Combinatorics - part 2

Notes of Alessandra Micheletti

Graph Theory and Discrete Mathematics
Data Science and Economics

PERMUTATIONS

DEFINITION. Let A be a finite set, composed by n distinct elements.
Suppose to order the elements of A.
A permutation is a reordering of the elements of A.

Example: $A = \{a, b, c\}$

permutations: abc
acb
bca
bac
cab
cba

6 permutations are possible

In general, if A is composed by n objects, how many permutations can we form with its elements?

elements	1	2	3	----	$n-1$	n
n. of ways in which we can fill the position	n	$(n-1)$	$(n-2)$		2	1

Thus the number of permutations of n different objects is

$n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1 = n!$

n factorial = product of the first n integers

Assume now to have n different objects but you want to compute the permutations only of a subset of k of them.

Example: in genetics, DNA is composed by a sequence of the 4 bases A, G, C, T. Triplets, that is sequences of 3 bases, are responsible of the synthesis of amino acids. How many different triplets can be formed?

triplet $\square \quad \square \quad \square$

ways to be filled $\quad 4 \quad 3 \quad 2$

N. of triplets $= 4 \cdot 3 \cdot 2 = 24$

In general: if we have n different elements and we want to fill k $(< n)$ positions, we can fill them in

position	1	2	3	....	$k-1$	k
ways to fill the positions	$n \cdot$	$(n-1) \cdot$	$(n-2) \cdot$	...	$(n-(k-2)) \cdot$	$(n-(k-1))$

Thus the number of permutations of k objects chosen among n is

$$n(n-1)(n-2)\cdots(n-k+2)(n-k+1) = P_{n,k}$$

that is the product of k decreasing integers starting from n

Note that we can rewrite $P_{m,k}$ as follows:

$$P_{m,k} = m\cdot(m-1)\cdot(m-2)\cdot\ \dots\ \cdot(m-k+1) =$$

$$= \frac{m\cdot(m-1)\cdot\ \dots\ \cdot(m-k+1)\cdot(m-k)\cdot(m-k-1)\cdot\ \dots\ \cdot 2\cdot 1}{(m-k)(m-k-1)\cdot\ \dots\ \cdot 2\cdot 1} =$$

$$= \frac{m!}{(m-k)!}$$

$$= \frac{\text{n. of permutations if we had } m \text{ "boxes to fill"}}{\text{n. of permutations in the "missing boxes"}}$$

Suppose finally that we want to count the permutations of the elements of a set A, which contains repeated elements

Example : $A = \{a, a, b, c\}$

The sequences

$b\,a\,c\,a$

$b\,a\,c\,a$

are the same, since we only switched the position of two repeated elements.
Then the number of different permutations of the 4 elements of A here is

$$\frac{\text{N. of permutations of all elements of A}}{\text{N. of permutations of the repeated elements}} = \frac{4!}{2!}$$

General rule:

If there are n objects of which n_1 are equal, n_2 are equal, ... , n_k are equal, the number of different permutations of the n objects is

$$\frac{n!}{n_1!\, n_2! \dots n_k!}$$

Example: the number of anagrams of the word

MISSISSIPPI

is

$$\frac{11!}{\underbrace{4!}_{I}\ \underbrace{4!}_{S}\ \underbrace{2!}_{P}} = 34650$$

Number of I S P

How would you compute this number with a pocket calculator?

COMBINATIONS

Let A be a set composed by n different objects. Assume that we want to count how many sequences of k objects can be composed by the elements of A, but now two sequences are different only if they contain different elements, and not if they contain the same elements with a different order. We call these sequences combinations of k objects chosen among n: $C_{n,k}$

EXAMPLE.

In a class there are 12 students and the school has a laboratory which contains only 6 students at a time. Thus the teacher has to organize two working groups to attend the laboratory, Group A and Group B.

In how many different ways Group A can be formed?
(Note that Group B is automatically formed once you define the composition of Group A)

Clearly two compositions of Group A are different only if they are formed by different students, and not by the way the teacher lists their names. Thus we are counting the combinations of 6 elements among 12

$$C_{12,6}$$

In order to obtain this number we can compute the number of permutations of 6 objects chosen among 12, and then divide by the number of sequences of 6 objects which differ only for the order of their elements, that is 6!

$$C_{12,6} = \frac{P_{12,6}}{6!} = \frac{12!}{(12-6)!\,6!} = 924$$

General rule:

The number of combinations of k objects chosen among n is given by

$$C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$$

Binomial coefficient

What is an efficient way to compute $C_{n,k}$ with a pocket calculator or a computer?