have demand function and marginal revenue

Profit Maximization



Equilibrium will be the point in which marginal cost is equal to the marginal revenue. We see that the previous feature (we said in the slide before)is always valid marginal revenue crosses marginal cost from above.

Profit Maximization: Lerner Index

- Can we re-write the FOC in a more intuitive way? Yes.
  - Just take $\boxed{MR = p(q) + p'(q)q = p + \frac{\partial p}{\partial q}q \text{ and}}$ multiply by $\frac{p}{p}$,

$$\boxed{MR = p\frac{p}{p} + \underbrace{\frac{\partial p}{\partial q}\frac{q}{p}}_{1/\varepsilon_d}p = p + \frac{1}{\varepsilon_d}p}$$

$$\varepsilon_d = \frac{\partial q}{\partial p} \cdot \frac{p}{q}$$

  - In equilibrium, $MR(q) = MC(q)$. Hence, we can replace MR with MC in the above expression.

Another interesting way of setting profit maximisation condition.
The equality between marginal revenue and marginal cost is elasticity.
First thing to do is writing again the Marginal revenue multiply factors by p/p’.

The term der p/ der q * q/p is the 1/elasticity of demand.
Elasticity of demand is der q/ der p + p/q

We can replace the expression p + 1/eps * p in the expression MR(q) = MC(q)

The marginal reveue p(1 + 1/eps) = MC

$$p\left(1 + \frac{1}{\varepsilon_d}\right) = MC$$

WE CAN DIVIDE BOTH SIDE BY $\left(1 + \frac{1}{\varepsilon_d}\right)$

Profit Maximization: Lerner Index

- Rearranging yields

MARK UP

$$\frac{p - MC(q)}{p} = -\frac{1}{\varepsilon_d} \quad > 0$$

- This is the ***Lerner index*** of market power
  – The price mark-up over marginal cost that a monopolist can charge is a function of the elasticity of demand.
- Note:
  – If $\varepsilon_d \to \infty$, then $\frac{p-MC(q)}{p} \to 0 \implies p = MC(q)$
  – If $\varepsilon_d \to 0$, then $\frac{p-MC(q)}{p} \to \infty \implies$ substantial mark-up

This way of rewriting equality between MC and MR is called ***Lerner index*** of market power.
This allows us to see that if you look at the left end side the numerator you can see that this is the difference between p and MC and we can consider this as index of market power.
The reason is that if you have market power in a monopoly you can charge an higher price.
Mark up is how much you charge in addition to marginal cost to consumer (p - MC(q)).
Ratio between mark up and price depend negatively on elasticity of demand.
The term on the right is positive because elasticity is negative (demand negative slope so -1/eps is positive.

If the elasticity increases the market power decreases.
This is intuitive: if elasticity of demand is high, consumer are more verses prices. So if you charge a lot price, you will lose a lot of demand.

In this case the firm cannot afford to charge very high price because she will sell few units.

In the learner index if elasticity of demand goes to infinity, then the markup goes to 0. This imply not market power for the firm and the optimal condition is the same as perfect competition.

However in case elasticity tends to 0 the demand is not very sensitive to price, so monopolist can charge very high prices so this means the learner index will tend to infinity.

We can write learner as:

$$P = \frac{MC}{1 + \frac{1}{\varepsilon_d}}$$

IN THIS CASE ELASTICITY TAKES NAME OF INVERSE ELASTICITY PRICING RULE (IEPR)

Profit Maximization: Lerner Index

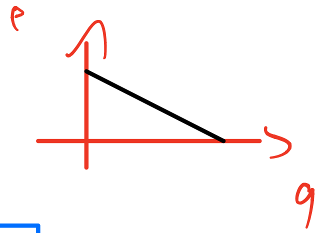
- The Lerner index can also be written as

$$p = \frac{MC(q)}{1+\frac{1}{\varepsilon_d}}$$

  which is referred to as the ***Inverse Elasticity Pricing Rule*** (IEPR).

- *Example* (Perloff, 2012):
  - Prilosec OTC: $\varepsilon_d = -1.2$. Then price should be $p = \frac{MC(q)}{1+\frac{1}{-1.2}} = 5.88MC$
  - Designed jeans: $\varepsilon_d = -2$. Then price should be $p = \frac{MC(q)}{1+\frac{1}{-2}} = 2MC$

Profit Maximization: Lerner Index



- ***Example 1*** (linear demand):
  - Market inverse demand function is

$$p(q) = a - bq$$

Slope

where $b > 0$

Intercept

  - Monopolist's cost function is $c(q) = cq$
  - We usually assume that $a > c \geq 0$
    - To guarantee $p(0) > c'(0)$
    - That is, $p(0) = a - b0 = a$ and $c'(q) = c$, thus implying $c'(0) = c$

Exercise

optional price for the monopolist under a linear demand function so we have a demand that is linear: it’s a inverse demand because on the left end side we have prices and in the right quantities.
So we have prices written as function of quantity.

We assume cost function is c that is a linear as well on q. So cost function is equal to c(q) and then we assume that the intercept of demand function is higher than the intercept of the total cost function —> so total cost function with slope equals to c.

Now we write the monopolist objective function: [next slide]

Profit Maximization: Lerner Index

- ***Example 1*** (continued):
  - Monopolist's objective function

$$\boxed{\pi(q) = (a - bq)q - cq}$$

FOC: $\underbrace{a - 2bq}_{MR} - \underbrace{c}_{MC} = 0$

$q^m = \dfrac{a-c}{2b}$

SOC: $-2b < 0$ (concave)

- Note that as long as $b > 0$, i.e., negatively sloped demand function, profits will be concave in output.
- Otherwise (i.e., Giffen good, with positively sloped demand function) profits will be convex in output.

Now we write the monopolist objective function: is the profit function.

Profit = p which depend on q - total cost.

Then we can compute FOC and SOC.
SOC < 0 and meet the assumption of b > 0.

You can find q

$$q^m = \frac{a-c}{2b}$$

This is the optimal quantity of the monopolist

Profit Maximization: Lerner Index

- ***Example 1*** (continued):
  - Solving for the optimal q^m in the FOC, we find monopoly output

$$\boxed{q^m = \frac{a-c}{2b}}$$

  - Inserting $q^m = \frac{a-c}{2b}$ in the demand function, we obtain monopoly price

$$p^m = a - b\left(\frac{a-c}{2b}\right) = \frac{a+c}{2}$$

  - Hence, monopoly profits are)

$$\pi^m = p^m q^m - cq^m = \frac{(a-c)^2}{4b}$$

Then you can replace the optimal quantity of the monopolist in demand.

$$q^m = \frac{a-c}{2b}$$

So you insert this expression in the demand function to find the monopoly price.

Demand function was be (a-b) * q, so we replace q with a - c / 2b.

After computation we get the optimal equilibrium price that is equal to $\frac{a+c}{2}$

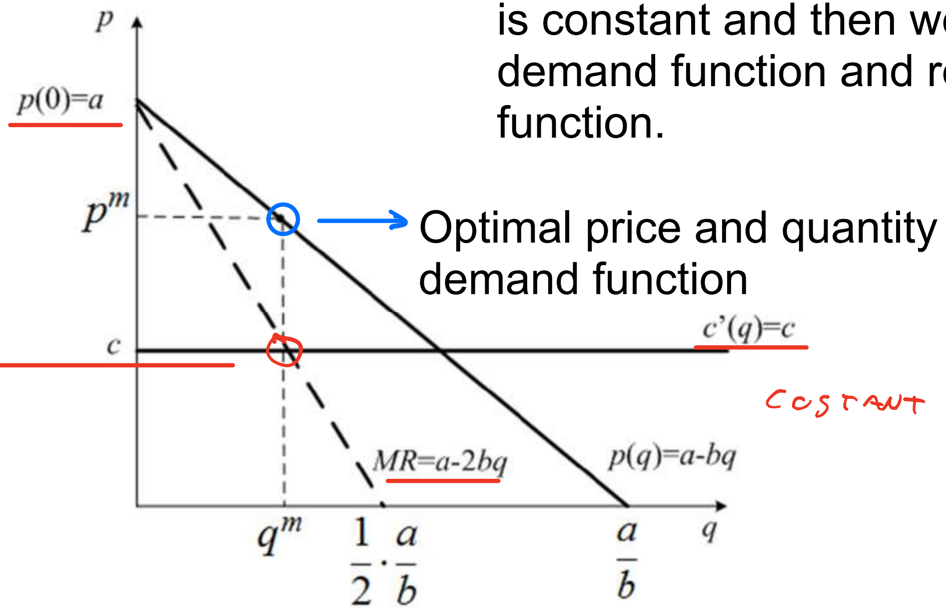
You can replace the optimal quantity and optimal prices in the profit function and you will obtain the form of the profit function.

$$\frac{(a-c)^2}{4b}$$

Profit Maximization: Lerner Index

- ***Example 1*** (continued):

Graphical representation of the example in which Marginal cost is constant and then we have demand function and revenue function.

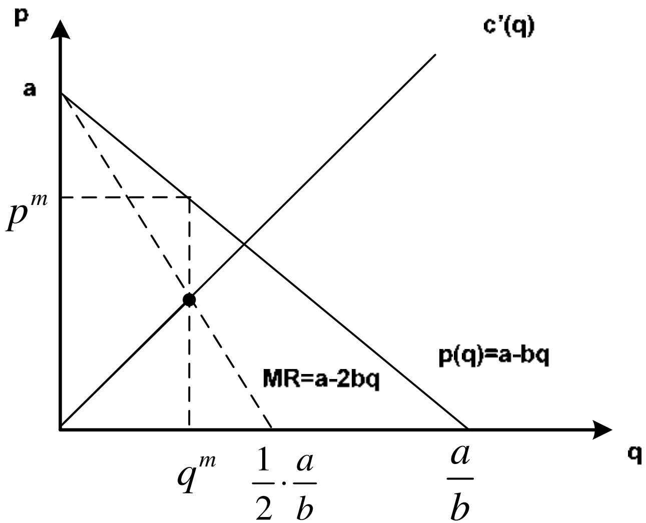


Optimal price and quantity in the demand function

Equilibrium:
MC cross MR

Profit Maximization: Lerner Index

- ***Example 1*** (continued):
  - Non-constant marginal cost
  - The cost function is convex in output
    $$c(q) = cq^2$$
  - Marginal cost is
    $$c'(q) = 2cq$$



As exercise solve the previous maximisation problem just considering another cost function in which the marginal cost is increasing so try to find the opt quantity and optimal price when the cost function cq^2

You can try also solve this example using constant elasticity demand: find optimal quantity and optimal prices in equilibrium

Profit Maximization: Lerner Index

- ***Example 2*** (Constant elasticity demand):
  - The demand function is

$$q(q) = Ap^{-b}$$

  - We can show that $\varepsilon(q) = -b$ for all q, i.e.,

$$\varepsilon(q) = \frac{\partial q(p)}{\partial p}\frac{p}{q} = \underbrace{(-b)Ap^{-b-1}}_{\frac{\partial q(p)}{\partial p}}\underbrace{\frac{p}{Ap^{-b}}}_{\frac{p}{q}}$$

$$= -b\frac{p^{-b}}{p}\frac{p}{p^{-b}} = -b$$

Profit Maximization: Lerner Index

- ***Example 2*** (continued):
  - We can now plug $\varepsilon(q) = -b$ into the Lerner index,

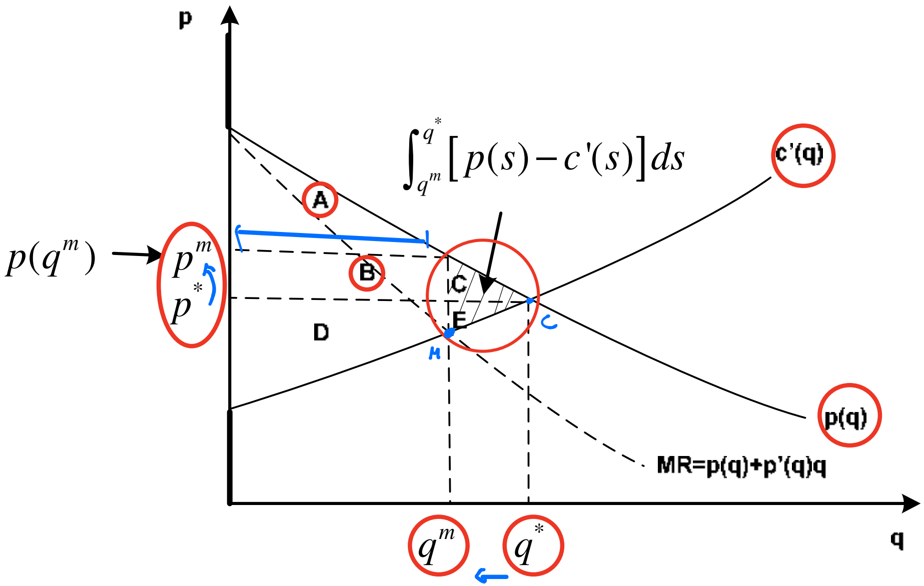
$$p^m = \frac{c}{1 + \frac{1}{\varepsilon(q)}} = \frac{c}{1 - \frac{1}{b}}$$

  - That is, price is a **constant mark-up over marginal cost.**

Welfare Loss of Monopoly

Welfare Loss of Monopoly

- Welfare comparison for perfect competition and monopoly.



In monopoly we have reduction in aggregate surplus compare to perfect competition and this reduction in aggregate surplus is called welfare loss of monopoly.

The starting point is to compare equilibrium in perfect competition and monopoly.
So we have the usual graph with price in vertical axes and quantity in the horizontal axes.
We have marginal cost curve increasing and we have demand curve that is decreasing and also we have marginal revenue curve defined in the previous lecture —> we can define competitive equilibrium which is the crossing point between demand and marginal cost curve and also we can define the monopoly equilibrium.
We know also that we can read the optimal quantity and competition in demand curve and we do the same in monopoly, while optimal demand in p^m
So we define the welfare loss of monopoly as the reduction aggregate surplus and we know that surplus is the area between demand curve and marginal cost curve.
The welfare loss going from q* to q^m and p to p^m is the area of this triangle.
In particular, you can rewrite this area in different subarea in C and E

Welfare Loss of Monopoly

- Consumer surplus
  - Perfect competition: $\boxed{A+B+C}$ $\longrightarrow$ $\Delta CS = B+C$
  - Monopoly: $\boxed{A}$
- Producer surplus:
  - Perfect competition: $\boxed{D+E}$ $\Delta PS = D+E-(D+B) =$
  - Monopoly: $\boxed{D+B}$ $E-B$
- **Deadweight loss of monopoly (DWL)**: *C+E*

$$DWL = \int_{q^m}^{q^*} [p(s) - c'(s)]ds$$

- **DWL decreases as demand and/or supply become more elastic**.

In perfect competition the consumer surplus is A + B + C, while in monopoly we have q^m and p^m so the consumer surplus is just the area A.
The difference between PC and Monopoly is B+C (which is the loss in consumer surplus just because we have monopoly.

We can also do something like that for produce surplus:
Is the area below equilibrium price and above marginal cost curve.
So in perfect competition is the sum of D+E, while in monopoly is D+B.

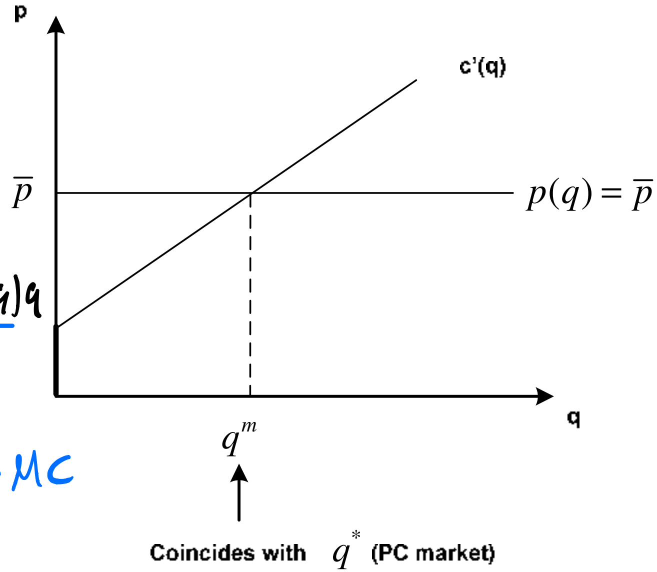
The total variation in aggregate surplus is equal to variation in consumer surplus + variation in producer surplus

$$\Delta AS = \Delta CS + \Delta PS$$

Also defined ad Deadwight loss of monopoly which is the are between demand and marginal cost OST curve that is included between optimal quantity in monopoly and optimal quantity in perfect competition ==> area in the triangle.
It’s possible to see that is the integral between the demand curve and marginal cost curve with q* and q^m as extreme of integration.
DWL decreases as demand and supply became more elastic ==> loss due to monopoly of wealth as the demand of supply became more elastic

Welfare Loss of Monopoly

- Infinitely elastic demand
$$\boxed{p'(q) = 0}$$
- The inverse demand curve becomes totally flat.
- Marginal revenue coincides with inverse demand:
$$MR(q) = p(q) + 0 \cdot q$$
$$= \boxed{p(q)} \qquad p(q) + \underbrace{p'(q)}_{0}q$$
- Profit-maximizing q
$$MR(q) = MC(q) \Longrightarrow$$
$$p(q) = MC(q) \longrightarrow P = MC$$
- Hence, $q^m = q^*$ and $DWL = 0$.



Example of infinitely elastic demand ==> demand horizontal and then we have marginal cost curve increasing. In this case you can check that if the price is constant the demand is flat so marginal revenue is equal to the price. It’s the same case of demand for individual firm in perfect competition.
We know that in general marginal revenue is $p(q) + p'(q) * q$ but since infinitely elastic, the derivative of p with respect to q means that $p'(q)$ is 0 and marginal revenue is equal to the cost.

The profit maximise condition is given by the equality between marginal revenue ad marginal cost but marginal revenue equals to price. To ew have same condition as in perfect completion.

In this example the DWL is equal to 0 since the equilibrium in this case is the same as the equilibrium in perfect competition in which P = MC. So in infinitely elastic demand is equal to 0.

Welfare Loss of Monopoly

- ***Example*** (Welfare losses and elasticity):
  - Consider a monopolist with constant marginal and average costs, $c'(q) = c$, who faces a market demand with constant elasticity

$q(p) = p^e$ with $e < -1$

where e is the price elasticity of demand ($e < -1$)

  - Perfect competition: $p_c = c$
  - Monopoly: using the IEPR

$$\boxed{p^m = \frac{c}{1 + \frac{1}{e}}}$$

$1 + \frac{1}{e}$ $e < -1$

Another case in which the monopolist has costant marginal cost: c’(q) = c.
We take the case of market demand not infinitely elastic but as the following form q(p) = p^e where e is the elasticity of demand.
Elasticity of demand must be negative and e < -1.
In perfect competition we know price = MC however in monopoly firm is price maker and can fix price.
Monopoly price is

$$p^m = \frac{c}{1 + \frac{c}{1+\frac{1}{e}}}$$

Since 1+ 1/e with e < -1 the denominator will be less than 1 so this mean that price in monopoly will be c over something less than one so is larger than c.

Welfare Loss of Monopoly

$q(p) = p\,a$

- ***Example*** (continued):
  - The consumer surplus associated with any price (p_0) can be computed as

PRIMITIVE

ONLY ONE FACTOR

SINCE $\infty^{-1} = \frac{1}{\infty} \Rightarrow 0$

$$CS = \int_{p_0}^{\infty} q(p)dp = \int_{p_0}^{\infty} p^e dp = \frac{p^{e+1}}{e+1}\Bigg|_{p_0}^{\infty} = -\frac{p_0^{e+1}}{e+1}$$

$$\frac{\infty^{e+1}}{e+1} - \frac{p_0^{e+1}}{e+1}$$

$\emptyset$

  - Under perfect competition, $\boxed{p_c = c,}$

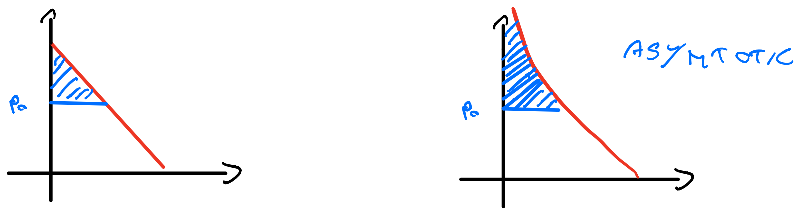
$$\boxed{CS} = -\frac{c^{e+1}}{e+1}$$

  - Under monopoly, $p^m = \frac{c}{1+1/e}$,

We want to computer consumer surplus

$$\boxed{CS_m} = -\frac{\left(\frac{c}{1+1/e}\right)^{e+1}}{e+1}$$

Now we compiute the consumer surplus associated with the demand
q(p) = p a
So this is the area below the demand curve and if we want to compute the consumer surplus. So area between the price p0 and price +infinity (infinity since the demand in this case crosses the vertical axes. If we have an asymptotic case we consider infinite.



If we take the integral of the function p^e.

Welfare Loss of Monopoly

- ***Example*** (continued):
  - Taking the ratio of these two surpluses

$$\boxed{\frac{CS_m}{CS} = \left(\frac{1}{1 + 1/e}\right)^{e+1}}$$

  - If $e = -2$, this ratio is ½
    - CS under monopoly is half of that under perfectly competitive markets

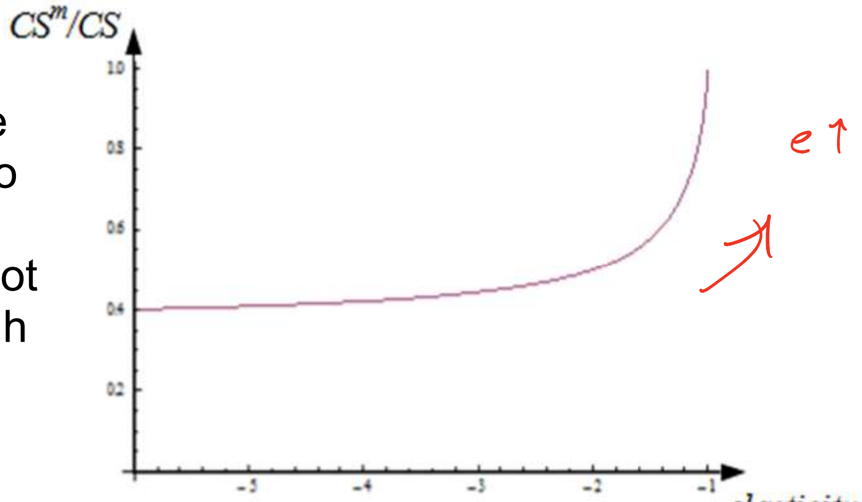
Ratio between Consumer surplus between perfect competition and monopoly increases with elasticity of demand!

