DEPARTMENT OF ECONOMICS, MANAGEMENT AND QUANTITATIVE METHODS

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B-74-3-B Time Series Econometics

Fabrizio Iacone

Discussion of Exercise Sheet 1

1.

i.

For a generic MA(2),

$\gamma_0 = Var(Y_t) = E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})^2\right]$

$= E\left(\varepsilon_t^2 + \theta_1^2\varepsilon_{t-1}^2 + \theta_2^2\varepsilon_{t-2}^2 + 2\varepsilon_t\theta_1\varepsilon_{t-1} + 2\varepsilon_t\theta_2\varepsilon_{t-2} + 2\theta_1\varepsilon_{t-1}\theta_2\varepsilon_{t-2}\right)$

$= \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 = (1 + \theta_1^2 + \theta_2^2)\,\sigma^2;$

$\gamma_1 = Cov(Y_t, Y_{t-1}) = E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})(\varepsilon_{t-1} + \theta_1\varepsilon_{t-2} + \theta_2\varepsilon_{t-3})\right] = (\theta_1 + \theta_2\theta_1)\,\sigma^2$

$\gamma_2 = Cov(Y_t, Y_{t-2}) = E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})(\varepsilon_{t-2} + \theta_1\varepsilon_{t-3} + \theta_2\varepsilon_{t-4})\right] = \theta_2\sigma^2$

$\gamma_{j\geq 3} = Cov(Y_t, Y_{t-j\geq 3}) = 0$

(these are, of course, particular applications of the formula $\gamma_j = \sum_{k=0}^{\infty}\psi_k\psi_{k+j}\sigma^2$ when we set $\psi_0 = 1$, $\psi_1 = \theta_1$, $\psi_2 = \theta_2$, $\psi_{j\geq 3} = 0$).

Thus,

$$\gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j = \gamma_0 + 2\times(\gamma_1 + \gamma_2)$$

as covariances γ_j for $i > 2$ are zero. We then rewrite

$$\gamma_0 + 2\times(\gamma_1 + \gamma_2) = \left(\left(1 + \theta_1^2 + \theta_2^2\right) + 2\left(\theta_1 + \theta_2\theta_1 + \theta_2\right)\right)\sigma^2 = (1 + \theta_1 + \theta_2)^2\,\sigma^2$$

ii. For $Y_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2}$,

$Y_{t-1} = \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} + \theta_2 \varepsilon_{t-3}$,
$Y_{t-2} = \varepsilon_{t-2} + \theta_1 \varepsilon_{t-3} + \theta_2 \varepsilon_{t-4}$, then

$$\begin{aligned}
Y_t + Y_{t-1} + Y_{t-2} &= \varepsilon_t + (1+\theta_1)\,\varepsilon_{t-1} + (1+\theta_1+\theta_2)\,\varepsilon_{t-2} + (\theta_1+\theta_2)\,\varepsilon_{t-3} + \theta_2 \varepsilon_{t-4} \\
\sum_{t=1}^{T} Y_t &= \varepsilon_T + (1+\theta_1)\,\varepsilon_{T-1} + (1+\theta_1+\theta_2) \sum_{t=1}^{T-2} \varepsilon_t + (\theta_1+\theta_2)\,\varepsilon_0 + \theta_2 \varepsilon_{-1}
\end{aligned}$$

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} Y_t = \frac{1}{\sqrt{T}} \varepsilon_T + (1+\theta_1) \frac{1}{\sqrt{T}} \varepsilon_{T-1} + (1+\theta_1+\theta_2) \frac{1}{\sqrt{T}} \sum_{t=1}^{T-2} \varepsilon_t + \frac{1}{\sqrt{T}} (\theta_1+\theta_2) \frac{1}{\sqrt{T}} \varepsilon_0 + \theta_2 \frac{1}{\sqrt{T}} \varepsilon_{-1}$$

