Notes Advanced Microeconomics

In economics we make a lot of assumption —> Main reason is that economics model are just a modellization of the reality. We need need something simpler and easier than the reality. Assumption should help us to simplify the problem.

Economics model are useful because when we analise data, we need to have some background theory to interpret the result.

Example
If i have some data on the sales or on the price of the good. Then we estimate a regression: you see how sales depends on the price. What kind of relation we are going to expect? Positive or negative? Negative.
How do we explain this negative relation? This relation is based on theory in which we assume we receive some satisfation(utility) from a given good. You can't compare satisfation with price that you're paying for that good. If price increase you buy less! If you have an income you can buy less unit of the good is lesser. Unit I can afford= Income/price. But there are example in which price increase and sells increase. The based theory is like this but is based on assumptions.

Ch.1 - Consumer Theory

First lectures will be manly on consumer choices.

Main topics: Consumer theory explains how consumer decides what to buy and how much of a good to buy.

In particolar, we will see the concept of preference and choice. We will main base the lecture on preference based-approach compared with the choice approach. We will see utilty function also and then introduce way to rationalise behaviours. In economics they care about their utilty but don't care about others so the concept of altruism will be implemented editing the concept of utilty function.

What is **preference**?
First, how consumer decides between two different goods.
Do you prefer an orange or an apple?
2 guys, one orange and one apple.
This choice are based on some preferences, so we will define is the preferences of individuals. This is an element with which can make choice. We will see the so called **consumption set** that is indicate with X.

Consumption set X: all set of alternatives that are available to the decision maker.
In this case the DM is the consumer.

So set of all possible choice means consumption set is very big. In the consumption set we will not only goods to buy but all possible combinations of quantities of these goods. (1 apple, 2 orange or 2 apples, 1 orange). In our exercise we will be mainly two goods with quantities to buy.

Preference based-approach: assume that I know the preference of consumers so according to the preference that i know i can predict what people will buy.
In the example i took i know she prefers oranges than apple and if i offer an orange or an apple she will decises to buy and orange.

Choice-based approach: the second approach is the choice-based approach. According to this approach i don't know her preference but I make her an offer and i offer her 1 orange and 1 apple. Based on the choice that she makes, she decides to buy the orange. So I infer that she prefer orange to apple. So I build the preference based on preferences on my observation of her consumption here. This is something closer to reality but it's harder to treat it analiticlaly. Sometime to change preference maybe. In the traditional theory we will use preference does not change over time. Main advantage of the choice-based approach is based on behaviours which is something I can observe while preference based approach rely on preferences we have to assume we know (even in the reality we don't) as assumption.

Preference-based approach

How preference defined? We should ask to the decision maker which are the possible alternatives which are all in the consumption set X.
Given two alternatives x and y that belong to the consumption set (X) what kind of ranking can you do? You can say that given this two, i can prefere an orange to an apple and the simple of preference is like a greater sign. I can say i prefer an apple to an orange o i could say I'm indifference (tilde symble).

A preference relation is an operator that allows you to do this ranking and the kind of presence relation we will be use must be complete.

Complete: Individuals must be able to compare every alternatives in the consumption set.
So for any two alternative you are able to choice between one of this.
Ex. I can compare even with good that i never bought.
In reality, If i ask you to compare one good to another you would say "i don't know".

A binary preference relation is a relation. In Mathematics a binary relation of a pair.
When we compare alternative we will compare what we called ordered pairs.

Binary relation: collection of ordered pairs(x,y) from a set x,y appartenete al consumption set X.

Esempio: this are two ordered pair.

THIS REFERS TO ORANGE GROUP

(1,2) (2,1)

N. ORANGE N. APPLES

BETTER BECAUSE I HAVE MORE QUANTITY

(2,1) (2,2)

THIS HOLDS IF ALL GOODS ARE BOTH DESIDERABLE

(POLLUTION SHOULD BE LESS)

BADS!

The symble of strict preferences.
Before we said that x is prefer to y but we don't say that x is strictly preferred to y. We are introducing the operator of **strict preferences**. Here we have three options: x strictly pref to y or y strictly pref to x or x ind to y. I can only choose one between the three.

