

# Combinatorics - part 2

Notes of Alessandra Micheletti

Graph Theory and Discrete Mathematics  
Data Science and Economics

# PERMUTATIONS

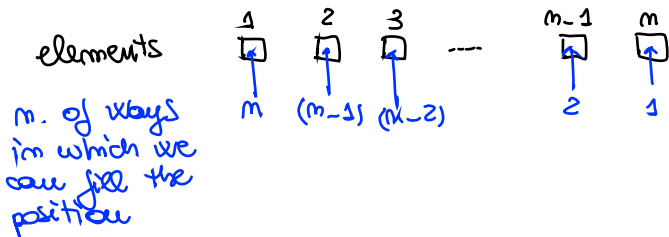
**DEFINITION.** Let  $A$  be a finite set, composed by  $n$  distinct elements.  
 Suppose to order the elements of  $A$ .  
 A permutation is a reordering of the elements of  $A$ .

Example:  $A = \{a, b, c\}$

permutations:

abc	}	6 permutations are possible
acb		
bca		
bac		
cab		
cba		

In general, if A is composed by  $n$  objects, how many permutations can we form with its elements?



Thus the number of permutations of  $n$  different objects is

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

$n$  factorial = product of the first  $n$  integers

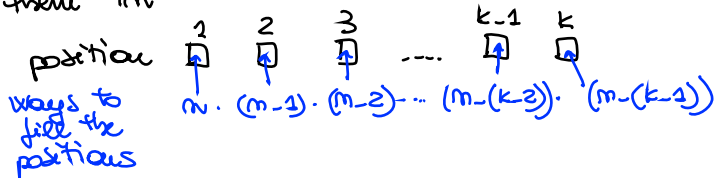
Assume now to have  $n$  different objects but you want to compute the permutations only of a subset of  $k$  of them.

Example: in genetics, DNA is composed by a sequence of the 4 bases A, G, C, T. Triplets, that is sequences of 3 bases, are responsible of the synthesis of amino acids. How many different triplets can be formed?



$$N. \text{ of triplets} = 4 \cdot 3 \cdot 2 = 24$$

In general: if we have  $m$  different elements and we want to fill  $k$  ( $k < m$ ) positions, we can fill them in



Thus the number of permutations of  $k$  objects chosen among  $m$  is

$$m(m-1)(m-2) \dots (m-k+2)(m-k+1) = P_{m,k}$$

that is the product of  $k$  decreasing integers starting from  $m$

Note that we can rewrite  $P_{m,k}$  as follows:

$$\begin{aligned}
 P_{m,k} &= m \cdot (m-1) \cdot (m-2) \cdot \dots \cdot (m-k+1) = \\
 &= \frac{m \cdot (m-1) \cdot \dots \cdot (m-k+1) \cdot (m-k) \cdot (m-k-1) \cdot \dots \cdot 2 \cdot 1}{(m-k) \cdot (m-k-1) \cdot \dots \cdot 2 \cdot 1} =
 \end{aligned}$$

$$= \frac{m!}{(m-k)!}$$

$$= \frac{\text{m. of permutations if we had } m \text{ "boxes to fill" }}{\text{m. of permutations in the "missing boxes"}}$$

Suppose finally that we want to count the permutations of the elements of a set A, which contains repeated elements

Example :  $A = \{a, a, b, c\}$

The sequences

$b$   $a$   $c$   $a$   
 $b$   $a$   $c$   $a$

are the same, since we only switched the position of two repeated elements.

Then the number of different permutations of the 4 elements of A here is

$$\frac{\text{N. of permutations of all elements of A}}{\text{N. of permutations of the repeated elements}} = \frac{4!}{2!}$$

## General rule:

If there are  $n$  objects of which  $n_1$  are equal,  $n_2$  are equal, ...,  $n_k$  are equal, the number of different permutations of the  $n$  objects is

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

Example: the number of anagrams of the word

MISSISSIPPI

is

$$\frac{11!}{4! 4! 2!} = 34650$$

Number of I S P

How would you compute this number with a pocket calculator?



# COMBINATIONS

Let  $A$  be a set composed by  $n$  different objects. Assume that we want to count how many sequences of  $k$  objects can be composed by the elements of  $A$ , but now two sequences are different only if they contain different elements, and not if they contain the same elements with a different order. We call these sequences **combinations of  $k$  objects chosen among  $n$** :  $C_{n,k}$

## EXAMPLE.

In a class there are 12 students and the school has a laboratory which contains only 6 students at a time. Thus the teacher has to organize two working groups to attend the laboratory, Group A and Group B.

In how many different ways Group A can be formed?  
(Note that Group B is automatically formed once you define the composition of Group A)

Clearly two compositions of Group A are different only if they are formed by different students, and not by the way the teacher lists their names. Thus we are counting the combinations of 6 elements among 12

$$C_{12,6}$$

In order to obtain this number we can compute the number of permutations of 6 objects chosen among 12, and then divide by the number of sequences of 6 objects which differ only for the order of their elements, that is 6!

$$C_{12,6} = \frac{P_{12,6}}{6!} = \frac{12!}{(12-6)! 6!} = 924$$

## General rule:

The number of combinations of  $k$  objects chosen among  $n$  is given by

$$C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{(n-k)! k!} = \binom{n}{k} \text{ Binomial coefficient}$$

What is an efficient way to compute  $C_{n,k}$  with a pocket calculator or a computer?













