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Time Series Econometics

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Chapter 9: Models of non-stationary time series

Topics: Models of non-stationary time series, Deterministic trends and other deterministic components, Unit roots / Integrated processes

Modelling nonstationarity, deterministic components

What if data have a deterministic trend (linear, quadratic...) or another deterministic component (such as a deterministic cycle or a seasonal effect)?

Consider

$$Y_t = d_t + u_t$$

where

d_t is a deterministic component

u_t is a stationary ARMA(p, q) model

Then we can:

1. estimate $\widehat{d}_t$,
2. compute $\hat{u}_t = Y_t - \widehat{d}_t$,
3. model $\hat{u}_t$ as if it was our observables.

For example, if we want to forecast $Y_{t+1|t,\ldots}$, we proceed as before and we

1. compute $\widehat{u}_t$.

2. choose an appropriate ARMA model for $\widehat{u}_t$

3. forecast $\widehat{u}_{t+1|t\ldots}$ using the standard approach to forecast ARMA processes

4. compute $\widehat{d}_{t+1}$

5. recombine the model and obtain

$$\widehat{Y}_{t+1|t,\ldots} = \widehat{d}_{t+1} + \widehat{u}_{t+1|t,\ldots}$$

Example, deterministic trend: $Y_t = \delta t + u_t$.

Estimate $\widehat{\delta} = \dfrac{\sum_{t=1}^{T} tY_t}{\sum_{t=1}^{T} t^2}$, then compute $\widehat{u}_t = Y_t - \widehat{\delta}t$; if you are interested in the forecast, this is $\widehat{Y}_{t+1|t,\ldots} = \widehat{\delta}(t+1) + \widehat{u}_{t+1|t,\ldots}$

Nonstationarity, stochastic components

Consider

$$Y_t = Y_{t-1} + \varepsilon_t,\ \varepsilon_t \text{ i.i.d}(0, \sigma^2) \text{ when } t \geq 1;\ Y_0 = 0$$

Clearly, the model is not stationary (notice the dependence on t in the definition). We can also notice this by checking, for example, the variance:

$$\begin{aligned} V(Y_t) &= V(Y_{t-1} + \varepsilon_t) = V(Y_{t-1}) + V(\varepsilon_t) + 2E(Y_{t-1}\varepsilon_t) \\ &= V(Y_{t-1}) + \sigma^2 = V(Y_{t-2} + \varepsilon_{t-1}) + \sigma^2 \\ &= V(Y_{t-2}) + V(\varepsilon_{t-1}) + 2E(Y_{t-2}\varepsilon_{t-1}) + \sigma^2 \\ &= V(Y_{t-2}) + 2\sigma^2 = \ldots = V(Y_0) + t\sigma^2 = t\sigma^2. \end{aligned}$$

In the same way, the covariances too depend on time: for $j > 0$,

$$\begin{aligned} Cov(Y_t, Y_{t+j}) &= Cov(Y_t, Y_t + \varepsilon_{t+1} + \ldots + \varepsilon_{t+j}) \\ &= Cov(Y_t, Y_t) + Cov(Y_t, \varepsilon_{t+1}) + \ldots + Cov(Y_t, \varepsilon_{t+j}) \\ &= V(Y_t) + 0 + \ldots + 0 = t\sigma^2. \end{aligned}$$

This model can also be rewritten, using recursive substitution t times,

$$Y_t = Y_0 + \sum_{j=0}^{t-1} \varepsilon_{t-j}$$

(notice here that the initial condition is not irrelevant, in the sense that it still affects Y_t and does not fade away; by setting $Y_0 = 0$, we do not "rule it out", but we "normalise for it").

This particular process is called "random walk": it is not stationary nor ergodic, it is not mean reverting, and all the properties we discussed for stationary ARMA(p, q) do not apply here.

Rearranging the indices, and replacing $Y_0 = 0$,

$$Y_t = \sum_{j=1}^{t} \varepsilon_j$$

this last notation motivates the fact that processes of this type are called "integrated", or, more precisely, "integrated of order 1", $I(1)$.

The concept of integration may be extended, to

$$Y_t = Y_{t-1} + u_t, \text{ when } t \geq 1, \text{ where}$$

$$u_t = \phi_1 u_{t-1} + \ldots + \phi_p u_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q},$$

u_t stationary and invertible

ε_t i.i.d$(0, \sigma^2)$;

$$Y_0 = 0$$

then $Y_t = \sum_{j=1}^{t} u_j$ is again an "integrated" $I(1)$ process, $Y_t \in I(1)$

(and it could be further generalised to u_t MA(∞), provided that $0 < \sum_{j=-\infty}^{\infty} \gamma_j < \infty$)

On the other hand, since $\Delta Y_t = u_t$, then $u_t \in I(0)$.

Notice that in our definition, $E(Y_t) = 0$ because we set $Y_0 = 0$.

When $Y_0 \neq 0$, say $Y_0 = \kappa$, using recursive substitutions,

$$Y_t = \kappa + \sum_{j=1}^{t} u_j$$

and $E(Y_t) = \kappa \neq 0$.

Most authors consider Y_t as defined here as $I(1)$, especially in empirical work. For our purpose, it may be convenient to refer to this as "$I(1)$ with non-zero mean", especially when the presence of a non-zero mean may change the limit distribution of estimators or test statistics.

Modelling nonstationarity, stochastic components

When it is known that $Y_t \in I(1)$, it means that it is known that $Y_t = Y_{t-1} + u_t$.

Rearranging terms, it means that it is known that $Y_t - Y_{t-1} = u_t$, i.e. it is possible to compute

$$u_t = \Delta Y_t.$$

As $u_t \in I(0)$, we focus on modelling u_t instead.

for example, in order to forecast, $Y_{t+1|t,\dots}$ we would

1. compute $u_t = \Delta Y_t$
2. using standard approach, forecast $\hat{u}_{t+1|t,\dots}$
3. recombine the model, i.e. $\hat{Y}_{t+1|t,\dots} = Y_t + \hat{u}_{t+1|t,\dots}$

.

Combining stochastic and non stochastic forms on non stationarity.

One model of particular interest is

$$Y_t = c + Y_{t-1} + u_t, \text{ when } t \geq 1, \text{ where}$$

$$u_t = \phi_1 u_{t-1} + \ldots + \phi_p u_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q},$$

u_t stationary and invertible

ε_t i.i.d$(0, \sigma^2)$;

$$Y_0 = 0$$

In this case,

$$Y_t = ct + \sum_{j=1}^{t} u_j$$

so both the linear trend and the integration are present. Some authors consider Y_t as defined here as $I(1)$: we will be more specific, and we will refer to this as "$I(1)$ and with a deterministic trend".

★ If $Y_0 = \kappa$ instead,

$$Y_t = \kappa + ct + \sum_{j=1}^{t} u_j$$

★ Recalling that $Y_t = c + Y_{t-1} + u_t$, the forecast of $Y_{t+1|t,\ldots}$ is

$$\widehat{Y}_{t+1|t,\ldots} = c + Y_t + \widehat{u}_{t+1|t,\ldots}.$$