The same relation could be define using another operator which is called **weak preference** operator where x is as good as y or x is weakly prefered to y.
Weak preference operator we will not have three options but only two. We could say that x is al least as good as y or y is at least as good as x. In this case, the difference in respect to the strict preference is that we can choose both. This mean the two goods could have the same value. In case you choice both option this mean that the two good are indifferent.

The same preferences can be describe using the two symbles. So to say that x is ind to y, in stric preference we said only check x ind y, while with the weak preference i check the two boxes.

This part is about: a way to define how to make choice and we introduce this operator to define preference between alternatives. So what is prefered to what and what is indifferent to what.

Reflexivity: any alternative x can be in a set of thing that I could buy and every alternative must be indifference to itself. (X is as good as X —> it's like a tautology.

Reflexivity implies that X ind to itself or using weak preference is as good as y.

In order to analise consumer behaviour we have to verify that relation is relational.
A preference relation if weak preference:
- Completeness: we are able to compare (to rank) every alternatives.
- Transitivity: implies 3 possible alternatives that are also referred as bundles of goods.

Bundle could be 2 orange and an apple. Or two oranges and two apples.

$$\underbrace{\underset{O\quad A}{(2,1)}}_{x} \qquad \underbrace{\underset{O\quad A}{(2,2)}}_{y}$$

If bundle like this we would have 3 goods:

$$\underset{O\quad A\quad S}{(2,1,3)}$$

But for the majority we will thread pair of goods.

Transitivity implies that you weakly prefered x to y and you weakly prefer y to z then implies you weakly prefer x to z. Even this is a stronger assumption so this mean if I ask you: you prefer orange to apple? You say orange. Then, you prefer apple to strawberry ? Then we can conclude that orange is weakly prefered to strawberries.
In theory holds but in the reality you may prefer strawberry to orange (but in the course we will use transitivity).

$$O \succsim A \quad A \succsim S \overset{\text{TRANSITIVITY}}{\Longrightarrow} O \succsim S$$

Rational preference mean that preference relation satisfy completeness (i can compare all possible alternatives) and transitivity.

One bundle is prefered to another but we don't say how to make choice so why x is prefered to y. One possible way of define preferences is one of the following and this is how preferences are described. X weakly prefered to y, we have to define a decision rule in which we can predic which choice will be made by the consumer.

X is weakly prefered to y if and only if:
I defined the components of the bundles. According to this decision rule, for this individual the first bundle prefered to the second if summing the quantity of the goods in the bundle I obtain a sum that is greater or equal to the sum of the components of the second bundles.

BUNDLES :

x (x_1, x_2) y (y_1, y_2)

$x \succsim y$ IF SUMMING THE COMPONENTS OF 1° BUNDLE I GET A BIGGER VALUE THAN THE SUM OF 2° BUNDLE COMPONENTS

x $(1,2)$ y $(2,1)$ $\longrightarrow 2+1 \geq 2+1 \longrightarrow 3 \geq 3$? ✓ THEN WE CAN CONCLUDE $x \succsim y$

$(1,2)$ $(3,1)$ $\longrightarrow 2+1 \geq 3+1 \longrightarrow 3 \geq 4$? ✗ THEN WE <u>CAN'T</u> CONCLUDE $x \succsim y$

Imaging the guys that we propose, choices the bundle with the highest quantity respected tot he two goods. We have to prove wherever a preference satisfy some preferences.
How can we test if this relation is complete and transitive?

Let's start with completeness. You have to be able to decide if x is weakly pref to y or y weakly pref to x of both (indifferent)?
This preference relation satisfy completeness? If i give you two bundles, i will always be able to compare this two bundle? This is the definition of completeness. Yes, it's complete because we can always compare two real number.

Transitivity: this satisfy transitivity. If i found that x is weakly pref to y, y strictly pref to z then x weakly pref to z? YES.

$$x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$$

$$\sum x_i \geq \sum y_i \wedge \sum y_i \geq z \Rightarrow \sum x_i \geq \sum z_i$$

We can always compare real number. Just compare the quantity in a bundle to verify the preference relation and verify is the preference relation is rational.

Although we did this assumption, there's a branch of economics which is called experimental eco monomials: takes individual and bring individuals to lab and test some assumption about Microeconomics theory. There are quite a lot of examples that individuals choice violate this assumptions.