Notice that e is negative so it becomes smaller in absolute value.
So if |e| increases (in absolute value) the ratio goes down.

Welfare Loss of Monopoly

- ***Example*** (continued):
  - The ratio $\frac{CS_m}{CS} = \left(\frac{1}{1+1/e}\right)^{e+1}$ decreases as demand becomes more elastic.

If consumer are very sensitive to prices, then monopoly cannot charge very high prices



$e \uparrow \quad \frac{cs^m}{cs} \uparrow$

Welfare Loss of Monopoly

USING ELASTICITY

$p^m = \frac{c}{1+\frac{1}{e}}$

- ***Example*** (continued):
  - Monopoly profits are given by

p^m

$$\pi^m = \underbrace{p^m q^m}_{TR} - \underbrace{c q^m}_{TC} = \left(\frac{c}{1+1/e} - c\right) q^m$$

where $q^m(p) = p^e = \left(\frac{c}{1+1/e}\right)^e$.

  - Re-arranging,

$$\pi^m = \left(\frac{-c/e}{1+1/e}\right)\boxed{\left(\frac{c}{1+1/e}\right)^e}$$

$$= -\left(\frac{c}{1+1/e}\right)^{e+1} \cdot \frac{1}{e}$$

Profits are positive in monopoly!
e is negative

We can also compute the measure of welfare transferred in monopoly to consumer surplus to the produce surplus.

Produce surplus is also called profit.

First thing to do i to compute profits of the monopolist = differences between total revenue and total cost.
We replace to p^m the opt price.

If you want compute the welfare transfer from consumer to producer we can compute the ratio of profit monopoly over consumer surplus.

Welfare Loss of Monopoly

- ***Example*** (continued):
  - To find the transfer from CS into monopoly profits that consumers experience when moving from a perfectly competition to a monopoly, divide monopoly profits $\left(\pi^m = -\left(\frac{c}{1+\frac{1}{e}}\right)^{e+1} \cdot \frac{1}{e}\right)$ by the competitive CS $\left(CS = -\frac{c^{e+1}}{e+1}\right)$

$$\frac{\pi^m}{CS} = \left(\frac{e+1}{e}\right)\left(\frac{1}{1+1/e}\right)^{e+1} = \left(\frac{e}{1+e}\right)^e$$

Decreasing if |e| value of elasticity of demand is increasing

  - If $e = -2$, this ratio is ¼
    - One fourth of the consumer surplus under perfectly competitive markets is transferred to monopoly profits

Welfare Loss of Monopoly

- More social costs of monopoly:
  - Excessive R&D expenditure (patent race)
  - Persuasive (not informative) advertising
  - Lobbying costs (different from bribes)
  - Resources to avoid entry of potential firms in the industry

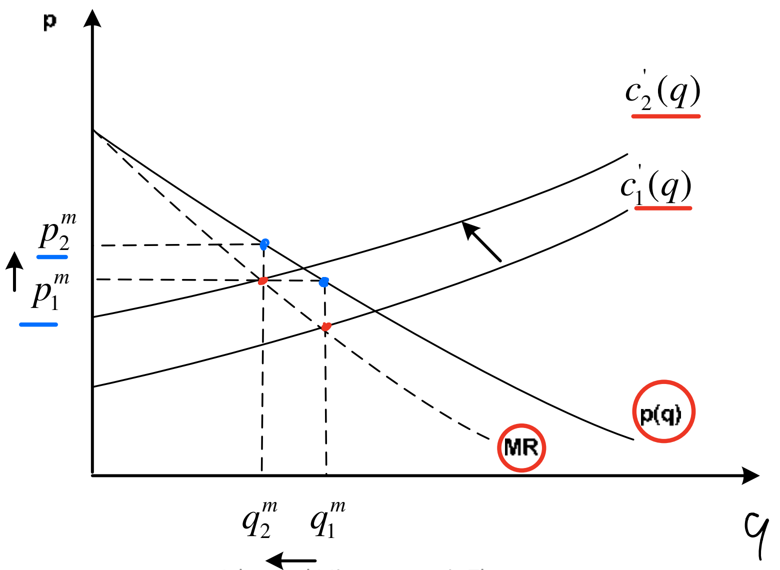
There are not only welfare losses that we have seen, but other potential welfare cost:

- one is what is called patent race (legal barrier) —> excessive R&D expenditure
- Monopolist using a lot of advertising (social network)
- Lobbying costs (cost that some firm may have to persuade to some politician to provide some specifics goal)
- Excessive amount of resource to be used by firm to prevent potential entry and reduce the potential amount of competitor.

Comparative Statics

Comparative Statics

- We want to understand how q^m varies as a function of monopolist's marginal cost



We are interested in understanding how equilibrium quantity in monopoly changes when monopoly marginal cost change.
We have usual graph with quantity and prices.

We have negatively slope demand curve and Marginal revenue curve.
Marginal cost function we have a lower cost function c1 and higher c2. Is immediate to see that going from lower to higher, the optimal quantity of the monopoly falls while the price increases.
Increase in marginal cost imply a shift to the left up of the marginal cost curve and produces an reduction in the optimal quantity of the monopoly and increases in the equilibrium price.

Comparative Statics

FOC PMP

MONOPOLY

$\pi = p(q)q - c(q)$

$q(c)$ $c'(q)=c$

- Formally, we know that at the optimum, $q^m(c)$, the monopolist maximizes its profits

$$\boxed{\frac{\partial \pi(q^m(c), c)}{\partial q^m} = 0}$$

- Differentiating wrt c, and using the chain rule,

COMPOSITE

2° DERIV

$$\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2} \frac{dq^m(c)}{dc} + \frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c} = 0$$

- Solving for $\frac{dq^m(c)}{dc}$, we have

$$\frac{dq^m(c)}{dc} = -\frac{\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}}{\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2}}$$

- In general we trying to get the sign of this number.

We can see the same thing with comparative statics analysis.
We start from the FOC of the profit maximisation problem of the monopolist
FOC implies Derivative of profit with respect to the quantity must be equal to 0 and we also know that profits depends on quantity produced.
The optimal quantity is in term function of marginal cost and profit depend directly on the marginal cost.
Indeed we know that profit are equal to Profit = p(q) * q - c(q).
We assume marginal costs are constant and we are considering how profit change when the c marginal cost changes.
Optimal quantity will be a function of the parameter c in the PMP.

What we do is totally differentiate the FOC with respect to the quantity so we have the 2° derivative of profit with respect to q^m.
Applying the rule of derivative for composite function we have to multiply by derivative of quantity with respect to marginal cost.
Plus, we have the second derivative of profit function so we are doing is computing derivative of profit with respect to q.
Then this must be equal to 0 since FOC was equal to 0.
We isolate the term d q^m/ d c.

Again this is another example of applying the implicit function theorem.

In general we trying to get the sign of this number —> how the optimal quantity change when marginal cost changes.

Comparative Statics

- ***Example***:
  - Assume linear demand curve $p(q) = a - bq$
  - Then, the cross-derivative is

$$\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c} = \frac{\partial \left(\frac{\partial [(a - bq)q - cq]}{\partial q} \right)}{\partial c}$$

$$\frac{\frac{\partial \pi}{\partial q}}{\partial c}$$

MR MC

$$= \frac{\partial [a - 2bq - c]}{\partial c} = -1$$

Denominator

$$\frac{\partial^2 \pi}{\partial q} = -2b$$

and

$$\frac{dq^m(c)}{dc} = -\frac{\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}}{\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2}} = \boxed{-\frac{-1}{-2b} < 0}$$

We can apply implicit function theorem in this example when inverse demand is linear. **Inverse because on the left prices and on the right quantity!**

- Before we compute the numerator:

the double derivative of profit function with respect to q and then to c. What we have to do is to compute der of profit function with respect to q and then derivative with respect to c.

- then we compute the denominator: compute the second derivative of profit function with respect to q.

So substituting the term in the formal we get $-\dfrac{-1}{-2b}$

This is < 0 since by assumption b was > 0 .
We have shown that when MC increase, the optimal quantity changes by
- 1/ 2b.

Comparative Statics

- ***Example*** (continued):
  - That is, an increase in marginal cost, c, decreases monopoly output, q^m.
  - Similarly for any other demand.
  - Even if we don't know the precise demand function, but know the sign of

$$\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}$$

In general you can also find other cases (the one before was with costant marginal cost) by applying other demand functions or other marginal cost function.
To sign the change is enough to sign numerator and denominator of this expression:

$$-\frac{\frac{\partial^2 \pi(q^m(c), c)}{\partial q \partial c}}{\frac{\partial^2 \pi(q^m(c), c)}{\partial q^2}}$$

Sign of variation in optimal quantity.

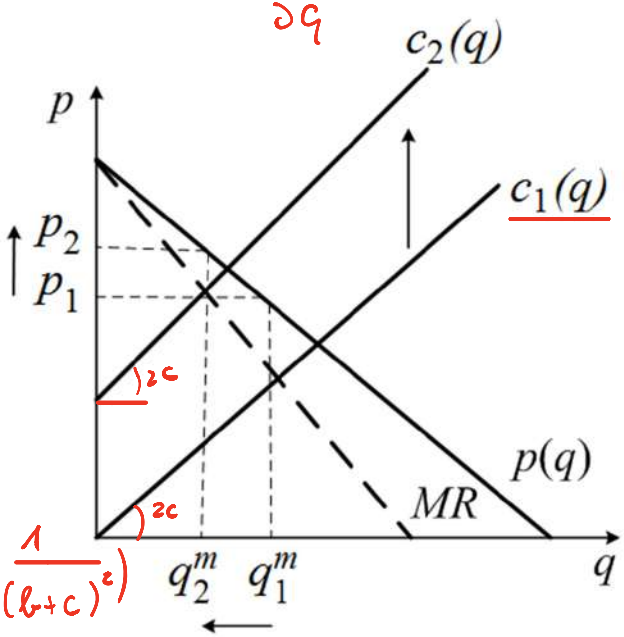
Comparative Statics (DbY)

$c(q) = cq^2$

$\frac{\partial c(q)}{\partial q} = 2cq$

- ***Example*** (continued):
  - Marginal costs are increasing in $\boxed{q}$
  - For convex cost curve $c(q) = cq^2$, monopoly output is
    $$\boxed{q^m(c) = \frac{a}{2(b+c)}}$$
  - Here,
    $$\frac{dq^m(c)}{dc} = -\frac{a}{2(b+c)^2} < 0$$

$\frac{a}{2}\left(-\frac{1}{(b+c)^2}\right)$



This is another example in which marginal cost is not constant and is increasing in quantity. There is a shift upward of the marginal cost so Marginal cost increases. Incidentally increasing in marginal cost correspond to a cost function that is convex. In this case the derivative of total cost with respect to q is equal to 2c q which is increasing in q and slope is 2c.
you can solve the problem (monopolist problem) when the function is this one and with algebra we can verify that optimal quantity is a/2 (b+c).
After having found expression we can compute the derivative of cost function with respect to c and in this case the parameter c. It’s easy to show that derivative is negative.

also in this case the optimal quantity is decreasing in the parameter c and what c determines is the magnitude of the marginal cost. As c increase, also the marginal cost increases.

$c \uparrow \Rightarrow c'(q) \uparrow$

Comparative Statics (DbY)

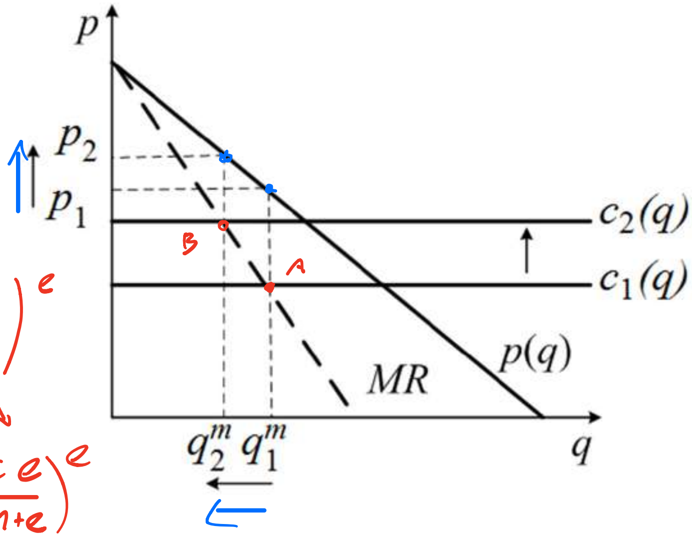
- ***Example*** (continued):
  - Constant marginal cost
  - For the constant-elasticity demand curve $q(p) = p^e$, we have $p^m = \frac{c}{1+1/e}$ and $q^m(c) = \left(\frac{ec}{1+e}\right)^e$

$$\left(\frac{c}{\frac{e+1}{e}}\right)^e$$

$$\left(\frac{ce}{1+e}\right)^e$$

  - Here,

$$\frac{dq^m(c)}{dc} = \frac{e}{c}\left(\frac{ec}{1+e}\right)^e$$
$$= \frac{e}{c} q^m < 0$$



We have a constant marginal cost that is a straight line. If price increases the marginal cost increases and shift to the up.
Equilibrium shift from A to B.
Optimal prices increases and optimal quantity decreases
We know that in monopoly the optimal price can be obtained using the inverse elasticity rule. So p^m = c / (1+1/e)
Demand in this case is constant elasticity demand p^e and is immediate to find also the equilibrium quantity by replacing p^m.

we have found the optimal quantity for monopoly using elasticity constant demand curve. What happen when marginal cost increases? Optimal quantity decreases and what we do is to compute the der of optimal quantity with respect to c.

$$\frac{\delta \frac{ce}{1+e}}{\delta c} = q^m = \left(\frac{e}{1+e}\right)^e \cdot c^e$$

$$\frac{\delta q^m}{\delta c} = \left(\frac{e}{1+e}\right)^e \cdot e\, c^{e-1}$$

YOU CAN WRITE THIS EXPRESSION IN THE FOLLOWING WAY

$$\frac{e}{c}\, q^m < 0 \longrightarrow$$ KNOWING THAT $e < 0$

Multiplant Monopolist

Multiplant Monopolist

N

- Monopolist produces output $\boxed{q_1, q_2, \ldots, q_N}$ across N plants it operates, with total costs $TC_i(q_i)$ at each plant $i = \{1,2, \ldots, N\}$. Demand $\underline{p = a - bQ}$ → LINEAR
- Profits-maximization problem Q

$$\max_{q_1,\ldots,q_N} \overbrace{\left[a - b\sum_{i=1}^{N} q_i\right]}^{Q} \underbrace{\sum_{i=1}^{N} q_i}_{TR} - \underbrace{\sum_{i=1}^{N} TC_i(q_i)}_{TC}$$

MULTIPLY TERMS

$a - b\sum q_i \cdot TR$

$\Downarrow$

$a\sum_i q_i - b\left(\sum_i q_i\right)^2$

- FOCs wrt production level at every plant j

$\frac{\partial \Pi}{\partial q_j}$

$$a - 2b\sum_{i=1}^{N} q_i - MC_j(q_j) = 0$$

$$\Leftrightarrow MR(Q) = MC_j(q_j)$$

for all j.

→ N FOC → ONE FOR EACH J

Multiplant monopoly is a monopolist firm that has several plants so this mean that the firm has to decide how to allocate production among different plants. So we have in this case a monopolistic firm which produce N different plants. We have quantity allocated which each of n plants. This plants has different TC function. Then we have an inverse demand which is linear.
We have to set up the PMP for the monopolist:

- we have total revenue as usual that is equal to prices which is equal to a-b Q and can be equal to the summation of production of all plants.

Then prices multiply by Q. Then we have the cost part: which are the sum of the total cost corresponding to the production allocate to each plan.

To find the max we have to compute the FOC with respect to the production of each plant 'j'.
Derivative of profit function then, we have to take in consideration index i and j. 'i' when we refer to all N plants and j is the quantity of the plants with respect to which we are computing the FOC. We will. Have N FOC —> one for each j

Compute derivative of profit function with respect to qj.

MULTIPLY TERMS $\rightarrow$ $\left[a - b\sum_{i=1}^{N} q_i\right]\sum_{i=1}^{N} q_i$

$$a \sum_i q_i - b\left(\sum_i q_i\right)^2$$

NOW EASIER COMPUTING DERIVATIVE

$\parallel$ IN CASE OF TWO PLANTS

$$a\,(q_1+q_2) - b\,(q_1+q_2)^2$$

$\parallel$

$$a - b\,2\,(q_1+q_2)$$

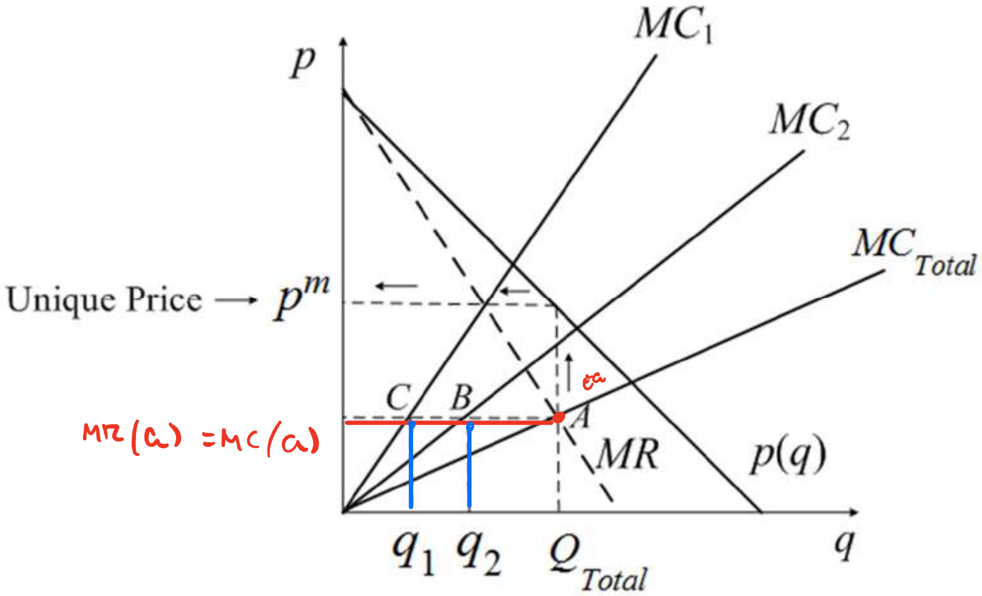
AS THE GENERAL CASE

$$a - b\,2\sum_i^N q_i$$

Then we have to compute the total cost derivative with respect to qj which is equal to - MCj. Derivative = 0 and we can rewrite at the **optimal choice** MR(Q) = MC(qj) moving marginal cost on the right-hand side.

Multiplant Monopolist

- Multiplant monopolist operating two plants with marginal costs MC_1 and MC_2.



This is the graph of equilibrium in multiplant monopoly. In this case we have the usual MR negatively sloped (decreasing), Mc of first and second plants. Total marginal cost is the sum between MC1 e MC2. The idea is to sum the quantity q1 and q2 offered at the given MC to find the total quantity that is offered at the MC. Ones you have obtain the total MC you are able to find the equilibrium: MC must be equal to the MR.
After finding the equilibrium you will also have also the equilibrium price and importantly you can find optimal production allocated to each plant just reading at the equilibrium MC = MR.
Looking at this two you can find the quantity allocated for 1° and 2° plant (linee azzurre) —> crossing point between MC functions and equilibrium total MC which is equal to the MR.

This is the description of the figure before

Multiplant Monopolist

- Total marginal cost is $MC_{total} = MC_1 + MC_2$ (i.e., horizontal sum)
- Q_{total} is determined by $MR = MC_{total}$ (i.e., point *A*)
- Mapping Q_{total} in the demand curve, we obtain price p^m (both plants sell at the same price)
- At the MC level for which $MR = MC_{total}$ (i.e., point *A*), extend a line to the left crossing MC_1 and MC_2.
- This will give us output levels q_1 and q_2 that plants 1 and 2 produce, respectively.

Exercise!

Multiplant Monopolist

- ***Example 1*** (symmetric plants):
  - Consider a monopolist operating N plants, where all plants have the *same* cost function $TC_i(q_i) = F + cq_i^2$. Hence, all plants produce the same output level $q_1 = q_2 = \cdots = q_N = q$ and $Q = Nq$ The linear demand function is given by $p = a - bQ$.
  - FOCs:

$$a - 2b\sum_{j=1}^{N} q = 2cq \quad \text{or} \quad a - 2bNq = 2cq$$

$$q = \frac{a}{2(bN + c)}$$

Multiplant Monopolist

- ***Example 1*** (continued):
  - Total output produced by the monopolist is
$$Q = Nq = \frac{Na}{2(bN + c)}$$
    and market price is
$$p = a - bQ = a - b\frac{Na}{2(bN + c)} = \frac{a(bN + 2c)}{2(bN + c)}$$
  - Hence, the profits of every plant j are $\pi_j = \frac{a^2}{4(bN+c)} - F$, with total profits of
$$\pi_{total} = \frac{Na^2}{4(bN + c)} - NF$$

Multiplant Monopolist

- ***Example 1*** (continued):
  - The optimal number of plants N^* is determined by
  $$\frac{d\pi_{total}}{dN} = \frac{a^2}{4}\frac{c}{(bN + c)^2} - F = 0$$
  and solving for N
  $$N^* = \frac{1}{b}\left(\frac{a}{2}\sqrt{\frac{c}{F}} - c\right)$$
  - N^* is decreasing in the fixed costs F, and also decreasing in c, as long as $a < 4\sqrt{cF}$.

Multiplant Monopolist

- ***Example 1*** (continued):
  - Note that when $N = 1$, $Q = q^m$ and $p = p^m$.
  - Note that an increase in N decreases q_j(=q) and π_j.

Multiplant Monopolist (DbY)

- ***Example 2*** (asymmetric plants): 7.6 book
  - Consider a monopolist operating two plants with marginal costs $MC_1(q_1) = 10 + 20q_1$ and $MC_2(q_2) = 60 + 5q_2$, respectively. A linear demand function is give by $\boxed{p(Q) = 120 - 3Q}$.
  - Note that $MC_{total} \neq MC_1(q_1) + MC_2(q_2)$
    - This is a vertical (not a horizontal) sum.
  - Instead, first invert the marginal cost functions

$$MC_1(q_1) = 10 + 20q_1 \Leftrightarrow q_1 = \frac{MC_1}{20} - \frac{1}{2}$$

$$MC_2(q_2) = 60 + 5q_2 \Leftrightarrow q_2 = \frac{MC_2}{5} - 12$$

Asymmetric plants: different plants have different cost functions

Symmetric plants: All plants have the same total cost function.

In this example we have two plants and MC1 is 10+20q, MC2 is 120-3Q

Find solution of the problem: find total marginal cost function: find MC1 and MC2.
IMPORTANT: to find total marginal cost function since we have to sums MC horizontally we cannot just compute the sums.
We have to explicitate the two expression with respect to quantity and then sum up the two quantities.
So we have to put quantity on the left-end side.