and notice that ε_T, ε_{T-1}, ε_0, ε_{-1} are all bounded in probability so $\frac{1}{\sqrt{T}}\varepsilon_T$, $\frac{1}{\sqrt{T}}\varepsilon_{T-1}$, $\frac{1}{\sqrt{T}}\varepsilon_0$, $\frac{1}{\sqrt{T}}\varepsilon_{-1}$ all go to 0 in probability as $T \to \infty$.
On the other hand,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T-2} \varepsilon_t = \frac{\sqrt{T-2}}{\sqrt{T}} \frac{\sqrt{1}}{\sqrt{T-2}} \sum_{t=1}^{T-2} \varepsilon_t$$

and, as $T \to \infty$,

$$\frac{\sqrt{T-2}}{\sqrt{T}} \to 1, \;\; \frac{1}{\sqrt{T-2}} \sum_{t=1}^{T-2} \varepsilon_t \to_d N\left(0, \sigma^2\right)$$

Combining all these results,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} Y_t \to_d N\left(0, \sigma^2 \left(1+\theta_1+\theta_2\right)^2\right)$$

2.
In this case we can use a Central Limit Theorem argument and derive

$$\sqrt{T}\left(\overline{Y} - \mu\right) \to_d N\left(0, \sum_{j=-\infty}^{\infty} \gamma_j\right)$$

and therefore

$$\sqrt{T} \frac{\left(\overline{Y} - \mu\right)}{\sqrt{\sum_{j=-\infty}^{\infty} \gamma_j}} \to_d N(0,1)$$

This statistic has known limit distribution but we cannot compute it because we do not know $\sum_{j=-\infty}^{\infty} \gamma_j$. However, we may replace it with an estimate: as $\sum_{j=-\infty}^{\infty} \gamma_j = \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j$, we can estimate the latter as

$$\widehat{\gamma}_0 + 2\sum_{j=1}^{T-1} k_j \widehat{\gamma}_j$$

where k_j is a weight called *kernel* and it is such that $k_j \to 0$ as $j \to T$. We will then test using

$$\sqrt{T}\frac{\left(\overline{Y} - \mu\right)}{\sqrt{\widehat{\gamma}_0 + 2\sum_{j=1}^{T-1} k_j \widehat{\gamma}}} \to_d N\left(0, 1\right)$$

for a suitable kernel, and reject the null hypothesis if the realisation of the absolute value of the test statistic exceeds the 5% critical value.
Two estimates of the long run variance are

$$\widehat{\gamma}_0 + 2\sum_{j=1}^{M} \widehat{\gamma}_j, \;\; M/T \to 0, \;\; \textit{rectangular kernel estimate}$$
$$\widehat{\gamma}_0 + 2\sum_{j=1}^{M} \frac{M-j}{M}\widehat{\gamma}_j, \;\; M/T \to 0, \;\; \textit{triangular kernel estimate}$$

The triangular kernel estimate is also known as Bartlett (kernel) estimate, or as Newey-West estimate. Typically we choose $M = \sqrt{T}$: in this case, with $T = 100$ we need $M = 10$. We only have four values for γ_j so that's all we can do, but we will bear in mind that the result of the test may be not so reliable.
Using the rectangular kernel, we get

$$\widehat{\gamma}_0 + 2\sum_{j=1}^{T-1} k_j \widehat{\gamma}_j = 2 + 2 \times (1 + 0.25 - 0.25) = 4$$

whereas using the triangular kernel we get

$$\widehat{\gamma}_0 + 2\sum_{j=1}^{T-1} k_j \widehat{\gamma}_j = 2 + 2 \times \left(\frac{3}{4}1 + \frac{2}{4}0.25 - \frac{1}{4}0.25\right) = 3.625$$

and the test statistic using the Rectangular kernel takes value

$$\sqrt{100} * \frac{4-3}{\sqrt{4}} = 5$$

whereas using the triangular kernel we get

$$\sqrt{100} * \frac{4-3}{\sqrt{3.625}} = 5.252.$$

As the critical value is 1.96, either way the absolute value of the test statistic excceds the critical value and the null is therefore rejected. Note: in the discussion I presented results using both kernels. In practice, you do not need to do so, discussing only one would be sufficient.

