Lecture 10 - 07-04-2020

1.1 TO BE DEFINE

1.2 MANCANO 20 MINUTI DI LEZIONE

$$\mathbb{E}\,[z] = \mathbb{E}\,[\,\mathbb{E}\,[z\,|\,x]\,] \qquad \longrightarrow \qquad \mathbb{E}\,[Z\,|\,X = x]$$

$$\mathbb{E}\,[X] = \sum_{t=1}^{m} \mathbb{E}\,[\,x \cdot \Pi\,(At)\,]$$ $A_1, ..., A_m$ **portion of sample law of total probability**

$x \in \mathbb{R}^d \qquad \mathbb{P}(Y_{\Pi(s,x)} = 1) = \mathbb{E}\left[\Pi Y_{\Pi(s,x)} = 1\right] =$ **Law of total probability**

$$= \sum_{t=1}^{m} \mathbb{E}\,(\,\Pi\{Y_t = 1\}\,\cdot\,\Pi\cdot\,\{\,\Pi(s,x) = t\}\,] \quad =$$

$$= \quad \sum_{t=1}^{m} \mathbb{E}\,[\,\mathbb{E}\,[\,\Pi\{Y_t = 1\}\,\cdot\,\Pi\cdot\,\{\Pi(s,x) = t\}\,|\,X_t\,]\,] =$$

given the fact that $Y_t \sim \eta(X_t) \Rightarrow$ give me probability

$Y_t = 1$ *and* $\Pi(s,x) = t$ *are independent given* X_Y $\quad$ (*e.g.* $\mathbb{E}\,[Z\,X] = \mathbb{E}\,[x] \cdot \mathbb{E}\,[z]$)

$$= \sum_{t=1}^{m} \mathbb{E}\,[\mathbb{E}\,[\,\Pi\{Y_t = 1\}\,|\,X_t\,] \cdot \mathbb{E}\,[\,\Pi(s,x) = t|Xt\,]\,] \;=\; \sum_{t=1}^{m} \mathbb{E}\,[\,\eta(X_t)\,\cdot\,\Pi\,\cdot\,\{\Pi(s,x) = t\}\,] =$$

$$= \quad \mathbb{E}\left[\,\eta\,(X_{\Pi(s,x)}\,)\right]$$

$$\mathbb{P}(Y_{\Pi(s,x)}|X = x = \mathbb{E}\,[\,\eta(X_{\Pi}(s,x))\,]$$

$$\mathbb{P}(Y_{\Pi(s,x)} = 1, y = -1) = \mathbb{E}\left[\,\Pi\{Y_{\Pi(s,x)} = 1\} \cdot \Pi\{Y = -1|X\}\,\right]] =$$

$$= \mathbb{E}\left[\,\Pi\{Y_{\Pi(s,x)} = 1\} \cdot \Pi\{y = -1\}\,\right] = \mathbb{E}\left[\,\mathbb{E}\left[\,\Pi\{Y_{\Pi(s,x)} = 1\} \cdot \Pi\{y = -1|X\}\,\right]\,\right] =$$

by independence i can split them

$$Y_{\Pi(s,x)} = 1 \qquad y = -1 \qquad \textbf{which is } 1 - \eta(x) \quad \textit{when} \quad X = x$$

$$= \mathbb{E}\left[\mathbb{E}\left[\Pi\{Y_\Pi(s,x)\} = 1|X\right] \cdot \mathbb{E}\left[\Pi\{y=-1\}|X\right]\right] = \mathbb{E}\left[\eta_{\Pi(s,x)} \cdot (1-\eta(x))\right] =$$

similarly:

$$\mathbb{P}\left(Y_{\Pi(s,x)} = -1, y = 1\right) = \mathbb{E}\left[(1-\eta_{\Pi(s,x)}) \cdot \eta(x)\right]$$

$$\mathbb{E}\left[\ell_D(\hat{h}_s)\right] = \mathbb{P}\left(Y_{\Pi(s,x)} \neq y\right) = \mathbb{P}\left(Y_{\Pi(s,x)} = 1, y = -1\right) + \mathbb{P}\left(Y_{Pi(s,x)} = -1, y = 1\right)$$
$$= \mathbb{E}\left[\eta_{\Pi(s,x)} \cdot (1 - eta(x))\right] + \mathbb{E}\left[(1-\eta_{\Pi(s,x)}) \cdot \eta(x)\right]$$