After having found the to MC with respect to quantity we can sum up the two quantity

Multiplant Monopolist (DbY)

- ***Example 2*** (continued): $MC_1 = MC_2 = MC$
  - Second,

$$Q_{total} = q_1 + q_2 = \frac{MC_{total}}{20} - \frac{1}{2} + \frac{MC_{total}}{5} - 12$$
$$= \frac{1}{4}MC_{total} - 12.5$$

  - Hence, $\boxed{MC_{total}} = 50 + 4Q_{total}$
  - Setting $MR(Q) = MC_{total}$, we obtain $Q_{total} = 7$ and $p = 120 - 3 \cdot 7 = 99$.
  - Since $MR(Q_{total}) = 120 - 6 \cdot 7 = 78$, then

$$MR(Q_{total}) = MC_1(q_1) \Longrightarrow 78 = 10 + 20q_1 \Longrightarrow q_1 = 3.4$$
$$MR(Q_{total}) = MC_2(q_2) \Longrightarrow 78 = 60 + 5q_2 \Longrightarrow q_2 = 3.6$$

After having found the to MC with respect to quantity we can sum up the two quantity.
We sum q1 + q2 and we know that in equilibrium MC in the two plants must be the same: so we replace MC1 and MC2 with MCtotal. Then we can sum up the terms.

Then explicitate this expression with the marginal cost so then we can equate MR to the MC total. We find MC total is 50 + 4 Qtot and then we can equate MC tot and MR.
MR can be found just looking at the demand function p(Q) = 120-3Q and then computing the marginal revenue function.
We will find Total quantity in monopoly (Qtot = 7) and equilibrium price by replacing in the demand (p = 99) and also you can find MR(Qtot) by replace quantity in the MR.
Also to find the quantity produced by the two plant is to equate MR to the marginal cost of the first plant and you can find the quantity in the 1° plant and Idem for the 2° plant.

Now an alternative solution for this exercise

$$\max_{q_1, q_2} \Pi = \overset{120-3Q}{P} \cdot Q - TC(q_1) - TC(q_2) =$$

$$= (120 - 3Q)Q - TC(q_1) - TC(q_2) =$$

$$Q = q_1 - q_2$$

$$= [120 - 3(q_1 - q_2)](q_1 - q_2) - TC(q_1) - TC(q_2)$$

$$= 120(q_1 + q_2) - 3(q_1 + q_2)^2 - TC(q_1) - TC(q_2)$$

COMPUTE $\frac{\partial \pi}{\partial q_1}$ AND $\frac{\partial \pi}{\partial q_2}$

$$\frac{\partial \pi}{\partial q_1} = \underbrace{120 - 6(q_1+q_2)}_{MR} - \underbrace{10 - 20q_1}_{MC_1} = 0$$

$$\frac{\partial \pi}{\partial q_2} = \underbrace{120 - 6(q_1+q_2)}_{MR} - \underbrace{60 - 5q_2}_{MC_2} = 0$$

REWRITE AS EQUALITY BETWEEN MC AND MR

$$\underbrace{120 - 6Q}_{MR} = \underbrace{10 + 20q_1}_{MC_1}$$

$$\underbrace{120 - 6Q}_{MR} = \underbrace{60 + 5q_2}_{MC_2}$$

SINCE LEFT SIDE EQUAL, ALSO RIGHT SIDE MUST BE EQUAL

RELATIONSHIP q_1, q_2 AT OPTIMAL ALLOCATION OF PRODUCTION AMONG THE TWO PLANTS

$$\underbrace{10 + 20q_1}_{MC_1} = \underbrace{60 + 5q_2}_{MC_2} \longrightarrow q_1 = \frac{50 + 5q_2}{20}$$

FOC $MR = MC_2$ $\longrightarrow$ REPLACE q_1 IN THIS CONDITION

$$\underbrace{120 - 6Q}_{MR} = \underbrace{60 + 5q_2}_{MC_2}$$

$$120 - 6\underbrace{\left(q_2 + \overbrace{\frac{50 + 5q_2}{20}}^{q_1}\right)}_{Q} = 5q_2$$

$$60 - 6\left(\frac{20q_2 + 50 + 5q_2}{20}\right) = 5q_2$$

$$60 - 6\left(\frac{25q_2 + 50}{20}\right) = 5q_2$$

$$60 - \frac{\cancel{30}^{15}}{\cancel{4}_{2}} q_2 - 15 = 5q_2 \qquad 60 - 15 = \frac{15}{2} q_2 + 5q_2$$

$$45 = q_2\left(5 + \frac{15}{2}\right)$$

$$45 = q_2 \quad \frac{25}{2} \qquad q_2 = \frac{90}{25} = 3.6$$

NOW WE CAN ALSO FIND q_1

$$q_1 = \frac{50 + 5 \cdot 3{,}6}{20} = 3.4$$

THEN ALSO TOTAL QUANTITY

$$Q^m = 3.6 + 3.4 = 7$$

$$P^m = 120 - 3Q = 99$$

Advanced Microeconomic Theory

Chapter 7: Monopoly price discrimination

Outline

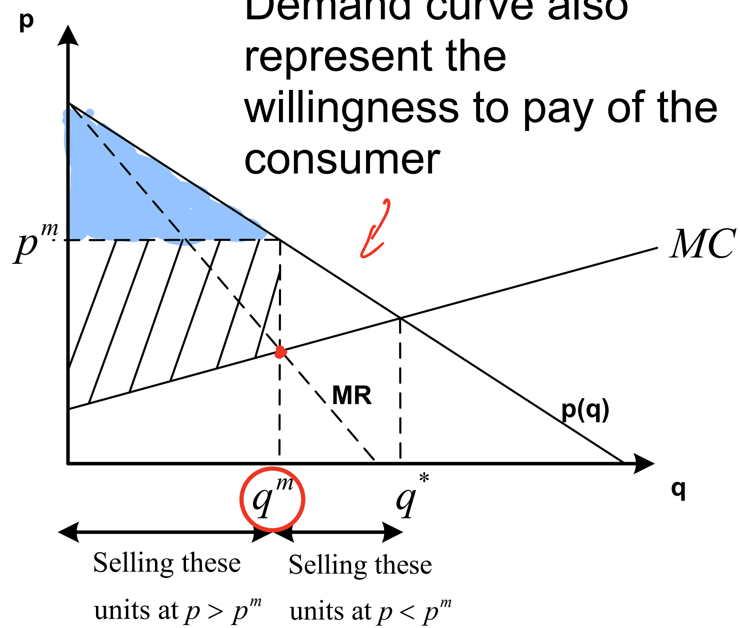
- Price Discrimination

Price Discrimination

Price Discrimination

- Can the monopolist capture an even larger surplus?
  - Charge $p > p^m$ to those who buy the product at p^m and are willing to pay more
  - Charge $c < p < p^m$ to those who do not buy the product at p^m, but whose willingness to pay for the good is still higher than the marginal cost of production, c.
  - With p^m for all units, the monopolist does not capture the surplus of neither of these segments.

Demand curve also represent the willingness to pay of the consumer



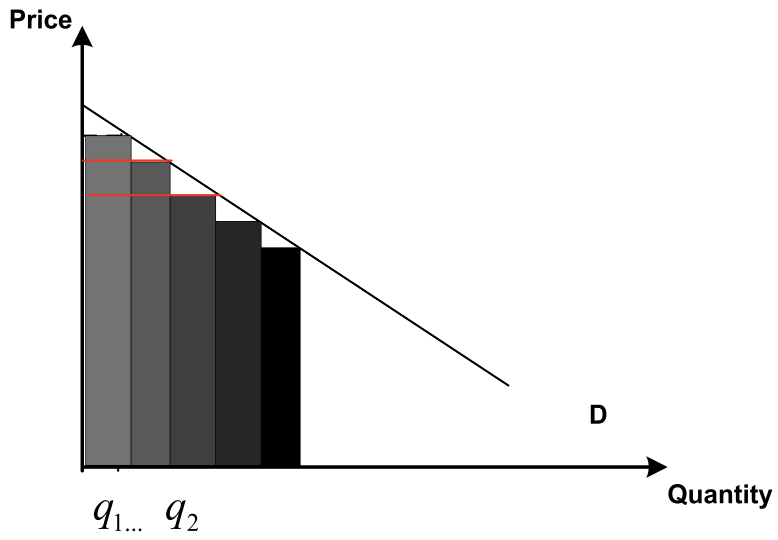
We have seen solution of monopoly profit maximisation problem and at the opt condition the monopolist produce at a point in which MR crosses MC. In this graph we have the demand curve negatively sloped.

This optimal price and quantity price for the monopolist will be the shade area which is below equilibrium price and above MC curve.

however this is not the maximum profit the monopolist might get, some consumer will be will to pay a price higher than p^m. In particular, this can be seen by looking at this portion of the demand.
Demand curve also represent the willingness to pay of the consumer.
In equilibrium price is equal to the marginal utility of consumption $p = u'(q)$.
Maximum price individual willing to pay is exactly the marginal utility.
The monopolist could get higher profit just charging higher price (blue area).
It also the case monopolist could get higher profit by charging prices that are above MC curve.
The idea is that the monopolist could gain extra profit by charging different prices to different consumer.

Price Discrimination: First-degree

- ***First-degree (perfect) price discrimination***:
  - The monopolist charges to every customer his/her maximum willingness to pay for the object.
  - *Personalized price*: The first buyer pays p_1 for the q_1 units, the second buyer pays p_2 for $q_2 - q_1$ units, etc.



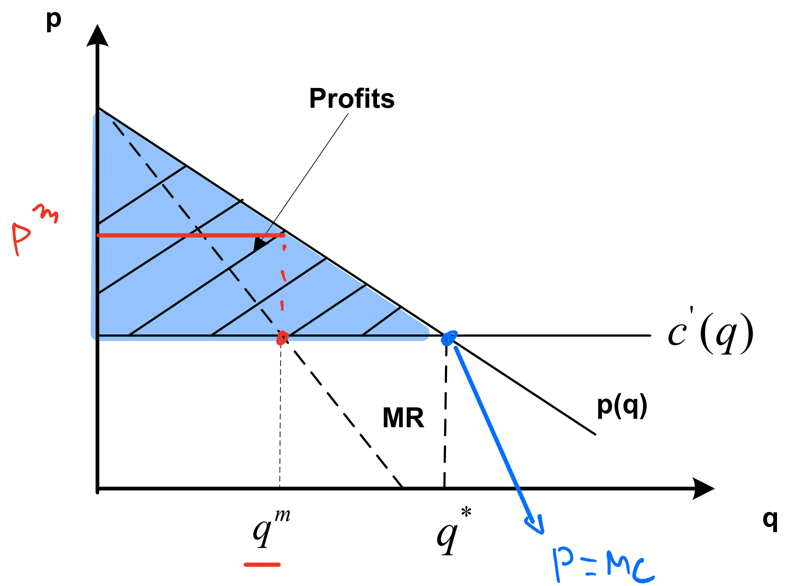
In case monopolist is able to charge to each consumer is willingness to pay —> 1° degree (perfect) price discrimination
The monopolist charges to every customer his maximum willingness to pay for the good.

So this is a visual representation in which the monopolist charges a certain price based on the unit q1, q2 ecc..

Is called also block pricing: for each block of quantities the monopolist change different prices. We assume that each unit charge different price we get the situation in which each block is a point in the demand curve.

Price Discrimination: First-degree

- The monopolist continues doing so until the last buyer is willing to pay the marginal cost of production.
- In the limit, the monopolist captures all the area below the demand curve and above the marginal cost (i.e., consumer surplus)



if you take this extreme case with perfect price discrimination. Monopolist can really capture all consumer surplus.
We have MR, demand negatively sloped and we assume MC are constants.
In the traditional solution will be crossing point between MR and Mc fucntion.
In the usual case we have p^m and q^m. But in the case of 1° the monopolist will produce up to the point in which the MC crosses the demand curve (p = MC). Now this is not the equilibrium price, each consumer or each quantity will be charge a single price.
The profits for the monopolist are gain by the are below the demand and above the MC curve (blue)
In perfect competition this would be also the consumer surplus(CS).
The monopolist can capture all consumer surplus and this became the monopolist profits.

In summary:
Perfect competition CS becomes the monopolist profit and equilibrium production with 1° degree is the same as in perfect competition. Indeed, in Perfect competition the equilibrium is characterise between the equality between price and MC (p = MC).

Price Discrimination: First-degree

F^*

- Suppose that the monopolist can offer a fixed fee, r^*, and an amount of the good, q^*, that maximizes profits.
- $\boxed{\text{PMP:}}$

Π $\qquad \frac{\partial TC}{\partial q} = c$

$$\max_{r,q} \left[r - cq \right]$$

$TC = cq$

$$s.t.\ \ u(q) \geq r$$

- Note that the monopolist raises the fee r until $u(q) = r$. Hence we can reduce the set of choice variables

$$\max_{q}\ \ u(q) - cq$$

- FOC: $\quad u'(q^*) - c = 0$ or $u'(q^*) = c$.
  - *Intuition*: monopolist increases output until the marginal utility that consumers obtain from additional units coincides with the marginal cost of production

Let’s do this analitically.
Imagine the monopolist can offered a fixed fee r* and a given amount of quantity q* that maximise the profits.
The problem of the monopolist —> have to offer a combination of fee and quantity of the good.
This is an example of mobile company that offer a given amount of Giga at a given price.
PMP for the monopolist firm is to maximise with respect to the choice variable (the fee and quantity).
We assume MC = c, so constant.
The constrain the monopolist faces is that utility of consumer must be greater of equal to the fee he paid (r).
Since profit is increasing in r and also constraint must hold, for the monopolist will be convenient to rise the fee to the point in which utilty of the consumer is equal to the fee.
In this condition the consumer will still buy the good: after having observed the constraint hold with equality, we can replace the constrain in the objective function so we replace into r an we get a new function of q—> u(q) =r.
So we can compute the FOC and we get that marginal utility u’ = c.
The monopolist will increase output until Marginal utility will be equal to the marginal cost.

Price Discrimination: First-degree

$p = u'(q)$

$$\boxed{r^*} = \int_0^{q^*} u'(q)dq = u(q^*) - u(0)$$

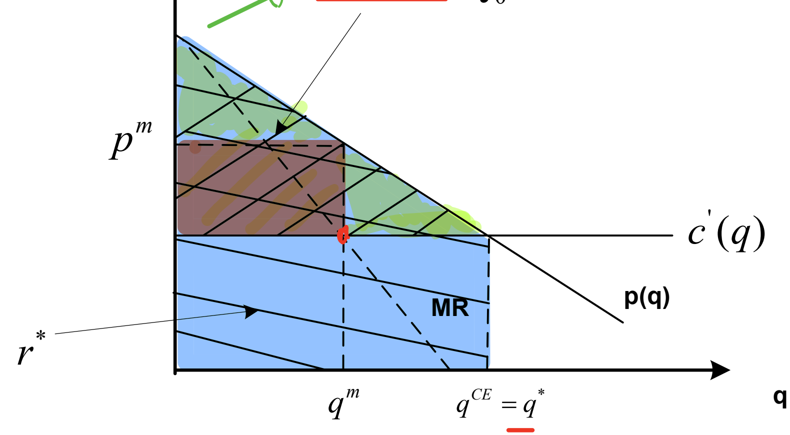
$cs = \tilde{\pi} = r^* - c q^*$

- Given the level of production q^*, the optimal fee is

$$r^* = u(q^*)$$

- *Intuition*: the monopolist charges a fee r^* that coincides with the utility that the consumer obtains from q^*

$$\boxed{\text{Profits}} = \int_0^{q^*} p(q)dq - c(q^*)$$



Graphical representation of solution of the problem. We can pick the profit in case of the traditional solution
Below the equilibrium price and above the Mc function.

While in case of 1° price discrimination the fee will be equal to the utilty of getting q* (which is define at u'(q*) = r) the max at the optimum of the constrain must be binding. So at the optimal the r* that coincide with utility that consumer obtains from q*.
r* represent the area below the demand curve and up to the point in which q = q*.
Is area below demand function with quantity 0 to q*.
By solving the integral we will have the integral of marginal utilty which is the utility function itself.

Profits are the area between r* - c q* so difference between the area of the trapezoids - c q* area.
Profits are consumer surplus in perfect competition.

Price Discrimination: First-degree

- ***Example***:
  - A monopolist faces inverse demand curve $p(q) = 20 - q$ and constant marginal costs $c = \$2$.
  - No price discrimination:

$$MR = MC \implies 20 - 2q = 2 \implies q^m = 9$$
$$p^m = \$11, \qquad \pi^m = \$81$$

  - Price discrimination:

$$p(Q) = MC \implies 20 - Q = 2 \implies Q = 18$$
$$\pi = \$162$$

Example 7.7 book

INVERSE FUNCTION

$P(q) = 20 - q \qquad c = 2$

② TRADITIONAL SOLUTION PMP

$\pi = TR - TC = p \cdot q - c\overset{=2}{q} = (20 - q)\, q - 2q$

FOC $\dfrac{\partial \pi}{\partial q} = \underbrace{20 - q}_{MR} = \underbrace{2}_{MC}$

MR HAS THE SAME INTERSEPT FUNCTION (I.E. 20) OF INVERSE FUNCTION, ALSO SLOPE DOUBLED (2q)

$q^M = \dfrac{18}{2} = 9 \qquad p^M = 20 - 9 = 11$

$\pi^M = 11 \cdot 9 - 2 \cdot 9 = 9 \cdot 9 = 81$

APPLY UNIFORM PRICE

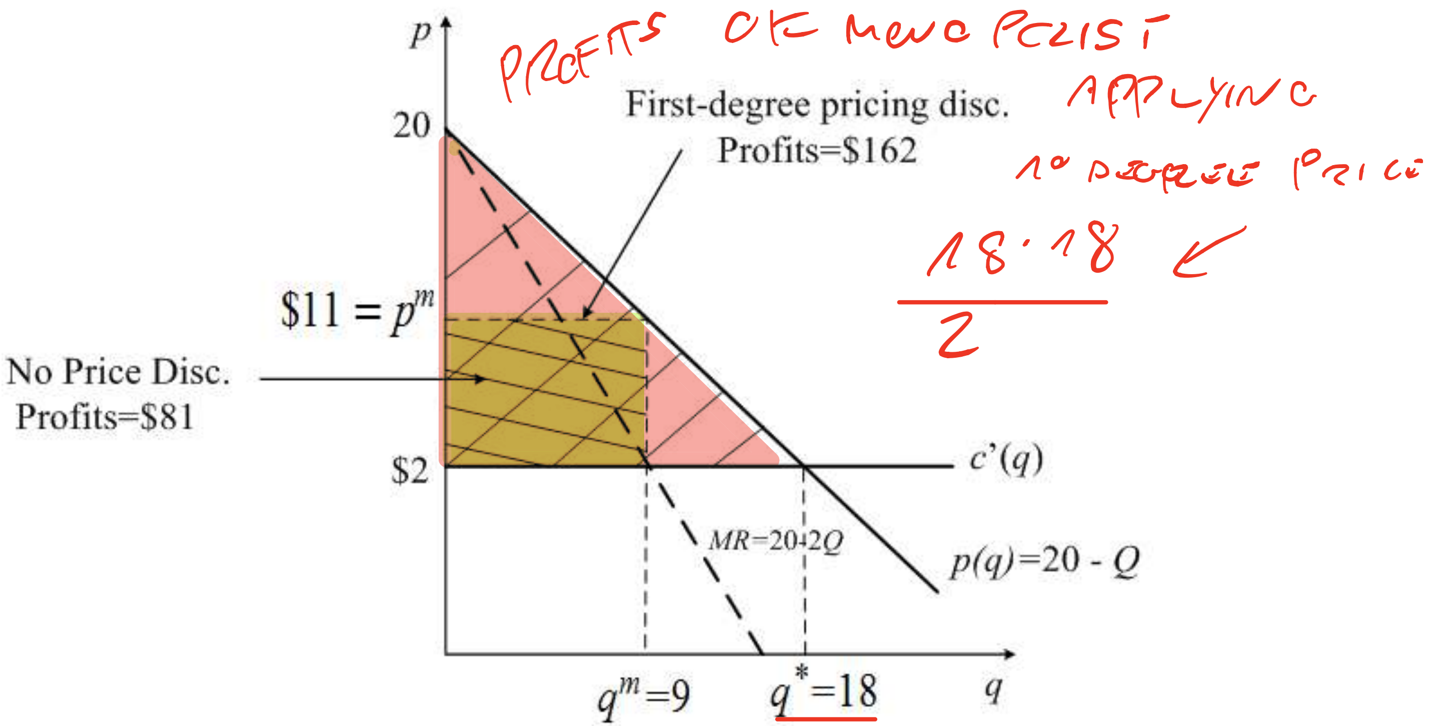
② CHECK SOLUTION IN CASE OF 1° DEGREE PRICE DISCRIMINATION

(SEE PICTURE BELOW)

$$\pi_{1ST}^{M} = \frac{18 \cdot 18}{2} = 162$$

Price Discrimination: First-degree

- ***Example*** (continued):



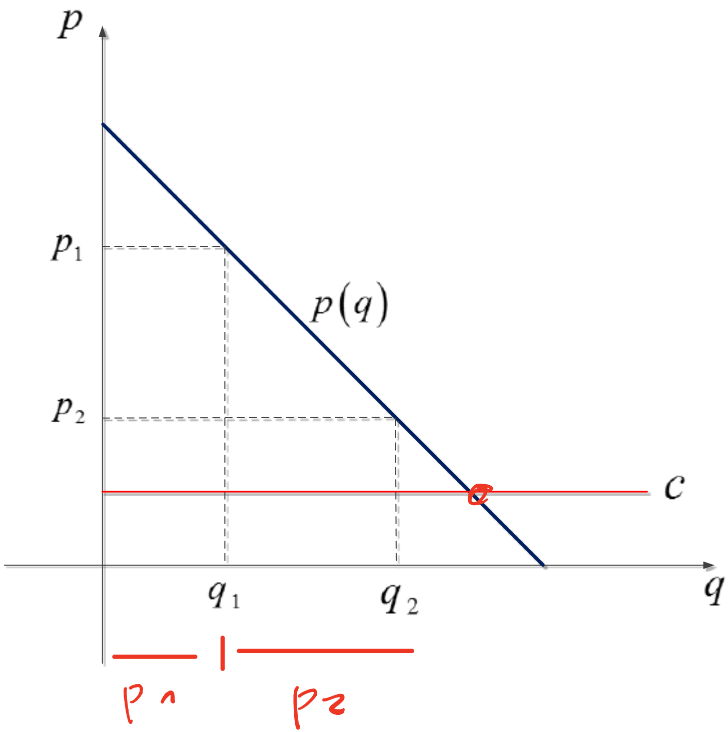
Price Discrimination: First-degree

- Summary:
  - Total output coincides with that in perfect competition
  - Unlike in perfect competition, the consumer does not capture any surplus
  - The producer captures all the surplus
  - Due to information requirements, we do not see many examples of it in real applications
    - Financial aid in undergraduate education ("tuition discrimination")

PRODUCER SHOULD KNOW WILLINGNESS TO PAY OF THE CONSUMER

Price Discrimination: First-degree

- ***Example*** (two-block pricing):
  - A monopolist faces a inverse demand curve $\boxed{p(q) = a - bq}$, with constant marginal costs $c < a$.
  - Under two-block pricing, the monopolist sells the first q_1 units at a price $p(q_1) = p_1$ and the remaining $q_2 - q_1$ units at a price $p(q_2) = p_2$.



