EX. 11th Oct 2013 [Sketch of the solution]

(1) We use the echelon form reduction

$$A=\begin{bmatrix}1&-2&3\\2&-5&1\\1&-4&-7\end{bmatrix}\ \begin{matrix}R_2\leftarrow R_2-2R_1\\R_3\leftarrow R_3-R_1\end{matrix}\ \begin{bmatrix}1&-2&3\\0&-1&-5\\0&-2&-10\end{bmatrix}\xrightarrow{R_3-2R_2}\begin{bmatrix}1&-2&3\\0&-1&-5\\0&0&0\end{bmatrix}$$

Number of non zero rows $=2 \Rightarrow \operatorname{rank}(A)=2$

$$B=\begin{bmatrix}1&2&-1&1\\2&4&-3&0\\1&2&1&\alpha\end{bmatrix}\xrightarrow[R_3-R_1\to R_3]{R_2-2R_1\to R_2}\begin{bmatrix}1&2&-1&1\\0&0&-1&-2\\0&0&2&\alpha-1\end{bmatrix}\xrightarrow{R_3+2R_2\to R_3}\begin{bmatrix}1&2&-1&1\\0&0&-1&-2\\0&0&0&(\alpha-5)\end{bmatrix}$$

$$\text{number of nonzero rows} = \begin{cases}\alpha-5=0 \Rightarrow \alpha=5 \Rightarrow \operatorname{rank}(B)=2\\ \alpha-5\neq 0 \Rightarrow \alpha\neq 5 \Rightarrow \operatorname{rank}(B)=3\end{cases}$$

(2) $h=1$

(a) $A=\begin{bmatrix}2&-1&0\\1&2&5\\-1&1&1\end{bmatrix}$, we have to consider the homogeneous system $Ax=b \Rightarrow$ we use the echelon form

$$\begin{bmatrix}2&-1&0\\1&2&5\\-1&1&1\end{bmatrix}\xrightarrow[R_3+1/2R_1]{R_2-1/2R_1}\begin{bmatrix}2&-1&0\\0&5/2&5\\0&1/2&1\end{bmatrix}\xrightarrow{R_3-1/5R_2}\begin{bmatrix}2&-1&0\\0&5/2&5\\0&0&0\end{bmatrix}$$

$\Rightarrow \infty$ - solution, 1 free variable $\Rightarrow \dim\{x : Ax=0\}=1$

(b) $h=-1$ $A=\begin{bmatrix}2&-1&0\\1&2&5\\1&-1&-1\end{bmatrix}\xrightarrow[R_3-1/2R_1]{R_2-1/2R_1}\begin{bmatrix}2&-1&0\\0&5/2&5\\0&-1/2&-1\end{bmatrix}\xrightarrow{R_3-\frac{1}{5}R_2}\begin{bmatrix}2&-1&0\\0&5/2&5\\0&0&0\end{bmatrix}$

$\operatorname{rank}(A)=2 \Rightarrow$ range of T is not $\mathbb{R}^3$ and is not one-to-one

(c) $h\in\mathbb{R}$ $A=\begin{bmatrix}2&-1&0\\1&2&5\\-h&h&h\end{bmatrix}\xrightarrow[R_3+\frac{h}{2}R_1]{R_2-1/2R_1}\begin{bmatrix}2&-1&0\\0&5/2&5\\0&h/2&h\end{bmatrix}\xrightarrow{R_3-\frac{h}{5}R_2}\begin{bmatrix}2&-1&0\\0&5/2&5\\0&0&0\end{bmatrix}$

$\Rightarrow \operatorname{Rank}(A)=2$

(3) $p=4$, $q=3$. In order to compute the matrix of L^2 the linear transformation:

$$T(e_1)=T\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-3\\1\\2\end{bmatrix} \quad T(e_2)=T\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}6\\-2\\-4\end{bmatrix} \quad T(e_3)=T\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\2\\5\end{bmatrix}$$

$$T(e_4)=T\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\\8\end{bmatrix}$$

$$A=\begin{bmatrix}-3 & 6 & -1 & 1\\ 1 & -2 & 2 & 3\\ 2 & -4 & 5 & 8\end{bmatrix}$$

The range of T is the space spanned by the column of A. We check the columns that are linearly independent.

We use echelon form

$$A=\begin{bmatrix}-3 & 6 & -1 & 1\\ 1 & -2 & 2 & 3\\ 2 & -4 & 5 & 8\end{bmatrix} \xrightarrow[R_3+\frac{2}{3}R_1]{R_2+\frac{1}{3}R_1} \begin{bmatrix}-3 & 6 & -1 & 1\\ 0 & 0 & 5/3 & 10/3\\ 0 & 0 & 13/3 & 26/3\end{bmatrix} \xrightarrow[R_3-\frac{13}{5}R_2]{} \begin{bmatrix}-3 & 6 & -1 & 1\\ 0 & 0 & 5/3 & 10/3\\ 0 & 0 & 0 & 0\end{bmatrix}$$

non zero pivots
columns 1, 3

$\dim(\text{range}(T))=2$

$$\text{basis}=\left\{\begin{bmatrix}3\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\5\end{bmatrix}\right\}$$