3.

Stationarity is defined by saying that $E\left(Y_t\right)$ and $E\left[\left(Y_t - E\left(Y_t\right)\right)\left(Y_{t-j} - E\left(Y_{t-j}\right)\right)\right]$ do not depend on time, so of course one way to check it is by looking at the first two moments.

Sometimes, it is easier to check if Y_t admits an MA(∞) representation $Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$, where the parameters ψ_j do not depend on time and are such that $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, and ε_t is white noise (notice that this also requires the existence of the second moment for ε_t).

Finally, stationarity can be checked on ARMA models by checking the roots of the polynomial equation associated to the AR part of the model are all outside the unit circle.

i. $Y_t = \varepsilon_t + 1.6\varepsilon_{t-1} + 0.48\varepsilon_{t-2}$.

This is MA(2), so it is stationary; alternative: notice that $\psi_0 = 1$, $\psi_1 = 1.6$, $\psi_2 = 0.48$ and $\psi_{j \geq 3} = 0$ so $\sum_{j=0}^{\infty} \psi_j^2 = 1 + 1.6^2 + 0.48^2 = 3.7904 < \infty$

ii. $Y_t = Y_{t-1} + \varepsilon_t$ $t > 0$, $Y_0 = 0$.

Replacing $Y_{t-1} = Y_{t-2} + \varepsilon_{t-1}$, we have $Y_t = Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t$.

Replacing $Y_{t-2} = Y_{t-3} + \varepsilon_{t-2}$, we have $Y_t = Y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t$.

Iterating, $Y_t = \sum_{s=1}^{t} \varepsilon_s$.

$E\left(Y_t\right) = E\left(\sum_{s=1}^{t} \varepsilon_s\right) = \sum_{s=1}^{t} E\left(\varepsilon_s\right) = 0$;

$E\left(Y_t^2\right) = E\left(\sum_{s=1}^{t} \varepsilon_s\right)^2 = E\left(\sum_{s=1}^{t} \varepsilon_s^2 + 2\sum_{s=1}^{t}\sum_{r=1}^{s-1} \varepsilon_s\varepsilon_r\right) =$
$\sum_{s=1}^{t} E\left(\varepsilon_s^2\right) + 2\sum_{s=1}^{t}\sum_{r=1}^{s-1} E\left(\varepsilon_s\varepsilon_r\right) = \sum_{s=1}^{t} \sigma^2 = t\sigma^2$
using $E\left(\varepsilon_s^2\right) = \sigma^2$, $E\left(\varepsilon_s\varepsilon_r\right) = 0$ because $s \neq r$.

Since the variance depends on time, the process is not stationary.

Notice, here, that writing this as $Y_t = \phi Y_{t-1} + \varepsilon_t$ and then concluding that the process is not stationary because $\phi = 1$ instead of $|\phi| < 1$ is not correct. It is indeed true that $\phi = 1$ and that the process is not stationary. What is not correct, is to treat $\phi = 1$ as a necessary condition for non-stationarity. In fact, in the lectures we only showed that if $|\phi| < 1$, then we have stationarity, but nothing about what happens if $|\phi| < 1$ fails. The condition $|\phi| < 1$ is a sufficient condition (to ensure stationarity), but we did not study $\phi = 1$ so we do not know.

iii. $Y_t = \varepsilon_t \varepsilon_{t-1}$.

$E\left(Y_t\right) = E\left(\varepsilon_t\varepsilon_{t-1}\right) = E\left(\varepsilon_t\right)E\left(\varepsilon_{t-1}\right) = 0$, where $E\left(\varepsilon_t\varepsilon_{t-1}\right) = E\left(\varepsilon_t\right)E\left(\varepsilon_{t-1}\right)$ follows using the fact that ε_t and ε_{t-1} are independent.