Make assumptions on D_x *and* η:

1. $\forall X$ drawn from D_x $\quad \max |X_t| \leq 1$
   Feature values are bounded in $[-1,1]$
   all my points belong to this:
   $$X = [-1,1]^d$$
2. η is such that $\exists c < \infty$:
   $$\eta(x) - \eta(x') \leq c \cdot \|X - x'\| \quad \forall\, x, x' \in X$$
   It means that η is **Lipschitz** in X $c < \infty \Leftrightarrow \eta$ is continous

using two facts:

$$\eta(x') \leq \eta(x) + c\,||X - x'|| \quad \longrightarrow \quad \textbf{euclidean distance}$$
$$1 - \eta(x') \leq 1 - \eta(x) + c\,||X - x'||$$

$X' = X_{\Pi(s,x)}$

$$\eta(X) \cdot (1 - \eta(x')) + (1 - \eta(x)) \cdot \eta(x') \leq$$
$$\leq \eta(x) \cdot ((1 - \eta(x)) + \eta(x) \cdot c\,||X - x'|| + (1 - \eta(x)) \cdot c\,||X - x'|| =$$
$$= 2 \cdot \eta(x) \cdot (1 - \eta(x)) + c\,||X - x'||$$

$$\mathbb{E}\left[\ell_d \cdot (\hat{h}_s)\right] \leq 2 \cdot \mathbb{E}\left[\eta(x) - (1 - \eta(x))\right] + c \cdot (E)\left[||X - x_{\Pi(s,x)}||\right]$$

where $\leq$ mean at most

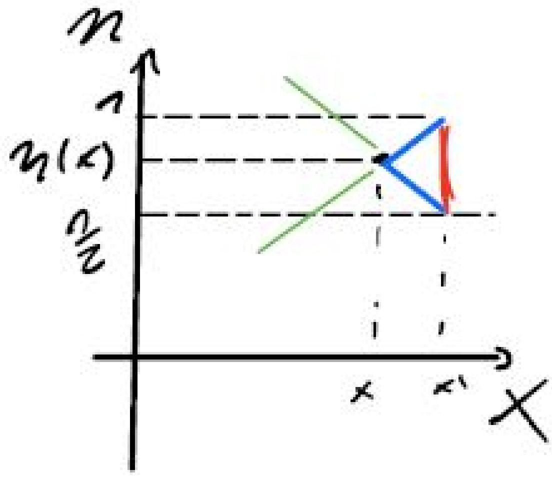

Figure 1.1: Point (2) - where $y = cx + q \qquad y = -cx + q$

1.3 Compare risk for zero-one loss

$$\mathbb{E}\left[\,\min\{\eta(x), 1-\eta(x)\}\,\right] = \ell_D(f^*)$$
$$\eta(x)\cdot(1-\eta(X)) \;\leq\; \min\{\,\eta(x), 1-\eta(x)\,\} \quad \forall x$$
$$\mathbb{E}\left[\,\eta(x)\cdot(1-\eta(x)\,\right] \;\leq\; \ell_D(f^*)$$
$$\mathbb{E}\left[\,\ell_d(\hat{l}_s)\,\right] \leq 2\cdot\ell_D(f^*) + c\cdot\mathbb{E}\left[\,\|X - X_{\Pi(s,x)}\|\,\right]\eta(x)\in\{0,1\}$$

Depends on dimension: **curse of dimensionality**

$\ell_d(f^*) = 0 \iff \min\{\eta(x), 1-\eta(x)\} = 0$ with probability $= 1$
to be true $\eta(x) \in \{0,1\}$

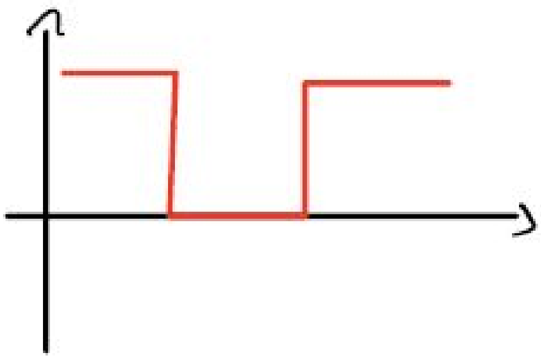

Figure 1.2: Point