1th column 3th column

(4) $T(e_1)=\begin{bmatrix}1\\2\\1\end{bmatrix}$; $T(e_2)=\begin{bmatrix}3\\\alpha\\4\end{bmatrix}$; $T(e_3)=\begin{bmatrix}\alpha\\0\\2\alpha\end{bmatrix}$; $T(e_4)=\begin{bmatrix}1\\\alpha\\0\end{bmatrix}$

$$A=\begin{bmatrix}1 & 3 & \alpha & 1\\ 2 & \alpha & 0 & \alpha\\ 1 & 4 & 2\alpha & 0\end{bmatrix},$$ we use the echelon form

$$\begin{bmatrix}1 & 3 & \alpha & 1\\ 2 & \alpha & 0 & \alpha\\ 1 & 4 & 2\alpha & 0\end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix}1 & 3 & \alpha & 1\\ 0 & \alpha-6 & -2\alpha & \alpha-2\\ 0 & 1 & \alpha & -1\end{bmatrix} \text{ swap } \begin{bmatrix}1 & 3 & \alpha & 1\\ 0 & 1 & \alpha & -1\\ 0 & \alpha-6 & -2\alpha & \alpha-2\end{bmatrix}$$

$$\xrightarrow{R_3+(6-\alpha)R_2} \begin{bmatrix} 1 & 3 & \alpha & 4 \\ 0 & 1 & \alpha & -1 \\ 0 & 0 & \alpha(4-\alpha) & 2(\alpha-4) \end{bmatrix}$$

$\forall \alpha$ the first two rows are non zero.

last row $\alpha = 0$

$[0 \quad 0 \quad 0 \quad -8]$ is non zero

$\alpha = 4 \to [0 \quad 0 \quad 0 \quad 0] \Rightarrow \text{rank}(A) = 2$

$\alpha = 0$ rank$(A) = 3 \Rightarrow$ three non zero rows, basis $= \{1°$ column, $2°$ column, $4°$ column$\} = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\0\\4\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$

$\alpha = 4$ rank$(A) = 2$ basis $= \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\4\\4\end{bmatrix} \right\}$

for $\alpha \neq 4 \Rightarrow \text{rank}(A) = 3 \Rightarrow \text{range}(A) = \mathbb{R}^3 \Rightarrow V = \begin{bmatrix}0\\1\\0\end{bmatrix} \in \text{range}(A)$

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(5) matrix A with columns v_1, v_2, v_3, v_4; and echelon form

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 5 & 4 & 8 \\ -1 & -3 & -2 & -5 \\ 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \xrightarrow[R_4-2R_2]{R_3+R_2}$$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ 2 non zero rows $\Rightarrow$ 2 vectors are linearly independent

$\Rightarrow \dim(\text{span}\{v_1, v_2, v_3, v_4\}) = 2$

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(6) $P = \dfrac{X \cdot Y}{\|Y\|^2} Y$ $\qquad X \cdot Y = 1 + 2 + 2 = 5$ $\qquad P = \dfrac{5}{9}\begin{bmatrix}1\\2\\2\end{bmatrix}$

$\|Y\|^2 = 1 + 4 + 4 = 9$

$$X - P = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{5}{9}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}4/9\\-1/9\\-1/9\end{bmatrix} \Rightarrow (X-P)\cdot Y = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0 \Rightarrow$$

$(X - P) \perp Y$

(7) $v_1 \cdot v_2 = 2 + 1 - 3 = 0$ $\quad v_1 \cdot v_3 = 4 - 5 + 1 = 0$ $\quad v_2 \cdot v_3 = 8 - 5 - 3 = 0$

[it is enough due to the symmetry of the dot product]

$\sqrt{v_1 \cdot v_1} = \|v_1\| = \sqrt{3}$ $\quad \|v_2\| = \sqrt{14}$ $\quad \|v_3\| = \sqrt{42}$ $\Rightarrow$

$\left\{ \bar{v}_1 = \frac{v_1}{\sqrt{3}} \,;\; \bar{v}_2 = \frac{v_2}{\sqrt{14}} \,;\; \bar{v}_3 = \frac{v_3}{\sqrt{42}} \right\}$ orthonormal basis of $\mathbb{R}^3$

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(8)

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^T A$$

$$Ax = b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x = A^{-1} b = A^T b = \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

orthogonal matrix

$$= \begin{bmatrix} 5/\sqrt{11} \\ 2/\sqrt{6} \\ 2/\sqrt{66} \end{bmatrix}$$

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(9)

$$\begin{cases} x_1 = b_1 \\ 2x_1 - x_2 = b_2 \\ -x_1 + x_2 + 3x_3 = b_3 \end{cases}$$

$Ax = b \Rightarrow$

$$\Rightarrow \begin{cases} x_1 = b_1 \\ x_2 = 2b_1 - b_2 \\ x_3 = \dfrac{-b_1 + b_2 + b_3}{3} \end{cases} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1/3 & 1/3 & 1/3 \end{bmatrix}$$