$E\left(Y_t^2\right) = E\left(\varepsilon_t^2\varepsilon_{t-1}^2\right) = E\left(\varepsilon_t^2\right)E\left(\varepsilon_{t-1}^2\right) = \sigma^2\sigma^2 = \sigma^4$, where $E\left(\varepsilon_t^2\varepsilon_{t-1}^2\right) = E\left(\varepsilon_t^2\right)E\left(\varepsilon_{t-1}^2\right)$ follows using the fact that ε_t and ε_{t-1} are independent.

$E\left(Y_t Y_{t-1}\right) = E\left[\left(\varepsilon_t \varepsilon_{t-1}\right)\left(\varepsilon_{t-1}\varepsilon_{t-2}\right)\right] = E\left(\varepsilon_t \varepsilon_{t-1}^2 \varepsilon_{t-2}\right) = E\left(\varepsilon_t\right) E\left(\varepsilon_{t-1}^2\right) E\left(\varepsilon_{t-2}\right) = 0 \times \sigma^2 \times 0 = 0.$

For $j > 1$, $E\left(Y_t Y_{t-j}\right) = E\left(\varepsilon_t \varepsilon_{t-1} \varepsilon_{t-j} \varepsilon_{t-j-1}\right) = 0.$

So both the mean and all the covariances do not depend on time, this process is stationary.

Notice, here, that since the process is stationary, the by Wold representation we can write it as $Y_t = \kappa_t + \sum_{j=0}^{\infty} \psi_j u_j$ for some white noise $\{u_t\}_{t=-\infty}^{\infty}$. In this particular case, $E\left(Y_t\right) = 0$ so set $\kappa_t = 0$ and representation is $Y_t = u_t$ where $E\left(u_t\right) = 0$, $E\left(u_t^2\right) = \sigma^4$.

iv. $Y_t = 0.8Y_{t-1} - 0.8Y_{t-2} + \varepsilon_t$

This is an AR(2), so look at $Y_t - 0.8Y_{t-1} + 0.8Y_{t-2} = \varepsilon_t$, $\left(1 - 0.8L + 0.8L^2\right) Y_t = \varepsilon_t$,

$\left(1 - 0.8z + 0.8z^2\right) = 0$, Solution is: $\frac{0.8 \pm \sqrt{0.8^2 - 4 \times 0.8}}{2 \times 0.8} = \frac{0.8 \pm \sqrt{-2.56}}{1.6} = \frac{0.8 \pm 1.6i}{1.6} = 0.5 \pm i$

to see if the roots are in absolute value outside the unit circle, remember that $|a \pm ib| = \sqrt{a^2 + b^2}$, so in this case $|0.5 \pm i| = \sqrt{0.5^2 + 1} = \sqrt{1.25} > 1$. Therefore, the process is stationary.

v. Y_t $NID\left(1, 1\right)$ for t odd, exponentially independently distributed for t even.

Regardless of wether t is even or odd, $E\left(Y_t\right) = 1$, $Var\left(Y_t\right) = 1$, $E\left[\left(Y_t - E\left(Y_t\right)\right)\left(Y_{t-j} - E\left(Y_{t-j}\right)\right)\right] = 0$ when $j \neq 0$ (because of the assumption of independence), so the process is stationary.

vi. Y_t is independent identically distributed Cauchy

The process has no finite first or second moment so it is not stationary.