There are potential **source of intransitivity** in preference:
1. Indistinguishable alternatives
2. framing effects
3. Aggregation of criteria
4. Change in preferences

This may violate the transitivity properties.

Example of framing effects.
Framing: phenomena in which your answer may depend on the order of the question.
Imaging that I bring you this and then ask you to decide this three alternative:
(Paris for $574)
We should actually say that a and b are the same because the holiday are the same. The holidays is a week in Paris for $574 so same offer. So we change the way we presenting it. If i compare a with c and b with c, if a > c also b>c. Instead, in the lab many individuals that violate this properties.

Another example:
Coffee paradigm (Paradigma del caffè). How many spoon of sugar you want? Maybe you cannot distinguished between 2 or 3 spoon maybe.
This is **also violation of rationality!**

If i giving you 70 orange and 70 apple and make you choose by majority. Then this violate transitivity.
This assumption help us to simplify the problems! But in reality is not like this.

The final goal is to define if a simple model can describe a reality and how well this can be predicted.

Ch. 2 - Utility function

Preference can be described in the way he showed before. We implicitly define a function for the preference relation that was the sum of the components of the bundle.

$$(x_1, x_2) \rightarrow \underbrace{y}_{\text{FUNCTION}} = x_1 + x_2$$ — THIS IS SUM OF COMPONENTS

THIS PREFERENCE RELATION CAN BE DEFINE BY THIS FUNCTION:

$$\underline{u(x_1, x_2) = x_1 + x_2}$$

A WAY TO DESCRIBE PREFERENCE RELATION

Utility function We can generalise:
Utility function is a function that is define the consumption set. So taking as input the bundle in the consumption set it give us a real number. This function can be called utility function representing the preferences if for any two alternatives we can say that x if weakly prefered to y if and of if the utility of the x is greater or equal than the utility of y.
We assume that we know the preference of the individuals in a sense that we know the utility function of the individuals.

So preference relation can be describe by utility function. If this is true, we can say if x weakly prefered to y or viceversa.
An important thing is that for our consumption theory is that we are able to rank the alternatives.

$$x \succeq y \iff u(x) \geq u(y)$$

20	10
30	10

The utility of bundle is greater than utility of bundle of y. I will choose x. So we want to predict if we will choose x instead of y. We don't care about cardinality, so the number of the utility function but we just care about the rank. So we want to allowed the consumer to rank (put in order) the different alternatives.

Any strictly increasing transformation of the utility function also give a utility function that describe the same preferences. If i apply an increasing transformation to the utility function which gives another utility function that describe the same preference.

Describe the same preference since it's a strictly increasing transformation.

$$u(x) = (\hat{x}_1, \hat{x}_2) = 2 \qquad u(y) = (\overset{2}{y}_1, \hat{y}_2) = 3$$

$$3 \geq 2 \text{ so } y \succsim x$$

IF I USE $3 \cdot u(x)$

$$3 \cdot u(x) = 6 \qquad 3u(y) = 9 \implies y \succsim x$$

In the example that we take we assume that all goods are desiderable. We will speak about monotonicity, strong monotonicity, satiation and non-satiation. We can include different goods in combination of n goods in which a set of vector with n component in which every components is a real number.

Advanced Microeconomics (EPS)

Chapter 1: Preferences

Outline

- Preference and Choice
- Preference-Based Approach
- Utility Function
- Indifference Sets, Convexity, and Quasiconcavity
- Special and Continuous Preference Relations
- Social and Reference-Dependent Preferences
- Hyperbolic and Quasi-Hyperbolic Discounting
- Choice-Based Approach
- Weak Axiom of Revealed Preference (WARP)
- Consumption Sets and Constraints

Preference and Choice

Preference and Choice

- We begin our analysis of individual decision-making in an abstract setting.
- Let $X \in \mathbb{R}_+^N$ be a set of possible alternatives for a particular decision maker.
  - It might include the consumption bundles that an individual is considering to buy.
  - *Example*:

$$X = \{x, y, z, \dots\}$$

$$X = \{\text{Apple}, \text{Orange}, \text{Banana}, \dots\}$$

Preference and Choice

- Two ways to approach the decision making process:
  1) ***Preference-based approach***: analyzing how the individual uses his preferences to choose an element(s) from the set of alternatives X.
  2) ***Choice-based approach***: analyzing the actual choices the individual makes when he is called to choose element(s) from the set of possible alternatives.