Another example is the two block pricing: example of **imperfect 1° degree price discrimination.**
Assume that monopolist is facing an inverse demand curve which is linear and we assume MC constant and below intercept demand curve (this assure equilibrium and a crossing point).
In general there will be n-block pricing. or the cost q unit firm apply price p1, for unit between q2 and q1 the firm apply p2. In this sense, the monopolist try to apply different prices to different consumers. Consumer is like divided in two groups.
We can apply the same thing of before to this groups.

Price Discrimination: First-degree

- ***Example*** (continued):
  - Profits from the first q_1 units

$$\pi_1 = p(q_1) \cdot q_1 - cq_1 = (a - bq_1 - c)q_1$$

while from the remaining $q_2 - q_1$ units

$$\pi_2 = p(q_2) \cdot (q_2 - q_1) - c \cdot (q_2 - q_1)$$
$$= (a - bq_2 - c)(q_2 - q_1)$$

  - Hence total profits are

$$\pi = \pi_1 + \pi_2$$
$$= (a - bq_1 - c)q_1 + (a - bq_2 - c)(q_2 - q_1)$$

Example 7.8 in the book —> 2 block pricing

$$P(q) = a - bq \qquad c'(q) = c \qquad a > c > 0$$

$$P(q_1) = p_1 \qquad P(q_2) = p_2$$

q_1 $\qquad$ $q_2 - q_1$

$$\max \pi = \pi_1 + \pi_2$$

Profits of 1° Block consumers

Profits of 1° Block consumers

$$\pi_1 = p(q_1)\,q_1 - c\,q_1 = (p(q_1) - c)\,q_1 = ((a - bq_1) - c)\,q_1$$

$$\pi_2 = p(q_2)(q_2 - q_1) - c(q_2 - q_1) = (p(q_2) - c)(q_2 - q_1) =$$
$$= (a - b q_2 - c)(q_2 - q_1)$$

$$\pi = \pi_1 + \pi_2 =$$
$$= (a - bq_1 - c)\,q_1 + (a - bq_2 - c)(q_2 - q_1)$$

Now maximise this

FOC $$\frac{\partial \pi}{\partial q_1} = \cancel{a} - bq_1 - \cancel{c} - \cancel{a} + bq_2 + \cancel{c} = 0$$

Interior solution

$$q_2 = \frac{2bq_1}{b} = 2q_1$$

FOC $\dfrac{\partial \pi}{\partial q_2} = -b(q_2 - q_1) + (a - bq_2 - c) = 0$

$-bq_2 + bq_1 + a - bq_2 - c = 0$

SEPARATING q_1 AND q_2

$-2bq_2 = a - c + bq_1 \qquad q_2 = \dfrac{a - c + bq_1}{2b}$

EQUATE TO FIND q_1

$$\frac{a - c + bq_1}{2b} = 2q_1$$

$$a - c + bq_1 = 4bq_1$$

$$3bq_1 = a - c \qquad q_1 = \frac{a-c}{3b}$$

$$q_2^* = \frac{2(a-c)}{3b}$$

$(q_2^* - q_1^*)$ BUT ALSO SUBSTITUTE q_1^* IN INVERSE DEMAND

$$p^*(q_1) = a - bq_1^* = a - b\,\frac{a-c}{3b} = \frac{3ab - ab + bc}{3b} =$$

$$= \frac{2ab + 3c}{3b} = \frac{\cancel{b}(2a+c)}{3\cancel{b}} = \frac{2a+c}{3}$$ OPTIMAL PRICE FOR q_1

OPTIMAL PRICE FOR q_2

$$p^*(q_2) = a - b\,\frac{2(a-c)}{3b} = \frac{3ab - 2ab + 2bc}{3b} =$$

$$= \frac{ab + 2bc}{3b} = \frac{b(a+2c)}{3b} = \frac{a+2c}{3}$$

FIND PROFIT IN 1° BLOCK AND 2° BLOCK

$$\pi_1 = \left(p(q_1^*) - c\right) q_1^* = \left(\frac{2a+c}{3} - c\right) \underbrace{\frac{a-c}{3b}}_{q_1} =$$

$$\frac{2a+c-3c}{3}\;\frac{a-c}{3b} = \frac{2a-2c}{3}\cdot\frac{a-c}{3b} = \frac{2(a-c)^2}{9b}$$

$$\pi_2 = \left(p(q_2^*) - c\right)(q_2^* - q_1^*) = \left(\frac{a+2c}{3} - c\right)\left(\frac{2(a-c)}{3b} - \frac{a-c}{3b}\right) =$$

$$= \frac{a+2c-3c}{3}\cdot\frac{a-c}{3b} = \frac{(a-b)^2}{9b}$$

$$\pi = \pi_1 + \pi_2 = \frac{3(a-c)^2}{9b} = \frac{(a-c)^2}{3b}$$

TOTAL PROFIT

YOU CAN ONE OR THIS PROFITS LARGER THAN THE PROFIT WITH UNIFORM PRICE

SO COMPUTE PROFIT IN UNIFORM PRICE

Price Discrimination: First-degree

- ***Example*** (continued):
  - FOCs:
$$\frac{\partial \pi}{\partial q_1} = a - 2bq_1 - c - a + bq_2 + c = 0$$
$$\frac{\partial \pi}{\partial q_2} = -b(q_2 - q_1) + a - bq_2 - c = 0$$
  - Solving for q_1 and q_2
$$q_1 = \frac{a-c}{3b} \qquad q_2 = \frac{2(a-c)}{3b}$$
which entails prices of
$$p(q_1) = a - b \cdot \frac{a-c}{3b} = \frac{2a+c}{3} \qquad p(q_2) = \frac{a+2c}{3}$$
where $p(q_1) > p(q_2)$ since $a > c$.

Price Discrimination: First-degree

- ***Example*** (continued):
  - The monopolist's profits from each block are

$$\pi_1 = (p(q_1) - c) \cdot q_1$$
$$= \left(\frac{2a+c}{3} - c\right) \cdot \frac{a-c}{3b} = \frac{2}{b}\left(\frac{a-c}{3}\right)^2$$

$$\pi_2 = (p(q_2) - c)(q_2 - q_1)$$
$$= \left(\frac{a+2c}{3} - c\right) \cdot \left(\frac{2(a-c)}{3b} - \frac{a-c}{3b}\right) = \frac{1}{b}\left(\frac{a-c}{3}\right)^2$$

  - Thus, $\pi = \pi_1 + \pi_2 = \frac{(a-c)^2}{3b}$, which is larger than those arising under uniform pricing , $\pi^u = \frac{(a-c)^2}{4b}$.

Price Discrimination: Third-degree

- ***Third degree price discrimination***:
  - The monopolist charges different prices to two or more groups of customers (each group must be easily recognized by the seller).
    - *Example*: youth vs. adult at the movies, airline tickets
  - Firm's PMP:

$$\max_{x_1,x_2} \; \overset{TR_1}{p_1(x_1)x_1} + \overset{TR_2}{p_2(x_2)x_2} - \underset{TC_1}{cx_1} - \underset{TC_2}{cx_2} \qquad \frac{\partial TC_1}{\partial x_1}=c$$

  - FOCs: $\overset{MR}{}$

$$\frac{\partial TC_2}{\partial x_2}=c$$

$$p_1(x_1) + p_1'(x_1)x_1 - c = 0 \implies MR_1 = MC$$
$$p_2(x_2) + p_2'(x_2)x_2 - c = 0 \implies MR_2 = MC$$

  - FOCs coincides with those of a regular monopolist who serves two completely separated markets practicing uniform pricing .

In this case, unlike the case with 1° degree price discrimination (monopolist has perfect information of each consumer and silliness to pay) there are different group of customer that can be recognise by the seller.

Now the PMP is to maximise profits with respect to quantity sold in first and second market which is define by different group of consumer.
We assume marginal cost is costant.

By computing the FOC we obtain that the MR - MC = 0, so is the same as optimal choice. So MR1 = MC
FOC with respect to x2 we obtain a similar condition MR2 = MC.
The solution of this problem can be found splitting the problem in two problem:

- $\max_{x_1} \Pi_1$
- $\max_{x_2} \Pi_2$

We obtain the same result as finding the maximum of the total profit. this is also shown in the example [next slide]

Price Discrimination: Third-degree

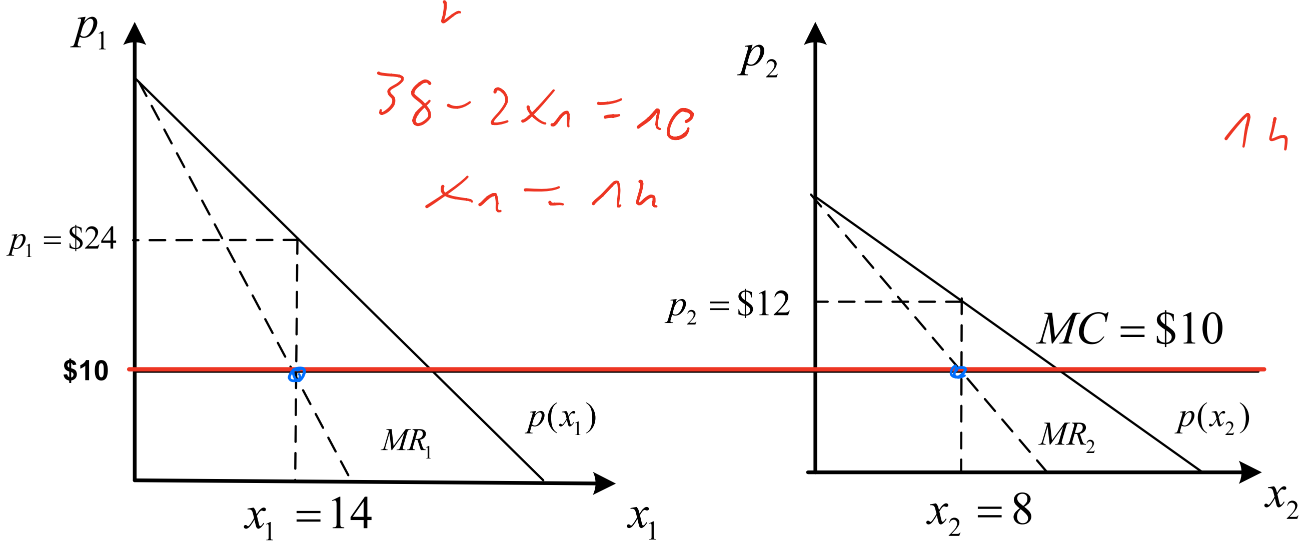
– ***Example***: $p_1(x_1) = 38 - x_1$ for adults and $p_2(x_2) = 14 - 1/4x_2$ for seniors, with $\boxed{MC = \$10}$ for both markets.

$$MR_1(x_1) = MC \implies 38 - x_1 = 10 \implies x_1 = 14 \quad p_1 = \$24$$
$$MR_2(x_2) = MC \implies 14 - 1/4x_2 = 10 \implies x_2 = 8 \quad p_2 = \$12$$

$38 - 2x_1 = 10$

$x_1 = 14$

$14 - \frac{1}{4}x_2$



Market 1
Adults at the movies

Market 2
Seniors at the movies

We have two markets characterised by two inverse demand function, price in 1° market and 2° market.

The equilibrium in the first market can be found equation MR with MC. Compute MR also for second market.

$$P_1 = 38 - x_1 \qquad MC = 10$$

$$38 - 2x_1 = 10 \qquad x_1 = 14 \qquad P_1 = 24$$

$$P_2 = 14 - \underbrace{\frac{1}{4} x_2 \cdot 2}_{MR} = \underbrace{10}_{MC}$$

$$14 - \frac{1}{2} x_2 = 10 \qquad x_2 = 8 \qquad P_2 = 12$$

We have found opt quantity and price for both market and we can see it in the graph.

In both cases the optimal choice is the crossing point between MR and MC curve. So monopolist sell 14 with 24 price in first and 8 in the second with 12 as price.

Price Discrimination: Third-degree

- Using the Inverse Elasticity Pricing Rule (IERP), we can obtain the prices

$$p_1(x_1) = \boxed{\frac{c}{1-1/\varepsilon_1}} \text{ and } p_2(x_2) = \frac{c}{1-1/\varepsilon_2}$$

  where c is the common marginal cost

- Then, $p_1(x_1) > p_2(x_2)$ if and only if

$$\frac{c}{\boxed{1-1/\varepsilon_1}} > \frac{c}{\boxed{1-1/\varepsilon_2}} \Longrightarrow 1 - \frac{1}{\varepsilon_2} < 1 - \frac{1}{\varepsilon_1}$$

$$\Longrightarrow \frac{1}{\varepsilon_2} > \frac{1}{\varepsilon_1} \Longrightarrow \varepsilon_2 < \varepsilon_1$$

- *Intuition*: the monopolist charges lower price in the market with more elastic demand.

A 3° degree price discrimination since the monopolist acts as if were serving two separate markets and maximise profit for each of the two markets then we can apply the inverse elasticity pricing rule to the two separeted market so buy IERP. So price = marginal cost / 1 - 1/elasticity and the same for the second marker. Under which condition the price charge in the 1° market > price in the 2° market. This holds if and only if the nominator is larger than the second denominator.

Monopolist charge higher price in the market in which elasticity of demand is lower.

If we do the reciprocal

Price Discrimination: Third-degree

- ***Example*** (Pullman-Seattle route):
  - The price-elasticity of demand for business-class seats is -1.15, while that for economy seats is -1.52
  - From the IEPR,

$$p_B = \frac{MC}{1 - 1/1.15} \implies 0.13p_B = MC$$

$$p_E = \frac{MC}{1 - 1/1.52} \implies 0.34p_E = MC$$

  - Hence, $0.13p_B = 0.34p_E$ or $p_B = 2.63p_E$
    - Airline maximizes its profits by charging business-class seats a price 2.63 times higher than that of economy-class seats

Price Discrimination: Second-degree

- ***Second-degree price discrimination***:
  - The monopolist cannot observe the type of each consumer (e.g., his willingness to pay).
  - Hence the monopolist offers a menu of two-part tariffs, (F_L, q_L) and (F_H, q_H), with the property that the consumer with type $i = \{L, H\}$ has the incentive to self-select the two-part tariff (F_i, q_i) meant for him.

2° degree price discrimination

There are different types of consumer and the problem is that the monopolist cannot really recognised the type of the consumer: there is an asymmetric information. Consumers know their own type, but firm doesn't know the type of each consumer.
We will see that monopolist can extract some surplus by imposing a so called **two part tariffs.**
This a tariff composed of two parts:

- fixed fee
- Quantity provided by the firm for the payment of that fees.

We have two menus (or offer) is provided by low type customer (L-type) and another offer that is proposed to high type customer (H-type)
We will see that the two part tariffs will be define in such way each customer self select into the tariffs that has been design for him.

Price Discrimination: Second-degree

- Assume the utility function of type i consumer
$$U_i(q_i, F_i) = \theta_i u(q_i) - F_i$$
where
  - q_i is the quantity of a good consumed
  - F_i is the fixed fee paid to the monopolist for q_i
  - θ_i measures the valuation consumer assigns to the good, where $\theta_H > \theta_L$, with corresponding probabilities p and $1 - p$.
- The monopolist's constant marginal cost c satisfies $\theta_i > c$ for all $i = \{L, H\}$.

We assume utility of consumer depend on quantity consumed and fees payed to consume that quantity.
Tetha is a parameters - FI which is the fees that enter negatively in the utilty function because is a cost.
The highest is theta i, the highest the marginal utitly of consumption.
Indeed, if you compute the derivative is

$$\frac{\partial U_i}{\partial q_i} = \theta_i u'(q_i) \longrightarrow \text{If } \theta_i \uparrow \Rightarrow u' \uparrow$$

Then, the good is provided by monopolist and constant marginal cost c.
Thetai is higher than the marginal cost for both consumer type L and H.

For solving the profit maximisation problem with asymmetric information we may want to solve the problem with perfect information.

With perfect information the profit will be equal to Fi - c qi

$$\max_{F_i, q_i} \tilde{\Pi} = F_i - c q_i$$

WE ALSO KNOW THAT UTILITY MUST BE EQUAL TO

THETA i · UTILITY − FEES

$$\text{s.t. } \underline{U_i(q_i, F_i) = \theta_i u(q_i) - F_i}$$

$$F_i = \theta u(q_i) \qquad CS = 0$$

PROFITS ARE INCREASING IN F_i, FOR MONOPOLIST IS CONVENIENT TO CHARGE THE HIGHEST POSSIBLE FEE

Consumer Surplus is 0 since the benefit buying the good is exactly equal the fee he paid. If we know the optimal Fi is equal to thetai u(qi) we can replace the profit function and in this case we have to solve the maximisation problem with respect to qi.

$$\max_{q_i} \Pi = \theta_i u(q_i) - c(q_i)$$

$$\frac{\partial \Pi}{\partial q_i} = \theta_i u'(q_i) - c = 0$$

$$\boxed{\theta_i u'(q_i) = c}$$

Optimal condition: optimal quantity is the quantity in which the marginal benefits of consuming quantity qi for consumer equal the marginal costs to provide the quantity qi. It's also an efficiency condition and in this symmetric information case is equivalent to third degree of perfect price discrimination because the monopolist manage to capture all the surplus of the consumer by charging Fl = willingness to pay fo the consumer making consumer surplus equal to 0.

Price Discrimination: Second-degree

- The monopolist must guarantee that
  1) both types of customers are willing to participate ("***participation constraint***")
     - the two-part tariff meant for each type of customer provides him with a weakly positive utility level
  2) customers do not have incentives to choose the two-part tariff meant for the other type of customer ("***incentive compatibility***")
     - type i customer prefers (F_i, q_i) over (F_j, q_j) where $j \neq i$

$H \nearrow H$, $H \searrow \cancel{L}$

Now we go back to asymmetric information
In the case of asymmetric information with respect to the two type of consumer is the value for the theta H is larger for theta low type. Although monopolist cannot recognise each type of consumer he knows in the market there is a fraction of consumer of high type equal to p and low type equal to 1-p. The sum of the two fraction must be equal to 1 which is the total population normalised to 1. [SLIDE 21]
We will see that the two part tariffs offered by monopolist must satisfy two types of constrain:

- participation constraint : Two part tariffs much give enough incentive to both consumer to buy the good from the monopolist. Each tariff must provide weakly positive utility level.
- Incentive compatibility: each consumer must self selecting in to the tariffs that have been design for him. He must have incentive to buy the tariffs that is meant for his type. Should have incentive to buy tariff of other type.

Price Discrimination: Second-degree

- The participation constraints (PC) are

$$\theta_L u(q_L) - F_L \geq 0 \qquad PC_L$$

$$\theta_H u(q_H) - F_H \geq 0 \qquad PC_H$$

- The incentive compatibility conditions are

$$\theta_L u(q_L) - F_L \geq \theta_L u(q_H) - F_H \qquad IC_L$$

$$\theta_H u(q_H) - F_H \geq \theta_H u(q_L) - F_L \qquad IC_H$$

Summarising we have two participation constrain: one for L type and one for H type. The both constraint state that net utility from buy the good must be weakly positive. Both L and H will buy good from the monopolist.

Second set of incentive are: incentive compatibility condition: this condition states that if L type buys his own tariff (design for him self) net utility must be weakly larger than the case the same guy of L type buy another type.

Price Discrimination: Second-degree

- Re-arranging the four inequalities, the monopolist's profit maximization problem becomes:

$$\max_{F_L, q_L, F_H, q_H} \; p\overbrace{[F_H - cq_H]}^{\pi_H} + (1-p)\overbrace{[F_L - cq_L]}^{\pi_L}$$

$$\theta_L u(q_L) \geq F_L$$

$$\theta_H u(q_H) \geq F_H$$

$$\theta_L[u(q_L) - u(q_H)] + F_H \geq F_L$$

$$\theta_H[u(q_H) - u(q_L)] + F_L \geq F_H$$

ALL IN A GRAPH

To maximise profit: we have profit for H and profit for L weighted by the proportion in the population of the H type and L type.
We have 4 constraints that we have just seen.
The F and H enter linearly so never multiply between them self and this maximisation problem can be decompose in two steps.
1) optimal Fl and Fh
2) replace Fl and Fh in the profit function and determine and maximise with respect with ql and qh.
We can decompose in two step.
We can do this because constraint are linear in Fl and Fh and ql and qh.