2.

i. Let $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ be white noise and assume that $\phi\left(L\right) Z_t = \theta\left(L\right) \varepsilon_t$. Invertibility means that we can invert $\phi\left(L\right) Z_t = \theta\left(L\right) \varepsilon_t$ as $\theta\left(L\right)^{-1} \phi\left(L\right) Z_t = \varepsilon_t$ (thus "inverting" $\theta\left(L\right)$) and (setting $\pi\left(L\right) = \theta\left(L\right)^{-1} \phi\left(L\right)$), we can write

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_j Z_{t-j}$$

ie given the history of Z_t, we can compute ε_t (we could also say that Z_t admits a AR(∞) representation, $Z_t = \sum_{j=1}^{\infty} \alpha_j Z_{t-j} + \varepsilon_t$).

ii.

$$
\begin{aligned}
Y_t - 2 &= \varepsilon_t + 1.6\varepsilon_{t-1} + 0.48\varepsilon_{t-2} = \left(1 + 1.6L + 0.48L^2\right)\varepsilon_t \\
&= \left(1 + 1.2L + 0.4L + (1.2 \times 0.4)\, L^2\right)\varepsilon_t = (1 + 1.2L)\,(1 + 0.4L)\,\varepsilon_t
\end{aligned}
$$

so $\lambda_1 = 1.2$, $\lambda_2 = 0.4$ and since $|\lambda_1| \geq 1$, then the process is not invertible.

iii.

For a generic MA(2),

$\gamma_0 = Var\,(Y_t) = E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})^2\right]$

$= E\left(\varepsilon_t^2 + \theta_1^2\varepsilon_{t-1}^2 + \theta_2^2\varepsilon_{t-2}^2 + 2\varepsilon_t\theta_1\varepsilon_{t-1} + 2\varepsilon_t\theta_2\varepsilon_{t-2} + 2\theta_1\varepsilon_{t-1}\theta_2\varepsilon_{t-2}\right)$

$= \sigma^2 + \theta_1^2\sigma^2 + \theta_2^2\sigma^2 = (1 + \theta_1^2 + \theta_2^2)\,\sigma^2$;

$\gamma_1 = Cov\,(Y_t, Y_{t-1}) = E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\,(\varepsilon_{t-1} + \theta_1\varepsilon_{t-2} + \theta_2\varepsilon_{t-3})\right] = (\theta_1 + \theta_2\theta_1)\,\sigma^2$

$\gamma_2 = Cov\,(Y_t, Y_{t-2}) = E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2})\,(\varepsilon_{t-2} + \theta_1\varepsilon_{t-3} + \theta_2\varepsilon_{t-4})\right] = \theta_2\sigma^2$

$\gamma_{j\geq3} = Cov\,(Y_t, Y_{t-j\geq3}) = 0$

(these are, of course, particular applications of the formula $\gamma_j = \sum_{k=0}^{\infty}\psi_k\psi_{k+j}\sigma^2$ when we set $\psi_0 = 1$, $\psi_1 = \theta_1$, $\psi_2 = \theta_2$, $\psi_{j\geq3} = 0$).

So

$$
\begin{aligned}
\rho_1 &= \frac{(\theta_1 + \theta_2\theta_1)}{(1 + \theta_1^2 + \theta_2^2)} = \frac{(1.6 + 0.48 \times 1.6)}{(1 + 1.6^2 + 0.48^2)} = 0.624\,74 \\
\rho_2 &= \frac{\theta_2}{(1 + \theta_1^2 + \theta_2^2)} = \frac{0.48}{(1 + 1.6^2 + 0.48^2)} = 0.126\,64 \\
\rho_{j\geq3} &= 0
\end{aligned}
$$

(notice, here, that the AC function of an MA(q) process only takes nonzero values if $j \leq q$).