Preference and Choice

- Advantages of the Choice-based approach:
  - It is based on observables (actual choices) rather than on unobservables (individual preferences)
- Advantages of Preference-based approach:
  - More tractable when the set of alternatives X has many elements.

Preference and Choice

- After describing both approaches, and the assumptions on each approach, we want to understand:

$$\text{Rational Preferences} \implies \text{Consistent Choice behavior}$$

$$\text{Rational Preferences} \impliedby \text{Consistent Choice behavior}$$

Preference-Based Approach

Preference-Based Approach

- ***Preferences***: "attitudes" of the decision-maker towards a set of possible alternatives X.

- For any $x, y \in X$, how do you compare x and y?
  - I prefer x to y $(x \succ y)$
  - I prefer y to x $(y \succ x)$
  - I am indifferent $(x \sim y)$

Preference-Based Approach

By asking:	We impose the assumption:
Tick one box (i.e., not refrain from answering)	*Completeness*: individuals must compare any two alternatives, even the ones they don't know.

Preference-Based Approach

- ***Completeness***:
  - For any pair of alternatives $x, y \in X$, the individual decision maker:
    - ❑ $x \succ y$, or
    - ❑ $y \succ x$, or
    - ❑ both, i.e., $x \sim y$
  - (The decision maker is allowed to choose one, and only one, of the above boxes).

Preference-Based Approach

- *A **binary relation** is a collection of ordered pairs (x,y) from a set $x,y \in X$.*

- ***Not all binary relations satisfy Completeness***.

Preference-Based Approach

- ***Weak preferences***:
  - Consider the following questionnaire:
  - For all $x, y \in X$, where x and y are not necessarily distinct, is x **at least as preferred** to y?

    ❑ Yes $(x \succsim y)$

    ❑ No $(y \succsim x)$
  - Respondents must answer yes, no, or both
    - Checking both boxes reveals that the individual is indifferent between x and y.
    - Note that the above statement relates to completeness, but in the context of weak preference $\succsim$ rather than strict preference $\succ$.

Preference-Based Approach

- ***Reflexivity***: every alternative x is weakly preferred to, at least, one alternative: itself.
- A preference relation satisfies reflexivity if for any alternative $x \in X$, we have that:
  1) $x \sim x$: any bundle is indifferent to itself.
  2) $x \succsim x$: any bundle is preferred or indifferent to itself.
  3) $x \nsucc x$: any bundle belongs to at least one **indifference set** (i.e. set of alternatives over which the consumer is indifferent), namely, the set containing itself if nothing else.

Preference-Based Approach

- The preference relation $\succsim$ is ***rational*** if it possesses the following two properties:

  a) *Completeness*: for all $x, y \in X$,
  either $x \succsim y$, or $y \succsim x$, or both.

  b) *Transitivity*: for all $x, y, z \in X$,
  if $x \succsim y$ and $y \succsim z$, then it must be that $x \succsim z$.

Preference-Based Approach

- ***Example 1.1*.**

Consider the preference relation

$$x \succsim y \quad \text{if and only if} \quad \sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$$

In words, the consumer prefers bundle x to y if the total number of goods in bundle x is larger than in bundle y.

In case of two goods $x_1 + x_2 \geq y_1 + y_2$

Preference-Based Approach

- ***Example 1.1*** (continues).
- *Completeness*:
  - either $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$ (which implies $x \gtrsim y$), or
  - $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} x_i$ (which implies $y \gtrsim x$), or
  - both, $\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i$ (which implies $x \sim y$).

- *Transitivity:*
  - If $x \gtrsim y$, $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} y_i$, and
  - $y \gtrsim z$, $\sum_{i=1}^{N} y_i \geq \sum_{i=1}^{N} z_i$,
  - Then it must be that $\sum_{i=1}^{N} x_i \geq \sum_{i=1}^{N} z_i$ (which implies $x \gtrsim z$, as required).

Preference-Based Approach

- The assumption of transitivity is understood as that preferences should not cycle.

- Example violating transitivity:

$$\underbrace{apple \succsim banana \quad banana \succsim orange}_{apple \succsim orange \ \text{(by transitivity)}}$$

  but $orange \succ apple$.