We solve the first step : find optimal amount of the fees.
We know that profit for h and l type are both increasing in the fee (which is revenue for the monopolist).
So the monopolist would like to charge a very high fee but at the same time he cannot do that since for instance if increases Fl this constraint is less likely to hold

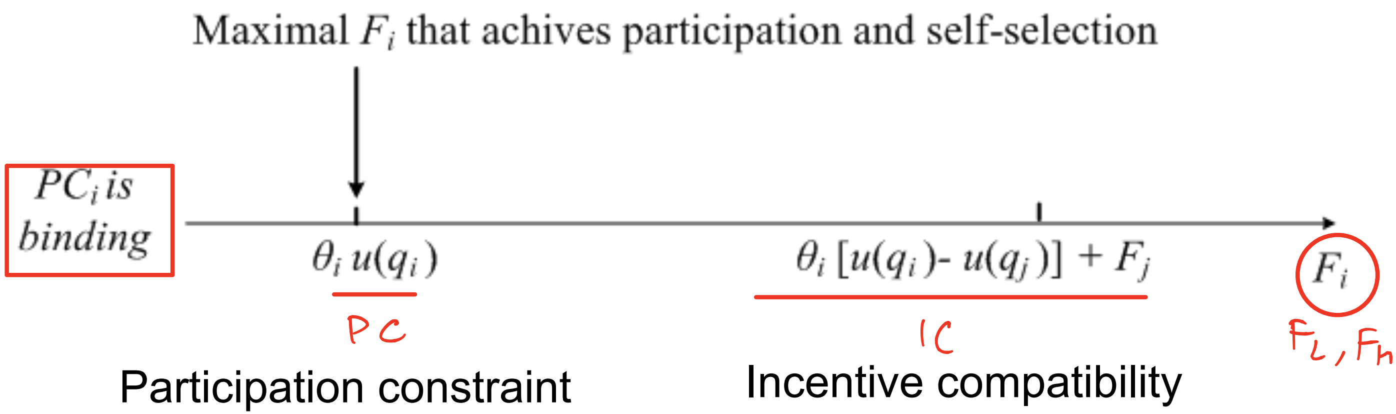
$$\theta_L u(q_L) \geq F_L$$

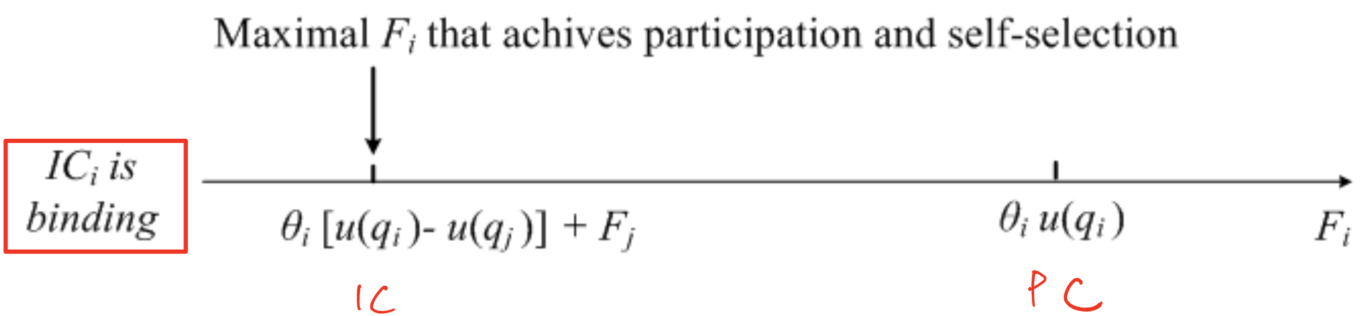
What the monopolist can do is to charge the highest possible fees in both market that is consistent with this four constraint.
We can represent the left-hand side of all the constraints in a graph.

Price Discrimination: Second-degree

- Both PC_H and IC_H are expressed in terms of the fee F_H
  - The monopolist increases F_H until such fee coincides with the lowest of $\theta_H u(q_H)$ and $\theta_H[u(q_H) - u(q_L)] + F_L$ for all $i = \{L, H\}$
  - Otherwise, one (or both) constraints will be violated, leading the high-demand customer to not participate

Price Discrimination: Second-degree





We do this graph here.
You see that for each type we have participation constraint and incentive compability constraint.
We may have two cases:

- the left-hand side of the participation constraint is Lower than the left hand side of incentive compatibility condition. [FIRST in the graph]

If the optimal amount of the fees will be the left-hand side of the participation constrain so PC is binding.

- the left-hand side of incentive compatibility constrain is the lowest of the two and in this case the optimal fees will be the left hand side of the IC and the IC will be the one binding.

Price Discrimination: Second-degree

- ***High-demand customer***:
  - Let us show that $\boxed{IC_H \text{ is binding}}$
  - An indirect way to show that

$$F_H = \theta_H[u(q_H) - u(q_L)] + F_L$$

  is to demonstrate that $F_H < \theta_H u(q_H)$

  - Proving this by contradiction, assume that

$$\boxed{F_H = \theta_H u(q_H)}$$

Now we prove the decompose of the solution to find Fl* and Fh* that is the first part of the problem. First, we prove the IC is binding so this mean that for the H type we will in the second condition [slide of graph] and for L type we will be in the first condition [slide graph].
Let's start from IC binding proof.
To prove this is demonstration other condition like PC for h is not binding. So this will mean proving by contradiction: we assume that the partecipation constraint for H type is binding. $\rightarrow$ $\boxed{F_H = \theta_H u(q_H)}$

Price Discrimination: Second-degree

$\theta_H > \theta_L$

– Then, IC_H can be written as

$$\cancel{F_H} - \theta_H u(q_L) + F_L \geq \cancel{F_H}$$
$$\Longrightarrow F_L \geq \theta_H u(q_L)$$

– Combining this result with the fact that $\theta_H > \theta_L$,

$$\underline{F_L} \geq \underline{\theta_H u(q_L)} > \underline{\theta_L u(q_L)}$$

which implies $\underline{F_L > \theta_L u(q_L)}$

THIS CONTRADICTS IC CONSTRAINT FOR L TYPE

– However, this violates PC_L

- We then reached a contradiction
- Thus, $F_H < \theta_H u(q_H)$
- IC_H is binding but PC_H is not.

$$\theta_H u(q_H) \geq F_H$$

Then we can replace for Fh into the IC constraint. So we replace

So we replace tetha u.. binding in Fh.

So we have prove that PCh is not binding.
So what must be binding is the IC h.

Price Discrimination: Second-degree

- ***Low-demand customer***:
  - Let us show that PC_L binding
  - Similarly as for the high-demand customer, an indirect way to show that
  $$F_L = \theta_L u(q_L)$$
  is to demonstrate that $F_L < \theta_L[u(q_L) - u(q_H)] + F_H$
  - Proving this by contradiction, assume that
  $$F_L = \theta_L[u(q_L) - u(q_H)] + F_H$$

We do something similar with **low demand customer.**
What is binding here is the PC whine IC is not binding.
A way to procede is to prove that ICl is not binding. Also in this case we prove this by contraction and IC is binding so $F_L = \theta_L[u(q_L) - u(q_H)] + F_H$

Price Discrimination: Second-degree

– Then, $\boxed{IC_H}$ can be written as

$$\theta_H[u(q_H) - u(q_L)] + \theta_L[u(q_L) - u(q_H)] + F_H = F_H$$
$$\Longrightarrow \theta_H[u(q_H) - u(q_L)] = \theta_L[u(q_L) - u(q_H)]$$
$$\Longrightarrow \theta_H = \theta_L$$

which violates the initial assumption $\theta_H > \theta_L$

- We reached a contradiction
- Thus, $F_L < \theta_L[u(q_L) - u(q_H)] + F_H$
- PC_L is binding but IC_L is not

Having FI we can replace in IC h and at the end we obtain tetaH = tetaL which is know for assumption that tetaH > teta L so we reach contradiction.
So we prove that ICl is not binding while PC l must binding

Price Discrimination: Second-degree

- In summary:
  - From PC_L binding we have

$$\boxed{\theta_L u(q_L) = F_L^*}$$

IC_L HOLDS (NOT BINDING)

  - From IC_H binding we have

$$\boxed{\theta_H[u(q_H) - u(q_L)] + F_L = F_H^*}$$

PC_H HOLDS

  - In addition,
    - PC_L binding implies that IC_L holds, and
    - IC_H binding entails that PC_H is also satisfied,
    - That is, all four constraints hold.

SO NOW WE CAN REPLACE INTO THE PROFIT FUNCTION THE OPTIMAL VALUE OF THE FEES

Price Discrimination: Second-degree

- The monopolist's expected PMP can then be written as unconstrained problem, as follows,

$$\max_{q_L,q_H \geq 0} \; p\,[F_H^* - cq_H] + (1-p)[F_L^* - cq_L]$$

$$= p\left\{\underbrace{\theta_H[u(q_H) - u(q_L)] + F_L}_{F_H} - cq_H\right\}$$

$$+(1-p)\left\{\underbrace{\theta_L u(q_L)}_{F_L} - cq_L\right\}$$

$$= p\left\{\theta_H[u(q_H) - u(q_L)] + \underbrace{\theta_L u(q_L)}_{F_L} - cq_H\right\}$$

$$+(1-p)\{\theta_L u(q_L) - cq_L\}$$

$$= p[\theta_H u(q_H) - (\theta_H - \theta_L)u(q_L) - cq_H]$$

$$+(1-p)[\theta_L u(q_L) - cq_L]$$

NOW MAX PROBLEM IN ONLY TWO VARS q_L, q_H

SO WE SOLVE MAXIMISATION PROBLEM FOR q_L AND q_H

$$\max_{q_L, q_H} \quad p \cdot (\underline{F_H} - c\, q_H) + (1-p)(\underline{F_L} - c\, q_L) =$$

REPLACE OPTIMALS

$$= p \cdot [\theta_H(u(q_H) - u(q_L)] + F_L - c\, q_H] + (1-p)(\theta_L \cdot u(q_L) - c\, q_L)) =$$

NOW REWRITE IN DIFFERENT FORM

$$= p \cdot [\theta_H(u(q_H) - u(q_L)] + F_L - c\, q_H] +$$
$$+ \theta_L u(q_L) - c\, q_L - p(\theta_L u(q_L) - c\, q_L) =$$

COLLECT P

$$= p[\theta_H(u(q_H) - u(q_L)) + \cancel{F_L} - c\, q_H - \underbrace{\cancel{\theta_L u(q_L)}}_{F_L} + q_L] +$$
$$+ \theta_L u(q_L) - c\, q_L =$$
$$= p[\theta_H[u(q_H) - u(q_L)] - c\, q_H + c\, q_L] +$$
$$+ \theta_L u(q_L) - c\, q_L$$

NOW COMPUTE FOC

$$\frac{\partial \Pi}{\partial q_H} = p[\theta_H u'(q_H) - c] = 0 \implies \theta_H u'(q_H) = c$$

This mean that for H type, the marginal benefits consuming the quantity qh must be equal to the marginal cost for the monopolist to provide that quantity qh.
So this is also an efficiency condition —> same of the symmetric equilibrium

In that the monopolist know the type of each customer.

Now we compute:

$$\frac{\partial \Pi}{\partial q_L} = p\left(-\theta_H \underline{u'(q_L)} + c\right) + \theta_L \underline{u'(q_L)} - c = 0$$

COLLECT $u'(q_L)$

$$u'(q_L)(\theta_L - p\theta_H) = c(1-p)$$

$$u'(q_L)\left[\frac{\theta_L - p\theta_H}{1-p}\right] = c \longrightarrow u'(q_L)\left[\theta_L - \frac{p}{1-p}(\theta_H - \theta_L)\right] = c$$

$$\Downarrow$$

NOW WE ADD AND SUB $\frac{p}{1-p}\theta_L$ THEN

We have found optimal condition for H type and L

Price Discrimination: Second-degree

- FOC with respect to q_H:

$$p[\theta_H u'(q_H) - c] = 0 \quad \Longrightarrow \quad \theta_H u'(q_H) = c$$

  - which coincides with that under complete information.
  - That is, there is not output distortion for high-demand buyer
  - Informally, we say that there is "**no distortion at the top**".

- FOC with respect to q_L:

$$p(-(\theta_H - \theta_L)u'(q_L)] + (1-p)[\theta_L u'(q_L) - c] = 0$$

which can be re-written as

$$u'(q_L)[\theta_L - p\theta_H] = (1-p)c$$

Price Discrimination: Second-degree

- Dividing both sides by $(1-p)$, we obtain

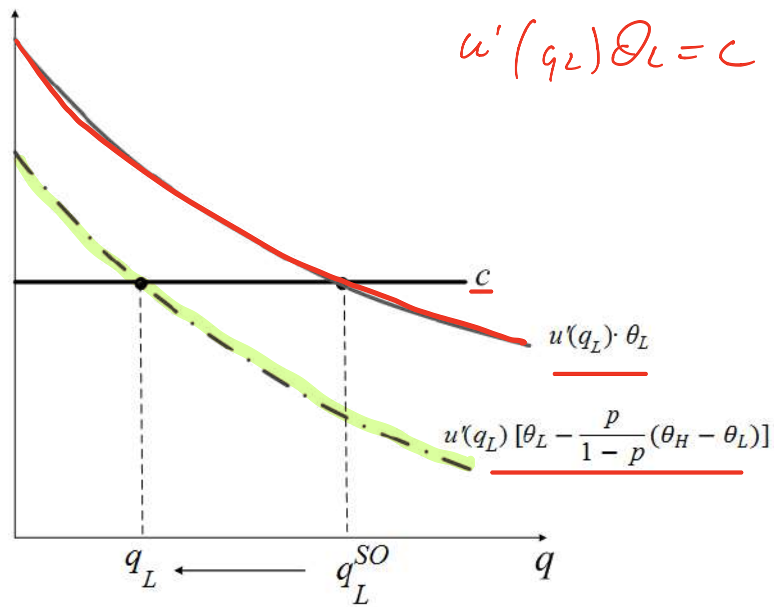
$$\boxed{u'(q_L)\left[\frac{\theta_L-\theta_H p}{1-p}\right]=c}$$

- The above expression can alternatively be written as

$$\boxed{u'(q_L)\left[\theta_L-\frac{p}{1-p}(\theta_H-\theta_L)\right]=c}$$

Price Discrimination: Second-degree

- $u'(q_L) \cdot \theta_L$ depicts the socially optimal output q_L^{so}, i.e., that arising under complete information
- The output offered to high-demand customers is socially efficient due to the absence of output distortion for high-type agents
- The output offered to low-demand customers entails a distortion, i.e., $q_L < q_L^{so}$
- Per-unit price for high-type and low-type differs, i.e., $F_H \neq F_L$
  - Monopolist practices price discrimination among the two types of customers.



So we can represent this in a graph.
We draw marginal cost for monopolist and optimality condition for L type and also we draw the condition that is the optimal condition we would have for the L type in case of symmetric information. In case of symmetric information we should have the equality between marginal utility and marginal cost —> will be equilibrium in symmetric case.

- Red Curve is marginal benefit of consuming in the case of symmetric info
- Green is the case of optimality condition in asymmetric info

The optimality condition in case of asymmetric info is below optimality in case of symmetric. In the term below we have theta subtracted with something, so that’s the reason why.
In equilibrium the optimal quantity for L type is less than social optimum that would be the optimal quantity in the case of symmetric information.

Price Discrimination: Second-degree

We can summarise equilibrium:

- Since constraint PC_L binds while PC_H does not, then only the high-demand customer retains a positive surplus, i.e., $\theta_H u(q_H) - F_H > 0$.
- The firm's lack of information provides the high-demand customer with an "information rent."
  - Intuitively, the information rent emerges from the seller's attempt to reduce the incentives of the high-type customer to select the contract meant for the low type.
  - The seller also achieves self-selection by setting an attractive output for the low-type buyer, i.e., q_L is lower than under complete information.

We call this separating equilibrium

2. The high type customer exploit this information and are able to retain some positive surplus: seller wants to avoid that H custom select the L tariffs offering a quite small quantity to L type.

Do as exercise

Price Discrimination: Second-degree

- ***Example***:
  - Consider a monopolist selling a textbook to two types of graduate students, low- and high-demand, with utility function

$$U_i(q_i, F_i) = \frac{q_i^2}{2} - \theta_i q_i - F_i$$

where $i = \{L, H\}$ and $\theta_H > \theta_L$.

  - Hence, the UMP of student type i is

$$\max_{q_i} \quad \frac{q_i^2}{2} - \theta_i q_i - F_i \quad \text{s.t.} \;\; pq_i + F_i \leq w_i$$

where $w_i > 0$ denotes the student's wealth.

Price Discrimination: Second-degree

- ***Example*** (continued):
  - By Walras' law, the constraint binds

$$F_i = w_i - pq_i$$

  - Then, the UMP can be expressed as

$$\max_{q_i} \quad \frac{{q_i}^2}{2} - \theta_i q_i - (w_i - pq_i)$$

  - FOCs wrt q_i yields the direct demand function:

$$q_i - \theta_i - p = 0 \quad \text{or} \quad q_i = \theta_i - p$$

Price Discrimination: Second-degree

- ***Example*** (continued):
  - Assume that the proportion of high-demand (low-demand) students is γ ($1-\gamma$, respectively).
  - The monopolist's constant marginal cost is $c > 0$, which satisfies $\theta_i > c$ for all $i = \{L, H\}$.
  - Consider for simplicity that $\theta_L > \frac{\theta_H + c}{2}$.
  - This implies that each type of student would buy the textbook, both when the firm practices uniform pricing and when it sets two-part tariffs
    - Exercise.

Advanced Microeconomic Theory

Chapter 7: Natural monopoly; Monopsony

Outline

- Regulation of Natural Monopolies
- Monopsony

Regulation of Natural Monopolies

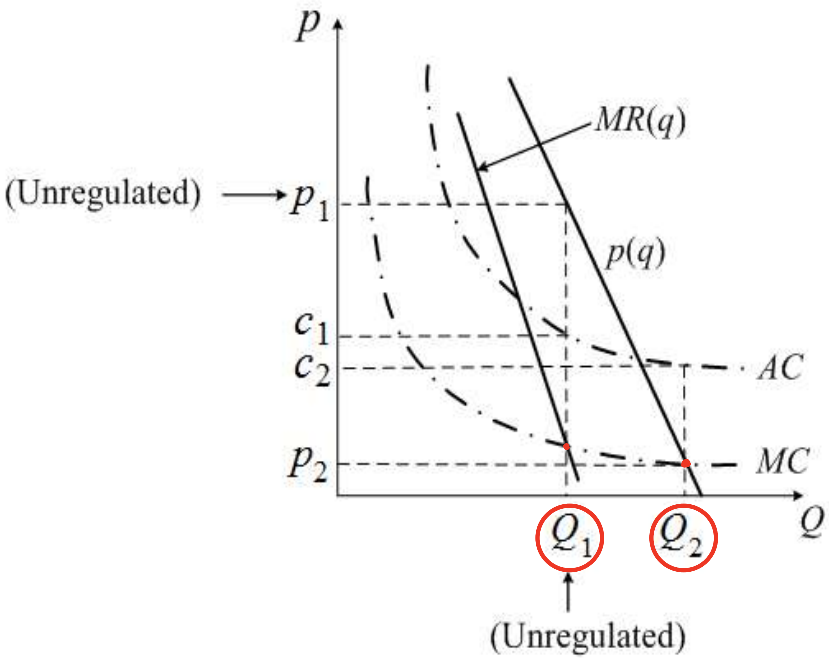
Regulation of Natural Monopolies

- ***Natural monopolies***: Monopolies that exhibit *decreasing* cost structures, with the MC curve lying below the AC curve.
- Hence, having a single firm serving the entire market is cheaper than having multiple firms, as aggregate average costs for the entire industry would be lower.

In some industries, average cost can be continuously decreasing with quantity: average cost curve is always decreasing, so marginal cost will lie below average cost curve.
So larger firm will be able to produce at lower cost than small firm and in those industry there will be a tendency to concentration to grow larger and larger.

Regulation of Natural Monopolies

- Unregulated natural monopolist maximizes profits at the point where MR=MC, producing Q_1 units and selling them at a price p_1.
- Regulated natural monopolist will charge p_2 (where demand crosses MC) and produce Q_2 units.
- The production level Q_2 implies a loss of $p_2 - c_2$ per unit.



Graphical representation: we have natural monopolist with average cost curve decreasing, demand curve and marginal revenue curve.
What would be equilibrium in monopoly? Crossing point between MR and MC.
Monopolist will tend to produce Q1 at price p1.
Different quantity from social optimum that you read in the crossing point between demand and marginal cost curve.
Social optimum will be a production equal to Q2 and so the price p2.

This level of production, the price is below the AC, this mean monopolist will make losses and would prefer to exit the market. This equilibrium is not sustainable in case of decreasing AC.
What solution do we have?

Regulation of Natural Monopolies

- Dilemma with natural monopolies:
  - abandon the policy of setting prices equal to marginal cost, OR
  - continue applying marginal cost pricing but subsidize the monopolist for his losses
- Solution to the dilemma:
  - A multi-price system that allows for price discrimination
  - Charging some users a high price while maintaining a low price to other users

What solution do we have?

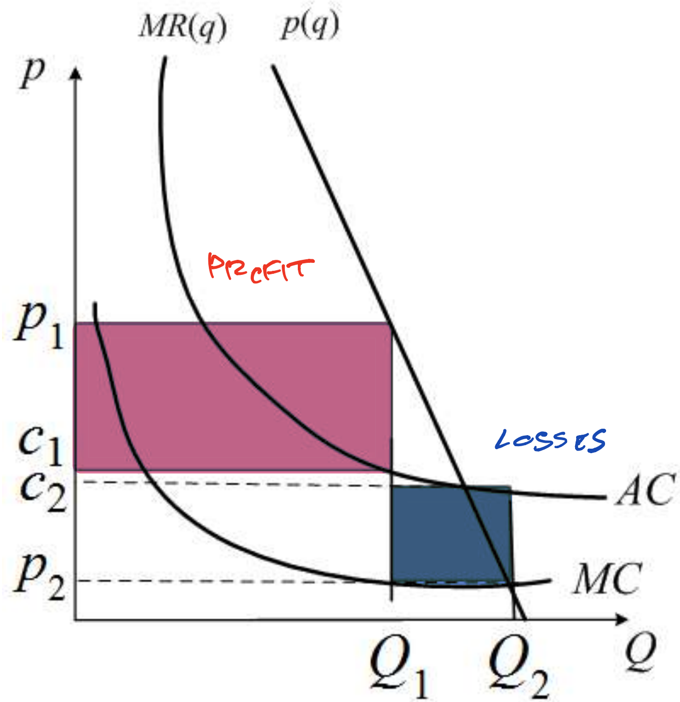
- We leave monopoly to charge a price which make MR = MC and let him making profit
- Produce social optimum but covering losses of the monopolist: this would be the case for instance of state on firms, which operate in industry and this industry will generally make continuously losses that must be covered by government
- Abandon uniform price and we can let the monopolist charge different prices to different consumers

Regulation of Natural Monopolies

- Multi-price system:
  - a high price p_1
  - a low price p_2
- *Benefit*: $(p_1 - c_1)$ per unit in the interval from 0 to q_1
- *Loss*: $(c_2 - p_2)$ per unit in the interval $(q_2 - q_1)$
- The monopolist price discriminates iff

$$\boxed{(p_1 - c_1)q_1 > (c_2 - p_2)(q_2 - q_1)}$$

$\Pi >$ LOSSES



Example in which price are differentiated, in this case we may lead the monopolist to charge two different prices.
We have two quantity Q1 and Q2, we may let the monopolist to charge the price p1 (above the MC) for the first Q1 quantity and then charge the price p2 that is below the AC, while p1 is above AC.
In particular, p2 is the social optimum price given by crossing point between the demand curve and MC. It’s immediate to say that profit will make this on first Q1 units (red) and this losses because the price below AC on the Q2- Q1 units (blu).
As long as the amount of profit is higher than the losses, so area of rectangle (profit) is > than area of box (losses), monopolist will continuos to produce. So condition allowing monopolist to charge different prices for first Q1 and the following Q2-Q1 is that profit must be larger than the losses.

In the book you can find other possible solution but we skip that part 🥰

Regulation of Natural Monopolies

- An alternative regulation:
  - allow the monopolist to charge a price above marginal cost that is sufficient to earn a “fair” rate of return on capital investments
- Two difficulties:
  - what is a “fair” rate of return
  - overcapitalization

Monopsony

Monopsony

- ***Monopsony***: A single buyer of goods and services.
- Monopsony (single buyer) is analogous to that of a monopoly (single seller).
- *Examples*: a coal mine, Walmart Superstore in a small town, etc.