The invertible representation of the same process is

$$
\begin{aligned}
Y_t - 14 &= (1 + (1/1.2)\,L)\,(1 + 0.4L)\,\varepsilon_t = \left(1 + (1/1.2)\,L + 0.4L + 0.4/1.2L^2\right)\varepsilon_t \\
&= \varepsilon_t + (1.48/1.2)\,\varepsilon_{t-1} + 1/3\varepsilon_{t-2}
\end{aligned}
$$

Check that the autocorrelations are the same:

$$
\begin{aligned}
\rho_1 &= \frac{(\theta_1 + \theta_2\theta_1)}{(1 + \theta_1^2 + \theta_2^2)} = \frac{(1.48/1.2 + 1/3 \times 1.48/1.2)}{\left(1 + (1.48/1.2)^2 + 1/3^2\right)} = 0.624\,74 \\
\rho_2 &= \frac{\theta_2}{(1 + \theta_1^2 + \theta_2^2)} = \frac{1/3}{\left(1 + (1.48/1.2)^2 + 1/3^2\right)} = 0.126\,64
\end{aligned}
$$

3.

In order to derive the best linear forecasts, we use the formula

$$\widehat{Y}_{t+1|t,...,t-m+1} = \alpha_1^{(m)} Y_t + \alpha_2^{(m)} Y_{t-1} + ... + \alpha_m^{(m)} Y_{t-m+1}$$

for $m = 1$ and for $m = 2$.

Recall that

$$\begin{pmatrix} \alpha_1^{(m)} \\ \alpha_2^{(m)} \\ ... \\ \alpha_{m-1}^{(m)} \\ \alpha_m^{(m)} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{m-2} & \gamma_{m-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{m-3} & \gamma_{m-2} \\ ... & ... & ... & ... & ... \\ \gamma_{m-2} & \gamma_{m-3} & \cdots & \gamma_0 & \gamma_1 \\ \gamma_{m-1} & \gamma_{m-2} & \cdots & \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ ... \\ \gamma_{m-1} \\ \gamma_m \end{pmatrix}$$

In order to compute $\alpha_m^{(m)}$ we then need to compute all the autocovariances up to γ_m first. Since we are interested in m up to $m = 2$, we compute

$$\begin{aligned} \gamma_0 &= \left(1 + \theta^2\right) \sigma^2 = \frac{5}{4}\sigma^2 \\ \gamma_1 &= \theta\sigma^2 = \frac{1}{2}\sigma^2 \\ \gamma_2 &= 0. \end{aligned}$$

So, when $m = 1$, (part i.)

$$\alpha_1^{(1)} = \left(\gamma_0 \right)^{-1} \left(\gamma_1 \right) = \frac{4}{5}\frac{1}{2} = \frac{2}{5}$$

and

$$\widehat{Y}_{t+1|t} = \alpha_1^{(1)} Y_t = \frac{2}{5} \times 0.8 = 0.32$$

and when $m = 2$ (part ii.)

$$\begin{pmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \frac{10}{21} \\ -\frac{4}{21} \end{pmatrix}$$

so

$$\widehat{Y}_{t+1|t,t-1} = \alpha_1^{(2)} Y_t + \alpha_2^{(2)} Y_{t-1} = \frac{10}{21} \times 0.8 - \frac{4}{21} \times 1.2 = 0.152\,38$$

iii. In order to compute the partial autocorrelation function, recall that this is $\alpha_1^{(1)}$, $\alpha_2^{(2)}$, $\alpha_3^{(3)}$. We already computed $\alpha_1^{(1)}$ and $\alpha_2^{(2)}$ so we only need $\alpha_3^{(3)}$. This is obtained solving

$$\begin{pmatrix} \alpha_1^{(3)} \\ \alpha_2^{(3)} \\ \alpha_3^{(3)} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}$$

so we need γ_3 as well, and it is easy to verify that $\gamma_3 = 0$. Therefore,

$$\begin{pmatrix} \alpha_1^{(3)} \\ \alpha_2^{(3)} \\ \alpha_3^{(3)} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{5}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{4} \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{42}{85} \\ -\frac{4}{17} \\ \frac{8}{85} \end{pmatrix}$$

Therefore, the partial autocorrelation function is

$\alpha_1^{(1)}$	$\alpha_2^{(2)}$	$\alpha_3^{(3)}$
$\frac{2}{5}$	$-\frac{4}{21}$	$\frac{8}{85}$