- Otherwise, we could start the cycle all over again, and extract infinite amount of money from individuals with intransitive preferences.

Preference-Based Approach

- Sources of intransitivity:
  a) Indistinguishable alternatives
  b) Framing effects
  c) Aggregation of criteria
  d) Change in preferences

Preference-Based Approach

- ***Example 1.2*** (Indistinguishable alternatives):
  - Take $X = \mathbb{R}$ as a share of pie and $x \succ y$ if $x \geq y - 1$ $(x + 1 \geq y)$ but $x \sim y$ if $|x - y| < 1$ (indistinguishable).
  - Then,

$$1.5 \sim 0.8 \text{ since } 1.5 - 0.8 = 0.7 < 1$$

$$0.8 \sim 0.3 \text{ since } 0.8 - 0.3 = 0.5 < 1$$

  - By transitivity, we would have $1.5 \sim 0.3$, but in fact $1.5 \succ 0.3$ (intransitive preference relation).

Preference-Based Approach

- *Other examples*:
  - similar shades of gray paint
  - milligrams of sugar in your coffee

Preference-Based Approach

- ***Example 1.3*** (Framing effects):
  - Transitivity might be violated because of the way in which alternatives are presented to the individual decision-maker.
  - What holiday package do you prefer?
    a) A weekend in Paris for $574 at a four star hotel.
    b) A weekend in Paris at the four star hotel for $574.
    c) A weekend in Rome at the five star hotel for $612.
  - By transitivity, we should expect that if $a \sim b$ and $b \succ c$, then $a \succ c$.

Preference-Based Approach

- ***Example 1.3*** (continued):
  - However, this did not happen!
  - More than 50% of the students responded $c \succ a$.
  - Such intransitive preference relation is induced by the framing of the options.

Preference-Based Approach

- ***Example 1.4*** (Aggregation of criteria):
  - Aggregation of several individual preferences might violate transitivity.
  - Consider $X = \{MIT, WSU, Home\ University\}$
  - When considering which university to attend, you might compare:
    a) Academic prestige (criterion #1)

       $\succ_1$: $MIT \succ_1 WSU \succ_1 Home\ Univ.$

    b) City size/congestion (criterion #2)

       $\succ_2$: $WSU \succ_2 Home\ Univ. \succ_2 MIT$

    c) Proximity to family and friends (criterion #3)

       $\succ_3$: $Home\ Univ. \succ_3 MIT \succ_3 WSU$

Preference-Based Approach

- ***Example 1.4*** (continued):
  – By majority of these considerations:

$$MIT \underbrace{\succsim}_{\text{criteria 1 \& 3}} WSU \underbrace{\succsim}_{\text{criteria 1 \& 2}} Home\ Univ \underbrace{\succsim}_{\text{criteria 2 \& 3}} MIT$$

  – Transitivity is violated due to a cycle.

  – A similar argument can be used for the aggregation of individual preferences in *group decision-making*:
    - Every person in the group has a different (transitive) preference relation but the group preferences are not necessarily transitive ("***Condorcet paradox***").

Preference-Based Approach

- Intransitivity due to a *change in preferences*
  - When you start smoking
    $$\text{One cigarette} \succsim \text{No smoking} \succsim \text{Smoking heavily}$$
    By transitivity,
    $$\text{One cigarette} \succsim \text{Smoking heavily}$$
  - Once you started
    $$\text{Smoking heavily} \succsim \text{One cigarette} \succsim \text{No smoking}$$
    By transitivity,
    $$\text{Smoking heavily} \succsim \text{One cigarette}$$
  - But this contradicts the individual's past preferences when he started to smoke.

Desirability

- monotonicity
- Strong monotonicity
- Non-satiation
- Local non-satiation

All x1,x2,x3 are defined on the set of real numbers.

Now we are going to define the first property.

Monotonicity

If i take any two bundles x and y and x != y.
If xk >=yk (quantity of good k in bundle x and y) then implies that x pref y.
If xk > yk then implies that x strictly pref to y.