Monopsony

Unlike monopoly in which we have single seller, here we have single buyers of good and services.
In particular, in this lecture we will consider monopsony in the labour market. This could be for instance situation in which we have large firms in a small town.

Monopsony

- Consider that the monopsony faces competition in the product market, where prices are given at $p > 0$, but is a monopsony in the input market (e.g., labor services).
- Assume an increasing and concave production function, i.e., $f'(x) > 0$ and $f''(x) \leq 0$.
  - This yields a total revenue of $pf(x)$.
- Consider a cost function $w(x) \cdot x$, where $w(x)$ denotes the inverse supply function of labor x.
  - Assume that $w'(x) > 0$ for all x.
  - This indicates that, as the firm hires more workers, labor becomes scarce, thus increasing the wages of additional workers.

We consider a firm that is facing perfect completion in the production market so price of the good sold by the firm is given and constant unlike monopoly.
We assume production function as marginal productivity is positive and decreasing, so production function is increasing and concave.
By p * f(x) is the value of the product sold in the market (TR of the firm).
We can consider a single factor production that is x neighbour and cost function of the firm is given by unit price of the labour which is w(x) which multiply the unit of labour.
For the single firm, the wage rate is not given so is a function of the amount of labour since price is a monopsony in the labour market, so this means that the firm faces an increasing labour supply: if firm wants to hire more worker it need to pay larger wages. Firm is a monopsonist in the market.
It is like firm is facing aggregate labour supply and the labour market.
Aggregate labour supply increasing because if you want more worker to supply labour you have to offer larger wages.
If labour supply increasing, then derivate of labour supply is positive

Monopsony

- The monopsony PMP is

$$\max_x \quad pf(x) - w(x)x$$

- FOC wrt the amount of labor services (x) yields

$$\frac{\partial \pi}{\partial x} = pf'(x^*) - w(x^*) - w'(x^*)x^* = 0 \qquad \text{INTERIOR OPT.}$$

$$\Longrightarrow \underbrace{pf'(x^*)}_{A} = \underbrace{w(x^*) + w'(x^*)x^*}_{B}$$

  – A: "marginal revenue product" of labor.
  – B: "marginal expenditure" (ME) on labor.
    - The additional worker entails a monetary outlay of $w(x^*)$.
    - Hiring more workers make labor become more scarce, ultimately forcing the monopsony to raise the prevailing wage on all inframarginal workers, as captured by $w'(x^*)x^*$.

PMP of the firm: difference between TR and TC.
Firm want to maximise profit with respect with total amount of labour x.
We have to compute the FOC.

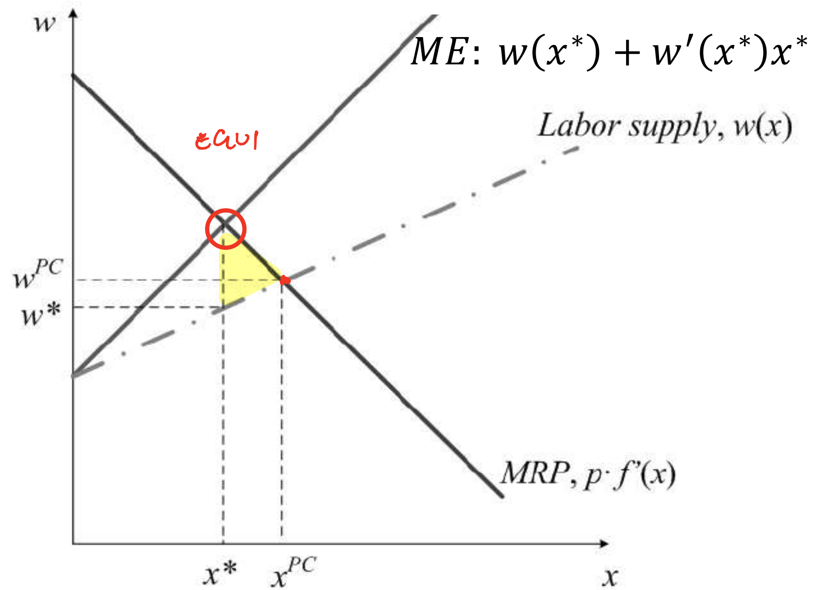
$$\frac{\partial \pi}{\partial x}$$

Marginal revenue product ‘A’ is how many quantity marginal worker is producing and A is the revenue of firm get selling those marginal productivity in the market.
While on the right we have the marginal expenditure of labour B (ME), is given by the increase in the wage multiply by the previous amount of worker firms was hiring.
If i want to increase x by 1 i will spent the wage of this guy but also the increase the wage of all worker that you are already highring —> depend to the fact that labour is increasing.

Monopsony

- Monopsonist hiring and salary decisions.
  - The marginal revenue product of labor, $pf'(x)$, is decreasing in x given that $f''(x) \leq 0$.
  - The labor supply, $w(x)$, is increasing in x since $w'(x) > 0$.
  - The marginal expenditure (ME) on labor lies above the supply function $w(x)$ since $w'(x) > 0$.
  - The monopsonist hires x^* workers at a salary of $w(x^*)$.

$w'(x) > 0$



We have quantity of labour on horizontal and wage rate on the vertical axes.

- MRP = p *f(x) which is decreasing by assumption since MR is decreasing
- Labour supply curve w(x) increasing by assumption (since w'(x) > 0)
- Marginal expenditure ME = w(x*) +w'(x*)x*

so wage and a factors since is larger than 0 so second term will be positive. ME lie above the labour supply curve and this imply the equilibrium in the presence of Perfect completion which is x^PC and w ^PC implies a lower demand and higher wage with respect to the monopolist equilibrium in which equilibrium will be given by the demand x* that is given by interception of ME with MRP

Monopsony

- The deadweight loss from monopsony is

$$DWL = \int_{x^*}^{x^{PC}} [pf'(x) - w(x)]dx$$

- That is, the area below the marginal revenue product and above the labor supply curve, between x^* and x^{PC} workers.

DWL for monopsony is the area below the MRP and above the labour supply function and we have to compute the area. In particular, we are integrating between Perfect competition equilibrium and monopsony equilibrium so in practise the area of the yellows triangle in the picture before.

Monopsony

monopsony optimal condition

- We can write the monopsony profit-maximizing condition, i.e., $\boxed{pf'(x^*) = w(x^*) + w'(x^*)x^*}$, in terms of labor supply elasticity, using the following steps:

$$pf'(x^*) = w(x^*) + \boxed{\frac{\partial w(x^*)}{\partial x^*} x^*}$$
$$= w(x^*)\left(1 + \frac{\partial w(x^*)}{\partial x^*} \frac{x^*}{\underline{w(x^*)}}\right)$$

- And rearranging,

$$pf'(x^*) = w(x^*)\left(1 + \frac{1}{\frac{\partial x^* w(x^*)}{\partial w \quad x^*}}\right)$$

Inverse of price elasticity of labour supply

Monopsony

- Since $\frac{\partial x^*}{\partial w}\frac{w(x^*)}{x^*}$ represents the elasticity of labor supply ε, then

$$pf'(x^*) = \underline{w(x^*)}\left(1 + \frac{1}{\varepsilon}\right)$$

- Intuitively, as $\underline{\varepsilon \to \infty}$, the behavior of the monopsonist approaches that perfect competition (also in the labor market)

SOC

Monopsony

- The equilibrium condition above is also sufficient as long as neg pos

$$pf''(x^*) - 2w'(x^*) - w''(x^*)x^* < 0$$

- Since $f''(x^*) < 0$, $w'(x^*) > 0$ (by assumption), we only need that either:
  a) the supply function is convex, i.e., $w''(x^*) > 0$; or
  b) if it is concave, i.e., $w''(x^*) < 0$, its concavity is not very strong, that is

$$pf''(x^*) - 2w'(x^*) < w''(x^*)x^*$$

Monopsony

- ***Example***:
  - Consider a monopsonist with production function $f(x) = ax$, where $a > 0$, and facing a given market price $p > 0$ per unit of output.
  - Labor supply is $w(x) = bx$, where $b > 0$.
  - The marginal revenue product of hiring an additional worker is
$$pf'(x) = pa$$
  - The marginal expenditure on labor is
$$w(x) + w'(x)x = bx + bx = 2bx$$

Monopsony

- ***Example*** (continued):
  - Setting them equal to each other, $pa = 2bx^*$, yields a profit-maximizing amount of labor:
$$x^* = \frac{ap}{2b}$$
  - x^* increases in the price of output, p, and in the marginal productivity of labor, a; but decreases in the slope of labor supply, b.
  - Sufficiency holds since
$$pf''(x^*) - 2w'(x^*) = -2b < 0$$

Advanced Microeconomic Theory

Chapter 9: Externalities and Public Goods

Outline

- Externalities
- Pigouvian Taxation
- Public Goods

Externalities

Externalities

- ***Externality*** emerges when the well-being of a consumer or the production possibilities of a firm is directly affected by the actions of another agent in the economy.
  - Example: the production possibilities of a fishery are affected by the pollutants that a refinery dumps into a lake.
  - The effects from one agent to another are not captured by the price system.
- The effects transmitted through the price system are referred to as "***pecuniary externalities***."

Externality: emerge when the well-being (utility of consumer) or production possibilities of another agent in the economy are directed affected by action of another agents (like other consumer).

An example could be a production of fishery affected by a pollutants that a refinery dumps into a lake.

It’s not only utility depends on action of other agent, but this effect are not capture by price system. So no price for this activity and we can say market is incomplete.
If price system is effected we refer to as pecuniary externalities

Externalities

- Consider a polluting firm (agent 1) and an individual affected by such pollution (agent 2).
- The firm's profit function is

$$\pi(p, x)$$

where p is the price vector and x is the amount of externality generated.

- Assume that p is given (i.e., p is parameter). Then, the profit function becomes

$$\boxed{\pi(x)}$$

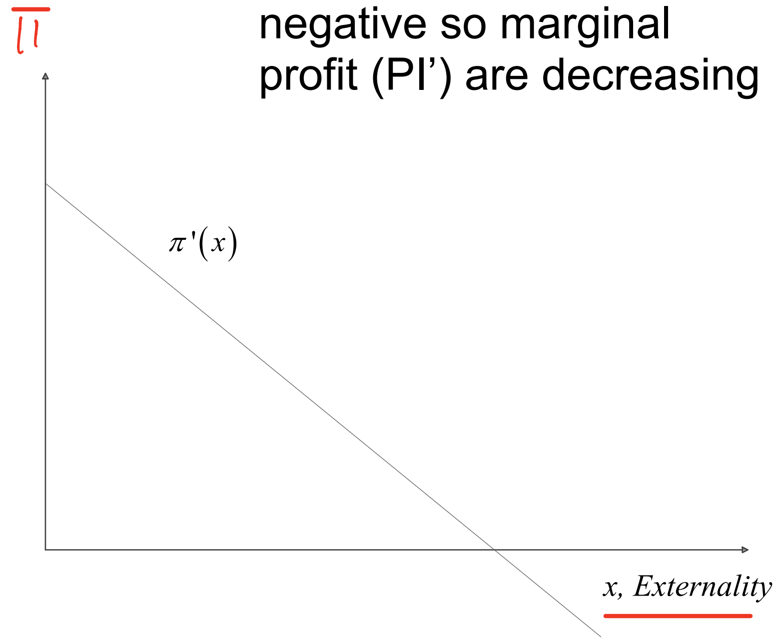
where $\pi'(x) > 0$ and $\pi''(x) < 0$.

An example. We assume a firm that pollute (agent 1) and agent 2 is affected. Profit function depends on price p and pollutant activity x. So x is the amount of externality generate.
Product market is competitive so price is given and firm is price taker and in this case we can represent the profit as function of the only one variable x. Profits are increasing by externalities but at the same time, is concave in the externalities.

Externalities

- The firm obtains a positive and significant benefit from the first unit of the externality-generating activity.
- But the additional benefit from further units is decreasing.

So derivative of profit is positive and PI'' is negative so marginal profit (PI') are decreasing



Externalities

- The individual's (i.e., agent 2's) utility is given by
$$u(q, x)$$
where $q \in \mathbb{R}^N$ is a vector of N-tradable goods and $x \in \mathbb{R}_+$ is the negative externality, $\boxed{\text{with } \frac{\partial u}{\partial x} < 0 \text{ and } \frac{\partial u}{\partial q_k} \geq 0}$ in every good k.

- Let $q^*(p, w, x)$ denote the individual's Walrasian demand. Then,
$$v(x) = u(q^*(p, w, x), x)$$
is the indirect utility function with $v'(x) < 0$ for all $x > 0$, where w is consumer wealth.

Consider the second agent of the consumer as a utility function that is u(q,x) where q are R^n good and x is the amount of externalities. We assume externality is bad so the marginal utility is negative (since is pollution). While marginal utility of all other goods i positive.
We can then define the Walrasian demand by solving UMP for the consumer and indirect utility will be value utility computed at the optimal amount of optimal demand for the goods that are sold in the market. While, we can consider x from the point of view of the consumer as parameter since the consumer cannot really affects the amount of the pollution and indirect utility is decreasing in the amount of externalities and w is consumer wealth.
We know Walrasian demand depend on the price, wealth and externalities (in this case)

Externalities

- ***Example***:
  - Consider the firm's profit function is given by $\pi = py - cy^2$, where $y \in \boldsymbol{R}_+^L$ is output and $p > c > 0$.
  - If every unit of output generates a unit of pollution, i.e., $x = y$, the profit function becomes $\pi(x) = px - cx^2$.
  - FOC wrt x yields $\pi'(x^*) = p - 2cx^* = 0$, producing $p = 2cx^*$ or $x^* = \frac{p}{2c}$.

Externalities

- ***Example*** (continued):
  - If every unit of output y generates $\frac{1}{\alpha}$ units of pollution, i.e., $\boxed{y = \frac{1}{\alpha}x,}$ where $\alpha > 0$, the profit function becomes

$$\pi(x) = p\frac{x}{\alpha} - c\left(\frac{x}{\alpha}\right)^2.$$

$c(y) = cy^2$

  - Taking FOC with respect to x yields

$$\pi'(x^*) = \frac{p}{\alpha} - 2c\frac{x^*}{\alpha}\frac{1}{\alpha} = 0,$$

with a competitive equilibrium level of pollution of

$$\boxed{x^* = \alpha\frac{p}{2c}.}$$

We assume that for every unit of output y (production of the firm) an amount of 1/a unit of pollution is produce. So production is y = 1/a x where alpha > 0. Production depends on the level of activity of the firm and the level of activity is produce in some pollution.
Direct relationship between production and pollution. We can rewrite profit function as: function of production

$$\Pi(y) = py - c(y)$$

WE CAN REPLACE Y

$$\Pi(x) = p \cdot \frac{x}{\alpha} - c\left(\frac{x}{\alpha}\right)^2$$

$$c(y) = \underbrace{cy^2}_{\text{ASSUMPTION}} \longrightarrow \text{COST FUNCTION IS CONVEX}$$

The compute the FOC.
We get the optimal level of externalities but also compiute the optimal level of production:

$$x^* = \alpha \frac{p}{2c} \qquad \boxed{y^*} = \frac{1}{\alpha} x$$

Externalities

- ***Competitive equilibrium***: All agents independently and simultaneously solve their PMP (for firms) or UMP (for consumers).
  - The firm independently chooses the level of the externality-generating activity, x, that solves its PMP

$$\boxed{\max_{x} \; \pi(x)}$$

  - Taking FOC with respect to x yields

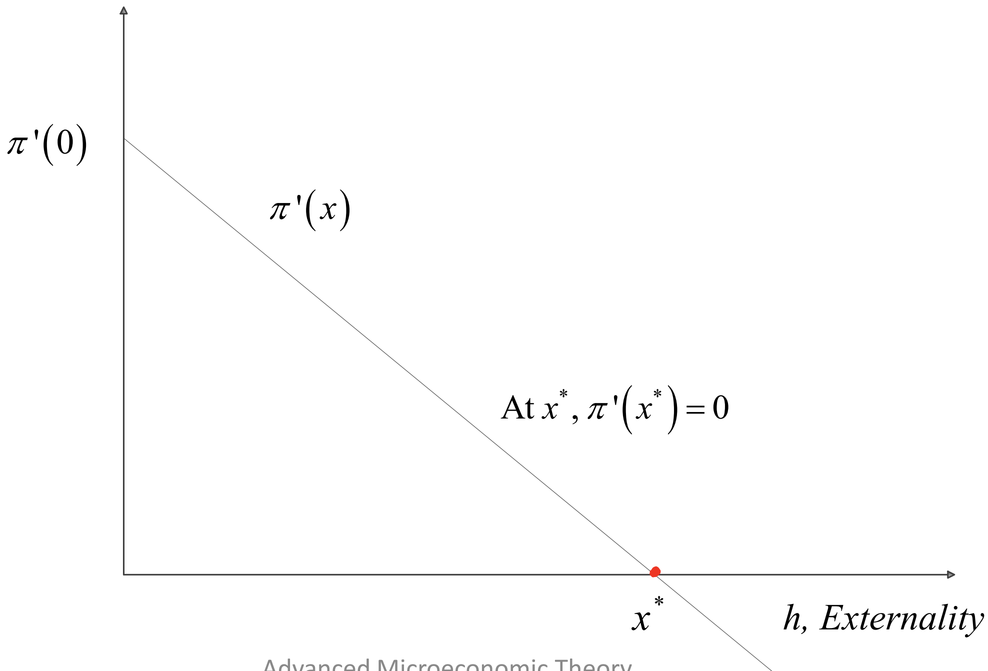
$$\pi'(x^*) \leq 0$$

    with equality if $x > 0$ (interior solution).

Let’s check competitive equilibrium in which agent independently and simultaneously solve their PMP (if firm) or UMP (if consumer).
The firm independently chooses the level of externality-generation activity, x, that solves it’s PMP.
In interior solution, the marginal profit must be equal to 0.

Externalities

– Firm increases the externality-generating activity until the point where the marginal benefit from an additional unit is exactly zero, i.e., $\pi'(x^*) = 0$.



The solution of firm profit maximisation problem is the crossing point between marginal profit curve (decreasing) and horizontal axes.
For the firm optimal condition is: $\pi'(x) = 0$

So marginal profit function must cross the horizontal axes. We have found the optimal production that is x*.

Externalities

– The UMP of the individual affected by pollution is

$$\max_{q} \; u(q,x) \quad \text{s.t.} \quad pq \leq w$$

where $p \in \mathbb{R}^N_+$ is the given price vector.

– Notice that $q \in \mathbb{R}^N$ does not include pollution as one of the N-tradable goods.

– Hence the individual cannot affect the level of the externality generating activity x.

- Uninteresting case
- This assumption is later relaxed

We know go for optimal solution of each consumer. Each consumer has to decide the optimal quantity of price bought in the market. Conditional on the budget constrain so p * q must be equal to the total wealth.
In this case the guy will solve the profit maximisation problem and we have already seen before what is the equilibrium of this maximisation problem.
x is a parameter so there will be no demand for x in this function, there will be only the opt demand for the goods and goods sold in the market.

Externalities

- ***Pareto optimum***:
  - The social planner selects the level of x that maximizes social welfare
$$\max_{x \geq 0} \ \pi(x) + v(x)$$
  - Taking FOC with respect to x yields
$$\pi'(x^0) \leq -v'(x^0) \text{ with equality if } x^0 > 0$$
    where x^0 is the Pareto optimal amount of the externality.
  - Intuitively, at a Pareto optimal (and interior) solution, the marginal benefit of the externality-generating activity, $\pi'(x^0)$, is equal to its marginal cost, $-v'(x)$.

Let’s compare the centralised equilibrium that is the equilibrium that arise when firm maximise their profit and consumer their utility with the social planner problem: maximise the aggregate surplus.
Social planner want to find allocation of good that mamximise aggregate surplus : sum of produce surplus and the consumer surplus (indirect utilty).
The social planner maximise the sums between the profits.
Social planner will compute the optimal x that maximise the aggregate surplus.
We compute derivative of profits with respect to x.

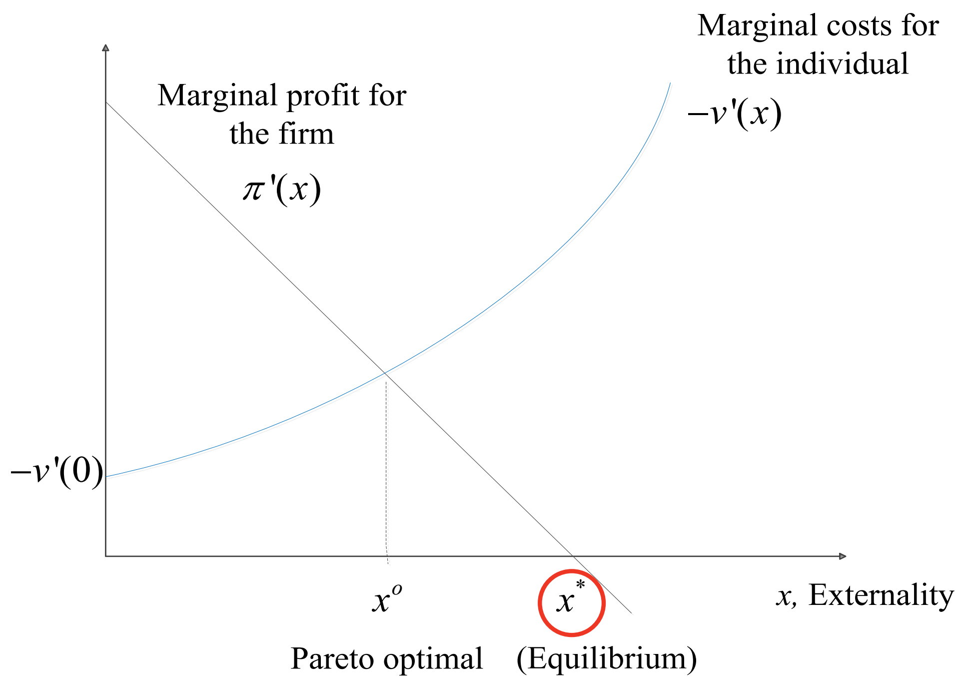
$\pi'(x) + v'(x) = 0 \qquad \pi'(x^0) = -v(x)$

SOCIAL OPTIMUM HOLDS WITH EQUALITY WITH INT. SOLUTION

Externalities

- Pareto optimal and equilibrium externality level (negative externality).

- Too much externality x is produced in the competitive equilibrium relative to the Pareto optimum, i.e., $x^* > x^0$.



Centralised equilibrium with social optimum.
In case of social we have -v’ which is increasing in x. This depend on the fact that the equilibrum is the crossing point between marginal utility for individual and marginal profit so the optimal socially amount for externality is x°. X° is less than x* so less than the externality in centralised equilibrium .

