**(1)** Find the eigenvalues and eigenvectors of $A$, $A^2$, $A^{-1}$, and $A+4I$ where:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Check that $det(A) = \lambda_1\lambda_2$.

**(2)** Find the two eigenvalues $\lambda_1$, $\lambda_2$, and the corresponding eigenvectors, of the matrix $A$,

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}.$$

let $\mathbf{V} \in \mathbb{R}^{\mathbf{2}}$ an eigenvector for the eigenvalue $\lambda_1$, and $\mathbf{W} \in \mathbb{R}^{\mathbf{2}}$ an eigenvector for the eigenvalue $\lambda_2$, show that the two vectors $\mathbf{V}$ and $\mathbf{W}$ are linearly independent.

**(3)** Find one eigenvector for the given matrix corresponding to the given eigenvalue,

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \lambda = 5, \quad \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}, \lambda = -1.$$

**(4)** Find the dimension of the subspace (eigenspace) of all eigenvectors corresponding to the eigenvalue $\lambda = 2$ for the matrix

$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 2 \\ 1 & 4 & 4 \end{bmatrix}.$$

Find a basis for this eigenspace.

**(5)** Let $M$ be the matrix

$$\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

Find the eigenvalues $\lambda_1, \lambda_2$, and a corresponding unit eigenvectors $\mathbf{V}_1$, and $\mathbf{V}_2$. Let $U$ the matrix with the same columns as these eigenvectors. Check that

$$M = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^T.$$

This example is a particular case of the following Theorem.

`If` $A$ `is a real symmetric matrix, then there is an orthogonal matrix` $Q$ `that diagonalizes` $A$`, that is,` $Q^TAQ = D$`, where` $D$ `is diagonal.`

Why?