1. So increasing the amount of some commodores cannot hurt x>=y.
2. $x = (x_1, x_2)$ $y = (x_1 + \varepsilon, x_2)$ $y \succeq x$ y PREF TO x

   $\varepsilon > 0$

   $y = (x_1 + \varepsilon, x_2 + \varepsilon)$ $y \succ x$

   THEN PREF RELATION SATISFY MONOTONICITY

Strong monotonicity

Two bundles in consumption set if xk >= yk for every good k then we conclude that x strictly pref to y (before was weakly preferred).

In the last example are monotone in which add eps only to x1 then we obtain a bundle strictly preference to x.
==> this means that is strong because we got a stronger condition. Even increasing the quantity of 1 good then you obtain a bundle that is strictly pref.
Also for the second in which we add eps to x1 and x2 then it hold because y>x comprende y>=x

Now we wander how this monotonicity can be translated in the characteristic of the utility function?
Monotonicity in preference implies that utility function is weakly monotonicity in its arguments.
If we increase all arguments we obtain a value that it is strictly increases its value.

If i have x1 and x2 and if ai multiply by a scalar alpha > 1.
Alpha x1 > x1 so is greater or equal than the initial utility of the bundle u(x1,x2).
Increasing quantity of x1 i get a greater utility so if weakly pref to the original one.

If alpha x1 and alpha x2 then the utility is strictly preferred than the original one.

If we change and we want to see what strong monotonicity imply in the utility function.
In case you increase only one good you obtain a strictly greater than the original one. U(ax1, x2) > u(x1,x2).

Examples

PREF CAN BE REPRESENTED BY THIS FUNCTION

$u(x_1, x_2) = \min\{x_1, x_2\}$

MONOTONE?

1. INCREASE QUANTITY OF 1 OF THE 2 GOODS

WE HAVE TO CHECK THE MINIMUM

$u(\alpha x_1, x_2) = \min\{\alpha x_1, x_2\} \overset{?}{\geq} \min\{x_1, x_2\}$

$\underline{\alpha > 1}$

ALWAYS TRUE?

$u(x_1, x_2) = \min\{x_1, x_2\}$ x_1 x_2

$(1, 2)$

$(2, 1)$

IF MIN == x_2

EQUAL

IF MIN == x_1

GREATER

$x_1 \leq x_2$?

$\alpha x_1 \leq x_2$?

IF $x_1 \leq x_2$ x_1 x_2 → MIN STILL INCREASE SO IT'S FINE

IF 1° CONDITION OF MONOTONICITY, THE SECOND CONDITION HOLDS FOR SURE

$u(\alpha x_1, \alpha x_2) = \min\{\alpha x_1, \alpha x_2\}$

THIS IS FOR SURE $\geq$ THAN MIN IN THE ORIGINAL BUNDLE

STRONG MONOTONICITY ?

IF WE JUST CHANGE ONE OF TWO ARG WE OBTAIN
A STRICLY PREFERENCE RELATION

1. ONE IS MONOTONICITY
2. STRONG MONOTONICITY

$u(\alpha x_1, x_2) > u(x_1, x_2)$

$\min\{\alpha x_1, x_2\} > \min\{x_1, x_2\}$

WE CAN OBTAIN MINIMUM AS x_1 OR x_2

IF x_1 MEANS x_2 LESS THAN x_1.
IF INCREASE THE x_1 AND MIN IS x_2

$x_2 > x_2$ IMPOSSIBLE SO STRONG MONOTONICITY
IS NOT SATISFIED

STRONG IS STRICTER THAN MONOTONICITY

MONOTONICITY $\Rightarrow$ STRONG MONO
$\cancel{\Leftarrow}$ NOT THE VICEVERSA

THEN WE WILL SPEAK ABOUT COMPLEMENTS
(PARADOSSO DEL CAFFE')

$(1,3)$ $(1,2)$
C S C S

Example2 —> linear utility function

$u(x_1, x_2) = x_1 + x_2$ → OR ANY SUM COMBINATION LIKE $2x_1 + 3x_2$ ...

- MONOTONICITY $\alpha > 1$

$u(\alpha_1 x_1, x_2) = \alpha x_1 + x_2$

LARGER THAN u SO IS STRICTLY

$u(\alpha x_1, \alpha x_2) = \alpha x_1 + \alpha x_2$

UTI IS STRICTLY LARGER SO RELATION IS STRICTLY MONOTONIC

SO SATISFY STRONG MONOTONICITY

WE SHOULD DO THE SAME FOR x_2 TO BE SURE

BUT IT IS CLEAR WE GET A STRONG MONOTONICITY

$u(x_1, \alpha x_2) = x_1 + \alpha x_2$

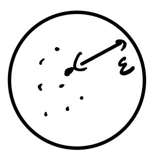
rational —> complete reflexives and transitivity. Transitivity assume completeness ??