Externalities

- ***Example***:
  – Consider a firm with marginal profits of
  $$\pi'(x) = a - bx, \text{ where } a, b > 0$$
  which is decreasing in x.

  – Assume a consumer with marginal damage function of
  $$v'(x) = c + dx, \text{ where } c, d > 0$$
  which is increasing in x. $\pi'(x) = -v'(x)$

We can do an example: we assume marginal profit linear in x with a, b >0. We assume the derivative of indirect utility is also linear. You see Marginal profit is decreasing in x while the marginal disutilty of x is increasing in x.

Externalities

- ***Example*** (continued):
  - The competitive equilibrium amount of externality x^* solves $\pi'(x^*) = 0$, i.e., $a - bx^* = 0$. Hence,
$$x^* = \frac{a}{b}$$
  - The socially optimal level of the externality x^0 solves $\pi'(x^0) = -v'(x^0)$, i.e., $a - bx^0 = c + dx^0$. Thus,
$$x^0 = \frac{a-c}{b+d}$$
which is positive if $\pi'(0) > -v'(0)$, i.e., $a > c$.

Before we do that we have a - bx° = c+ dx° since one is the profit and the other is the v. Then, what we find is $a - bx^{o} = c + dx^{o}$

$$a - c = bx + dx \qquad a - c = x \cdot (b + d)$$

$$x^{o} = \frac{a - c}{b + d}$$

This should be the social optimum

Since $\pi'(x) = 0 \longrightarrow a - bx^{*} = 0$

so $x^{*} = \dfrac{a}{b}$

Externalities

- Negative externalities are not necessarily eliminated at the Pareto optimal solution.
- This would only occur at the extreme case when $\boxed{-v'(0) > \pi'(0).}$
- In this setting, curve $\pi'(x)$ and $-v'(x)$ do not cross, and the Pareto optimal solution only occurs at the corner where $x^0 = 0$.

Externalities

- If firm's production activities produce a ***positive externality*** in the individual's wellbeing, then

$$\underline{v'(x) > 0} \text{ and } \underline{-v'(x) < 0}$$

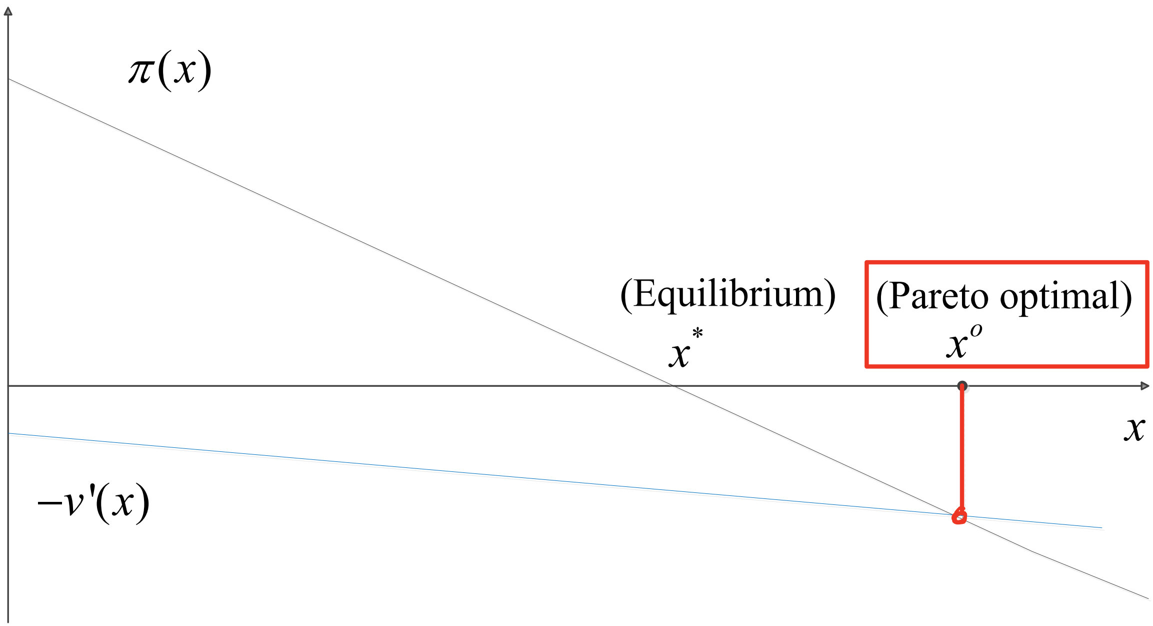
  - That is, $-v'(x) < 0$ lies in the negative quadrant.
  - $\pi'(x)$ remains unaffected.

- In this setting, there is an underproduction of the externality-generating activity relative to the Pareto optimum, i.e., $x^* < x^0$.

You can find a case with positive externalities in the market. So in this case the marginal disutility of consuming will be positive.
In this case x produced utitly is good and not bad (like case of negative externalities).
So if we take with the minus sign we got that $-v'(x) < 0$

Externalities

- Pareto optimal and equilibrium externality level (positive externality).



In this case we have marginal profit function and opposite of marginal utitily function and also in this case we assume it’s decreasing. In this case is decreasing since we have $-v'(x) < 0$

In this case, the social (or Pareto) optimum is given by the two crossing curve.

Social optimum is larger than centralise optimum and this is because we are in a positive externality case.

Solutions to the Externality Problem:
Pigouvian Tax

Pigouvian Taxation

- This policy sets a tax t_x per unit of the externality-generating activity x.
- What is the level of tax t_x that restores efficiency?
- Let us start by re-writing the firm's PMP

$$\max_{x \geq 0} \; \pi(x) - t_x \cdot x$$

- FOC with respect to x:

$$\pi'(x) - t_x \leq 0 \implies \boxed{\pi'(x) \leq t_x}$$

  or $\underline{\pi'(x) = t_x}$ for interior solutions.
- *Intuition*: the firm increases x until the point where the marginal benefit from an additional unit of x coincides with the per-unit tax t_x.

We focused on the Pigouvian tax, in the books there are other solutions.

Let’s assume that the government wants to find a solution for an efficiency cause by the presence of a negative externalities and think to propose a tax on that negative externality.
What could be the amount of the tax to restablish efficiency in the market?
We rewrite PMP of the firm and the problem was to maximise profits the amount with the amount negative externalities.
Now there’s another cost t which is the tax of the government.

The FOC became the derivative of profit - tax <= 0 and this mean that the derivative of the profit is <= than the tax.
This mean that a firm increase pollution up to the point in which the marginal profit of the firm of increasing pollution is exactly equal to the tax rate that the firm have to pay.

We have to find the tax rate!

Pigouvian Taxation

- We know that at the social optimum (i.e., x^0)
$$\pi'(x^0) = -v'(x^0)$$
- Hence, the tax t_x needs to be set at
$$t_x = -v'(x^0)$$
- This forces the firm to internalize the negative externality that its production generates on consumer's wellbeing at x^0.

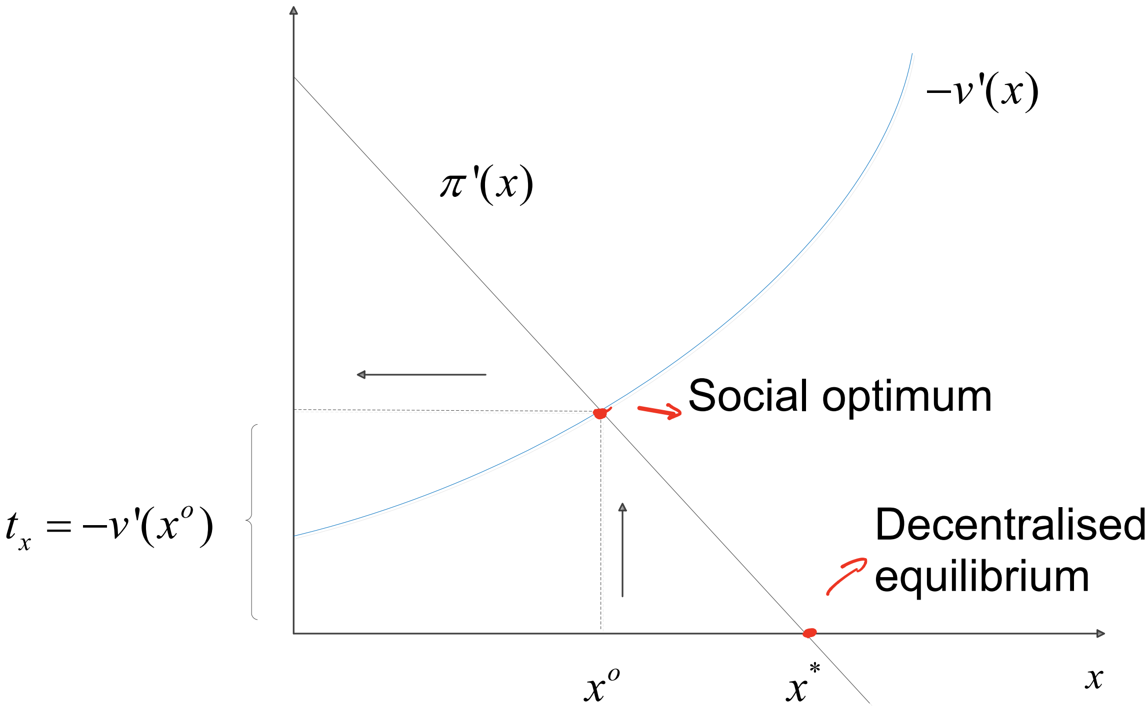
We’ have to find the tax rate!
We know from social welfare maximisation problem: social optimum satisfy this statement: $\pi'(x^0) = -v'(x^0)$

So it’s clear that tax = -v’(x°) to establish efficiency in the market. The level of tax rate that makes production equal to the social optimum level of production is equal to -v’ (x°).

We have marginal profit function that is decreasing and marginal disutility for externality increasing and decentralised equilibrium and social optimum.

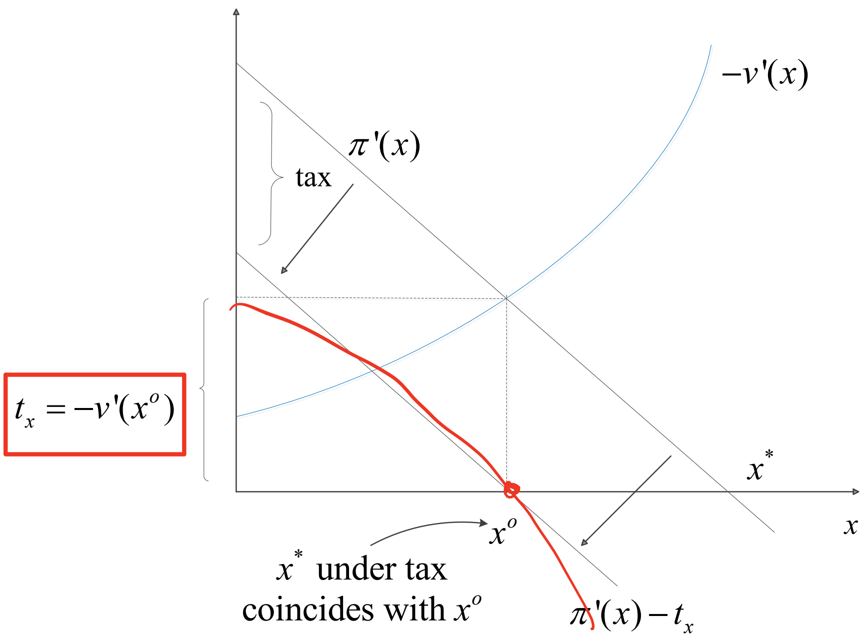
Pigouvian Taxation

- The tax t_x leads the firm to choose a level of x equal to x^0



Pigouvian Taxation

- The tax produces a downward shift in $\pi'(x)$.
- The new marginal benefit curve $\pi'(x) - t_x$ crosses the horizontal axis exactly at x^0.



Government can make production be equal to the social optimum. Simply by raising a tax equal to the marginal disutility of the negative externality and social optimum.
By leveraging this tax, what happen to profit function? Goes down exactly by the amount of the tax and the crossing point now and horizontal axes (level firm would choice) is the social optimum level.
This kind of tax is called Pigouvian tax.

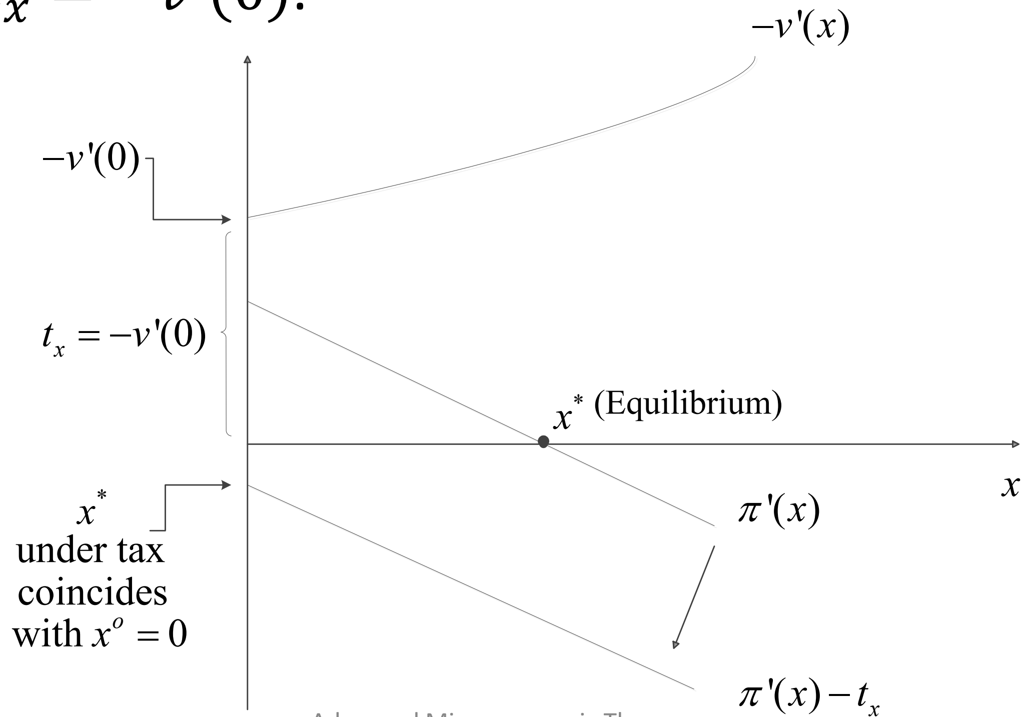
Pigouvian Taxation

- The optimality-restoring tax t_x is equal to the marginal externality at the optimal level x^0.
  - That is, it is equal to the amount of money that the affected individual would be willing to pay in order to reduce x slightly from its optimal level x^0.
- **The tax t_x induces the firm to internalize the externality that it causes on the individual.**

tx can also be interpret as amount of money individual is willing to pay in order to reduce x that is pollution from his optimum level x°.
Imposing Pigouvian tax make the firm internalise the negative externality.
For firm is government add taxes he will interiorise it on the PMP. The optimal production of externality decided by the firm will go down.

Pigouvian Taxation

- If the negative externality is very substantial (and the socially optimum is at $x^0 = 0$), the optimal Pigouvian tax is $t_x = -v'(0)$.



Another example in which optimal level of externality will be 0. Indeed, in this case the marginal disutility of negative externality is always higher than the marginal profit of the firm so even the first unit of externality produces an higher disutility of the consumer than profits that gives to the firms. In this case the socially optimal production will be 0, while decentralise equilibrium will be x*. In this case Marginal externality to 0 so the marginal profit curve will shift down and the optimal production will be 0.

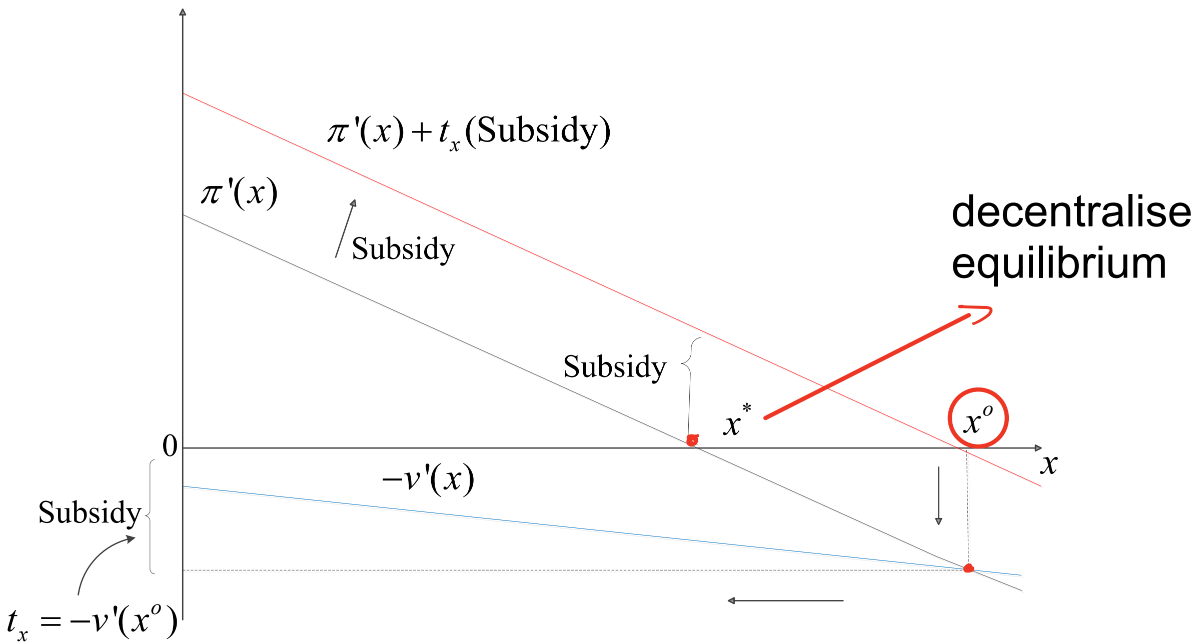
Pigouvian Subsidy

- Previous discussions can also be extended to *positive externalities*.
- Since now $v'(x^0) > 0$ (i.e., x increases individual's welfare), the optimality-correcting tax is
$$t_x = -v'(x^0) < 0$$
- We thus set "negative taxes" on the externality: a per-unit subsidy (s_x).
- The firm receives a payment of t_x for each unit of the positive externality it generates.

Another thing is that same solution can also helps in presence of positive externalities (positive marginal utility to the individual).
So in this case the government should gave to the firm a subsidy (sussidio) instead of a tax.

Pigouvian Subsidy

- The per-unit subsidy produces an upward shift in the marginal benefits of the firm.
- The firm has incentives to increase x beyond the competitive equilibrium level x^* until reaching the Pareto optimal level x^0.



This is an example of equilibrium in positive externality. We have usual profit function, and decentralise equilibrium.
What firm would choose in the absence of Pigouvian taxes? The Pigouvian subsidy!
$-v'(x)$ is the marginal utility now, since $-v'(x)<0 \Rightarrow v'(x)>0$
because of positive externality.
So $-v'$ curve will be below zero and social optimum will be crossing point between marginal profit function and marginal utility so x°.

The government has to impose a subsidy that is equal to the opposite of marginal utility of consumer when consuming the social optimum x°. This mean that marginal profit function will go up by the same measure that is tx. This produces a negative tax, so negative tax is a subsidy. This amount will shift the Marginal profit function upward and then new crossing point will be exactly the social optimum. So government in case of positive externality can reach the social optimum by imposing Pigouvian subsidy

Pigouvian Policy: Important Points

a) A tax on the negative externality is equivalent to a subsidy inducing agents to reduce the externality.

– Consider a subsidy $s_x = -v'(x^0) > 0$ for every unit that the firm's choice of x is below the (decentralized) equilibrium level of x^*.

Lump sum (fixed amount)

– The firm's PMP becomes:

$$\max_{x \geq 0} \pi(x) + s_x(x^* - x) = \pi(x) + \underbrace{s_x x^*}_{\text{subsidy}} - \underbrace{s_x x}_{\text{per unit tax}}$$

– FOC with respect to x yields

$$\pi'(x^0) - s_x \leq 0 \text{ or } \pi'(x^0) \leq s_x$$

A tax on a negative externality is equivalent to a subsidy in using agent agent to reduce externality. We can achieve the same result by taxing the firm or by giving some money incentive to reduce negative externality.
Why this is the case? Let's consider a subsidy.
Since we have negatively externality v'(x°) is negative so with minus will be > 0. Let's assume that what government does is to give a subsidy a set to each unit of the externality that is below the level x* (what firm would choose in the decentralise equilibrium).
For each unit of externality the firm receive a subsidy equal to sx.
So the problem of the firm become to maximise with respect to the externalities the profits + the revenue from this subsidy.
Fix amount of subsidy since x* is given by the individual for the firm. (Lump sum)

If we apply the FOC, we will have that marginal profit - per unit tax rate <= 0. So marginal profit must be <= per unit tax rate and holds with equality in case of interior solution.

It's the same solution of the Pigouvian tax.

Pigouvian Policy: Important Points

- This FOC coincides with that under the Pigouvian taxation (taxing the negative externality at t_x), plus a (negative) lump-sum tax of $t_x x^*$.

- Hence, a subsidy for the reduction of the externality can exactly replicate the outcome of the Pigouvian tax.

Pigouvian Policy: Important Points

b) *The Pigouvian tax levies a tax on the externality-generating activity (e.g., pollution) but not on the output that generated such pollution.*

- Taxing output might lead the firm to reduce output, but it does not necessarily guarantee a reduction in pollutant emissions.
- A tax on output can induce the firm to reduce emissions if emissions bear a constant relationship with output.

2° important feature of the Pigouvian taxes is that Activity that must to be taxed is the negative externality, while is not production that must be tax (production of goods).
The two are equivalent if there is a constant relationship between the level of production and pollution.

Pigouvian Policy: Important Points

c) *The quota and the Pigouvian tax are equally effective under complete information.*

   – They might not be equivalent when regulators face incomplete information about the benefits and costs of the externality for consumers and firms.

The Pigouvian tax is the quota equally effective under complete information, while in incomplete may produce different result

Public Goods

- Before defining public goods, let us define two properties:
  - ***Non-excludability***: If the good is provided, no consumer can be excluded from consuming it.
  - ***Non-rivalry***: Consumption of the good by one consumer does not reduce the quantity available to other consumers.

We can define 4 types of good from this properties

	Rivalrous	**Non-rivalrous**
Excludable	Private Good	Club Good
Non-excludable	Common property resource	Public good

Public Goods

- ***Private goods***, e.g., an apple. These goods are rival and excludable in consumption.

- ***Club goods***, e.g., golf course. These goods are non-rival but excludable in consumption.

You can exclude some guy paying price but (not rival) a lot of people can afford it.

- ***Common property resources***, e.g., fishing grounds. These goods are rival but non-excludable in consumption.

a lot can fish, the more i fish, less other guy can fish

- ***Public goods***, e.g., national defense. These goods are non-rival and non-excludable in consumption.

Non rival and non excludable.

Public Goods

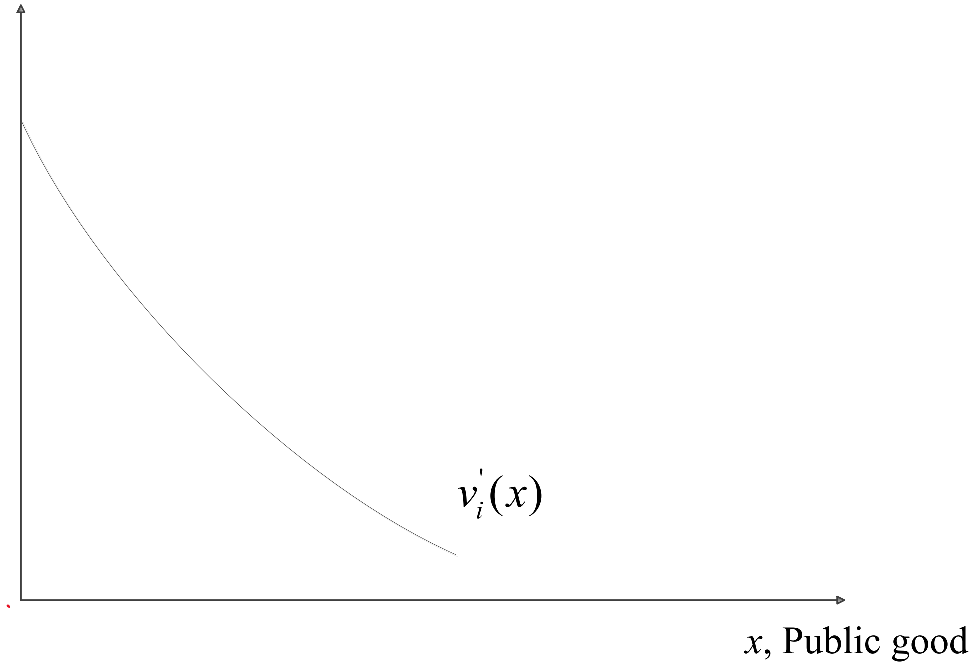
We assume in the economy that there are :

- Consider I consumers, one public good x and L traded private goods. This private are rival and excludable
- Every consumer i's marginal utility from the consumption of x units of a public good is $v_i'(x)$
  - Note that x does not have a subscript because of non-rivalry (every individual can enjoy x units of the public good)
- We consider the case of a public good, where $v_i'(x) > 0$ for every individual i
  - A "public bad" would imply $v_i'(x) < 0$ for every i
- We assume that $v_i''(x) < 0$, which represents a positive but decreasing marginal utility from additional units of the public good.

Marginal utilty for public good (positive and decreasing).

Public Goods

- Marginal benefit from the public good



Public Goods

- We assume that the marginal utility from the public good, $v_i'(x)$, **is independent of the private goods (separable utility, e.g. quasilinear).**

We assume from the producer that:

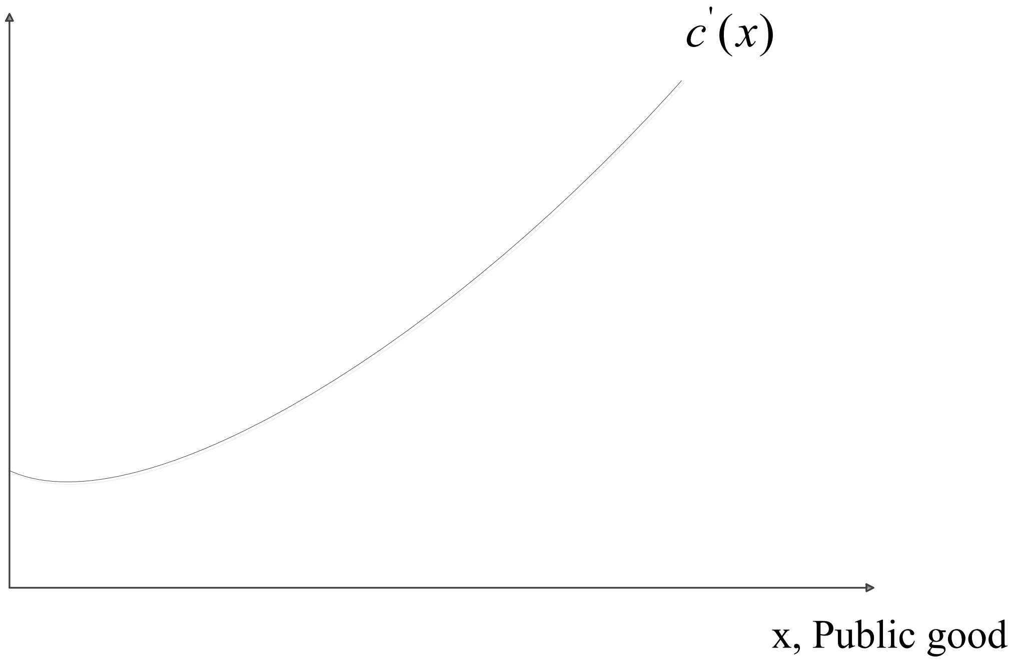
- The cost of supplying x units of the public good is $c(x)$, where $c'(x) > 0$ and $c''(x) > 0$ for all x
  - That is, the costs of providing the public good are increasing and convex in x.

A part depends on a given good and the second part of utility depend on all other good are called numer.

The marginal cost is in positive quadrant since c’ > 0 and c’’ >0

Public Goods

- Marginal costs from providing the public good



Public Goods

- Let us first find the Pareto optimal allocation

$$\max_{x \geq 0} \sum_{i=1}^{I} v_i(x) - c(x)$$

(it would be $\sum_{i=1}^{I} v_i(x) + \pi(x)$ but $\pi(x) = p_x x - c(x)$ but $p_x = 0$, public good is free – compare social welfare with externality)

- FOC with respect to x yields

$$\sum_{i=1}^{I} v_i'(x^o) - c'(x^o) \leq 0$$

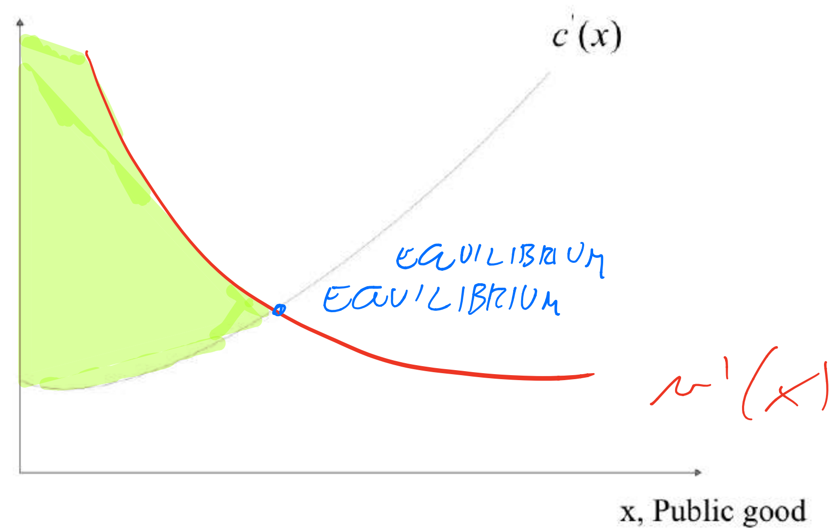
with equality if $x^o > 0$.

- SOCs are satisfied since

$$\sum_{i=1}^{I} v_i''(x^o) - c''(x^o) \leq 0$$

<0 $\quad -\!>0$ $\quad <0$

Now, optimal provision for the public good from social planner:
maximise the total welfare or the aggregate surplus and the aggregate surplus that would be



Social planner wants to maximise the area in green (under marginally utility curve and above marginal cost).

So maximise total welfare for aggregate surplus and this can also be seen in the following way: the total welfare is equal to the total sum of the utility from public good of i individual + the profit providing the public good. But at the same time since the good is public, there is no price for public good. Total revenue are zero and profit correspond to the minus the cost.
So it is equal to maximise the sum of the utility of the consumer + profit of producing public good.

We apply FOC. Also SOC is satisfied: second derivative less than zero because of the assumption we made on marginal utility of public good and marginal cost. So we assume marginal utility is 0 and v'' is < 0 and cost function is convex > 0 but with - sign is < 0.
So SOC is satisfied : objective function is concave.

Public Goods

- In case of an interior solution, the optimal level of public good is achieved for the level of x^o that solves

$$\sum_{i=1}^{I} v_i'(x^o) = c'(x^o)$$

Sum of marginal utilities must be equal to the marginal cost

- That is, the sum of the consumers' marginal benefit from an additional unit of the public good is equal to its marginal cost (***Samuelson rule***).

MANCA NELLE SLIDE

- The Pareto optimal level of public goods does not coincide with that of private goods, where, for interior solutions,

$$v_i'(x_i^*) = c_j'(x_j)$$

- That is, every individual i's private marginal benefit from the private good is equal to its marginal cost

The good in this case is rival and excludable must be equal to the marginal cost of producing the good.

Inefficiency of the Private Provision of Public Goods

Inefficiency of the Private Provision of Public Goods

- Let us consider the case in which a market exists for the public good and that each consumer i chooses how much of the public good to buy, denoted as $x_i \geq 0$, taking as given a market price of p (p_{x} in the previous slides).
- The total amount of the public good purchased by all I individuals is hence $x = \sum_{i=1}^{I} x_i$.
- Consider a single producer of the public good with a cost function $c(x)$.

Starting from this condition that will be the optimal condition in which social planner maximise social welfare.
We'll see in the case that we leave private firm to provide public goods. We will have level of production that is inefficient from a social point of view.

We consider a market in which there is a public good and this public good is traded in the market and so has a price. Each consumer decide the amount of public good to buy and the total amount is as the summation over all I consumer of the amount of public good demand by each consumer. Then we assume market provide of public good has a total cost of c(x).

Inefficiency of the Private Provision of Public Goods

- Formally, at a competitive equilibrium price p^*, each consumer i's purchase of the public good, x_i^*, must solve (assume quasi-linear utility)

$$\max_{x_i \geq 0} \; v_i\left(x_i + \sum_{k \neq i} x_k^*\right) + (w_i - p^* x_i)$$

  – The first term reflects that individual i benefits from both the x_i units of the public good he purchases and $\sum_{k \neq i} x_k^*$ units of the public good that all other individuals acquire;
  – In determining his purchases of the public good, individual i takes the purchases of all the other individuals as given;
  – consumer i pays $p^* x_i$ when acquiring x_i units of the public good.

Now we find the competitive equilibrium price in which each consumer i purchases the amount xi and this amount must hold the following condition:

$$\max_{x_i \geq 0} \; v_i(x_i + \textstyle\sum_{k \neq i} x_k^*) + (w_i - p^* x_i)$$

We assume the consumer function has a quasi-linear utility function and consumer maximise the utility deriving from consumption of public good + utility from other goods (the numerer).

$$v_i(x_i + \textstyle\sum_{k \neq i} x_k^*)$$

So the utility derived from consumption of public good is vi that depend amount of that good he buy + the amount of other consumer buy. Why? Because it's not rival.

The second term: $(w_i - p^* x_i)$

Once you buy the public good, the amount xi you have the expenditure p* xi so the income that is left after buying the public good xi is the total amount of wealth - the expenditure for buying xi.

Inefficiency of the Private Provision of Public Goods

- FOC with respect to x_i yields
$$v_i'(x_i^* + \sum_{k \neq i} x_k^*) - p^* \leq 0$$
with equality if $x_i^* > 0$ (interior solution).

- For compactness, let x^* denote the total purchases of the public good, that is,
$$x^* = x_i^* + \sum_{k \neq i} x_k^*.$$

- Hence, the above FOC can be expressed as
$$v_i'(x^*) - p^* \leq 0$$
with equality if $x_i^* > 0$ (interior solution)

If we compute the FOC, then compute the objective function derivative with respect with xi.
So we will have v'i on total good - p* <= 0 (with equality in case of interior opt).
For compactness we refer to x* to the total purchase of the public good.
So the FOC will be: $v_i'(x^*) - p^* \leq 0$

SO AT THE END FOR INT. OPT

$$v_i(x^*) = p^*$$

Inefficiency of the Private Provision of Public Goods

- On the other hand, the firm's PMP is

$$\boxed{\max_{x \geq 0} \quad \underbrace{p^* x}_{TR} - \underbrace{c(x)}_{TC}}$$

- $\boxed{\text{FOC}}$ with respect to x yields

$$p^* - c'(x^*) \leq 0$$

with equality if $x^* > 0$ (interior solution).

- Finally, the market clearing condition implies that the total amount of the public goods produced coincides with the amount consumed by all individuals.

Now we check the firm PMP and we assume prices are given.
So compute the FOC and get in equilibrium that: $p^* = c'(x^*)$

So we can put this condition together (the one in the slide before).

Inefficiency of the Private Provision of Public Goods

- Combining the FOCs for consumers and the firm, we obtain

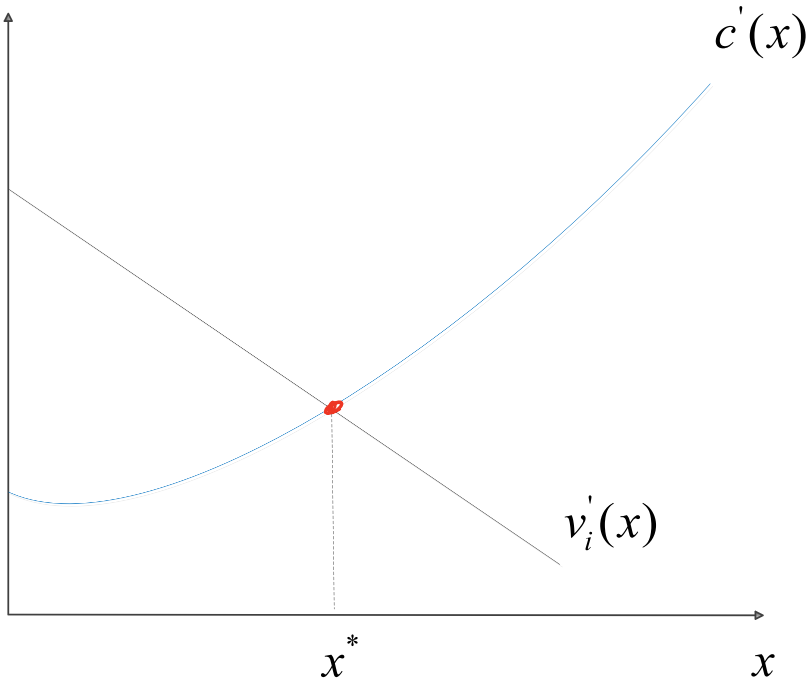
$$\boxed{\begin{aligned} v_i'(x^*) &= c'(x^*) \text{ if } x^* > 0, \\ v_i'(x^*) &< c'(x^*) \text{ if } x^* = 0 \end{aligned}}$$

CORNER SOLUTION

- Intuitively, individual i increases his consumption of the public good until the point in which his marginal benefit from the public good equals the marginal cost.

Inefficiency of the Private Provision of Public Goods

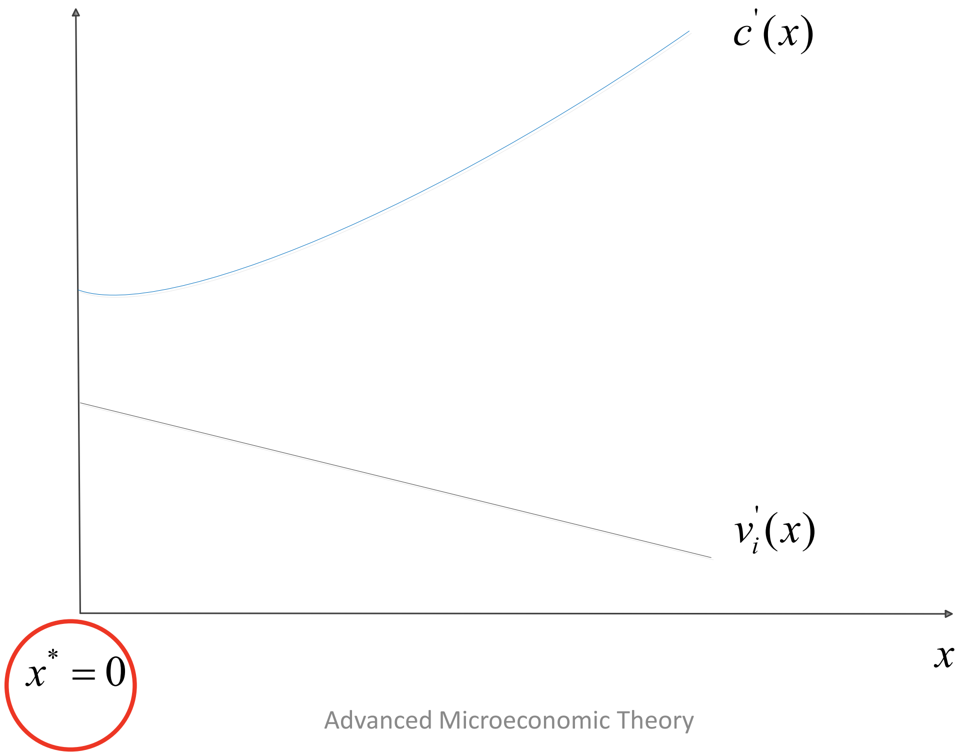
- Equilibrium level of public good (interior solution).



we leave the market provide the public good with equilibrium crossing point between marginal utility of consuming that good and marginal cost.
We have marginal cost increasing and marginal utility decreasing and x* optimal quantity produced by the market.

Inefficiency of the Private Provision of Public Goods

- Equilibrium level of public good (corner solution).



Example of corner solution. Marginal cost above the marginal utility so not crossing point and equilibrium will be at 0 —> corner solution.
You can immediately the difference between the optimal condition when we leave the market to provide the public good and the equilibrium condition in the case of social planner [next slide]

Inefficiency of the Private Provision of Public Goods

- However, at the Pareto optimality, we must have

$$\sum_{i=1}^{I} v_i'(x^o) = c'(x^o)$$

- That is, the summation of the marginal benefit that all individuals obtain from the public good must equal the marginal cost.
- Hence, there is an *underprovision* of the public good in the competitive equilibrium relative to the optimal allocation.
  - *Exception*: when the marginal cost curve is not vertical, i.e., $c''(x) \neq +\infty$.

You can immediately the difference between the optimal condition when we leave the market to provide the public good and the equilibrium condition in the case of social planner [next slide].
In particular, the difference between slide [46], in the left-hand side only marginal utility of the individual i while in the left-hand side we have the sum of all marginal utility of the consumer.
So we can see graphically, that the market provigion of the public good leads to under provigion of the public good —> so less quantity of public good in the market as described in slide [50]

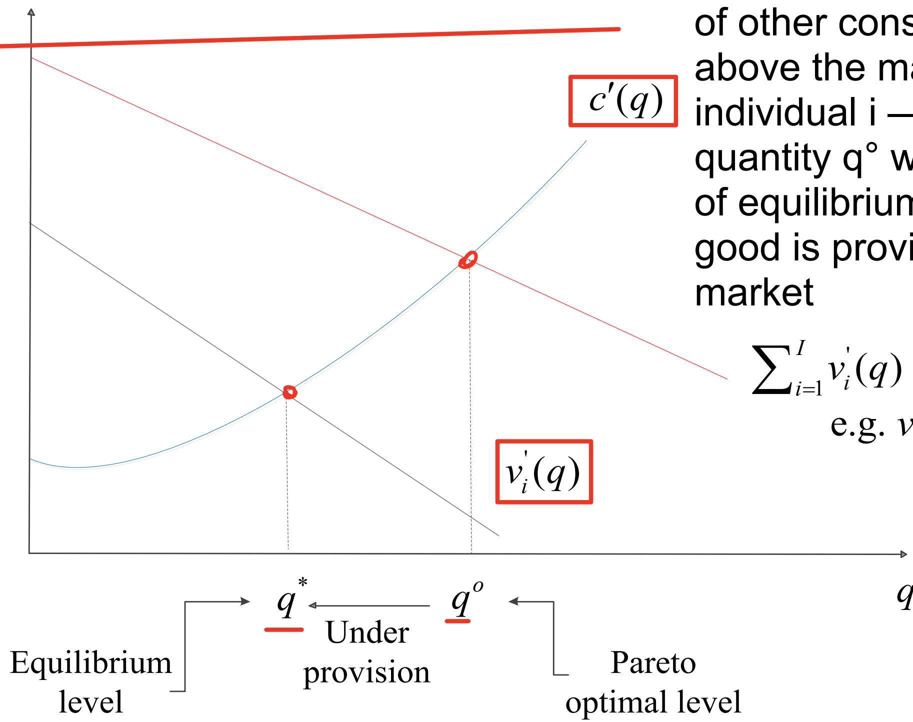
Inefficiency of the Private Provision of Public Goods

- Pareto optimal and equilibrium level of public good (two-consumer economy example)

- If good provided by the market we will have suboptimal provigion

$\Downarrow$

That's reason public goods are provided by the government



- Summation of marginal utility of other consumer will be above the marginal utility of individual i —> optimal quantity q° will be to the right of equilibrium level in the case good is provided by the market

$$\sum_{i=1}^{I} v_i'(q)$$

e.g. $v_1'(q) + v_2'(q)$

Inefficiency of the Private Provision of Public Goods

- *Intuition*:
  - Each individual's purchase of the public good benefits not only him, but also all other individuals in the economy.
  - Each individual does not internalize the positive externalities that his individual purchase of the public good generates on other individuals.
  - Hence, each individual does not have enough incentives to purchase sufficient amounts of the public good.
  - This leads to the *free-rider problem*, whereby the public good in underprovided.

The main reason why private provision of public good is inefficient is that each individual doesn't take in account that buying public good also benefits other consumer: each individual benefit from the total amount of public good not only on the he buys.

Each individual does not internalise the positive externalities: so each individual doesn't have enough incentive to purchase an sufficient amount of the public good.

This leads to free rider in which the public good is under provided.

Another example of public good in Covid19 could be a restriction in social activity: an individual has utility from going out but at the same time has some disutility on risking his own health —> individual decides to go out only considering his utility from leisure and risk for his own health. Will tend to go out much more in the case in which individual is also internalising the potential negative externality producing by going out for instance because it can be a vector of the virus (and damage people). The government have to internalise this public bad just by imposing some penalty and fees for going out. If you cough out you will have to pay or illegal complain.The objective is to get the optimal level of people going out.