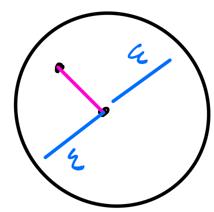
Non-satiation

You are never happy. You always find a bundle that is strictly pref than the original one. So this is not very usable. We will use more frequently local non-satiation

local non-satiation

We always find a bundle that is close to the original one, but we pref the original one.





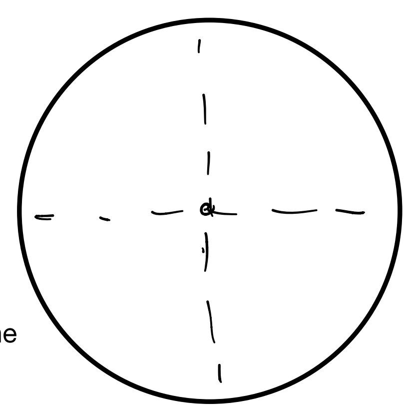
We always have an Euclidean distance < eps.
Euclidean distance is computed as
x = (x1,x2) Y(x1, x2)
take difference power of two and then rad.
So we compute the distance we got a point the in circle by increase for a small quantity. This must happen for any distance eps.

For instance you can compare very close alternatives that differ for a very small amount.

Application of definition of local association.
Two goods.
[slide cerchio]
In x1 we have quantity of first good in bundle x. In y we have the second quantity of bundle x (which is x2). The bundle (2,2) can be represented by a point, also for y.
Y2 contain a small quantity of x2 and y1 larger than x1. So the distance

In case we have two bad good [called bads](pollutions of water and air)

The more we are close to the origin
The more we are happy.



(0,0) can we find another bundle close to this and
Preferred to the original one?
We can't have negative pollution.
Drawing small circle x we don't find any bundle pref to the original one
So this violate the LNS.

Another situation is the thick indifference sets (or curve).
An indifference set is the set of all bundle that are indifferent to the consumer (same level of utility)
Imagine now we have an area then, so this mean we cannot draw arbitrarialy small circle, because all

circle in this area of the indifference curve are indifference. So we will not consider this case.

IF PREF $\succeq$ STATISFY MONOTONICITY

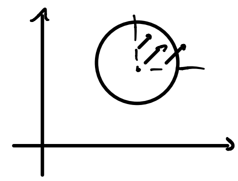
$\Rightarrow$ IT ALSO SATISFY LNS

LNS $\rightarrow$ DRAW CIRCLE TO FIND BUNDLE $>$ TO THE ORIGINAL ONE

a) $u(x_1, x_2)$ $\quad u(\alpha x_1, \alpha x_2) \rightarrow u(x_1+\alpha, x_2+\alpha)$

$\alpha > 1$ $\quad\quad \alpha > 0$

THIS $>$ $u(x_1, x_2)$



WE WILL ALWAYS FIND SOMETHING PREF TO THE ORIGINAL ONE

SO IF WE PROVE MONOTONICITY IMPLY WE ALSO HAVE NLS

Indifference set

A bundle x and the indifference bundle in the consumption sets are indifference to the respect to x.
Y ind to X
IND(x)

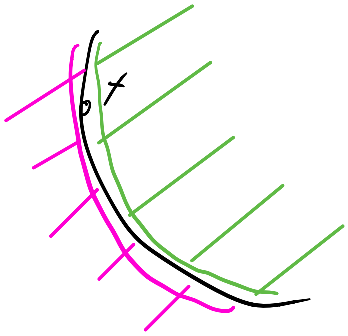
The upper-counter set

The set of all bundle in the consumption set such that bundle are strictly preferred to x
UCS(x)

Lower-counter set

The set of all bundle in the consumption set such that bundle are strictly preferred to x Such that x is strictly pref to y
LCS(x)

Graphically we can show it in this example in the following way.



ALL BUNDLES IN LINE ARE IND TO X

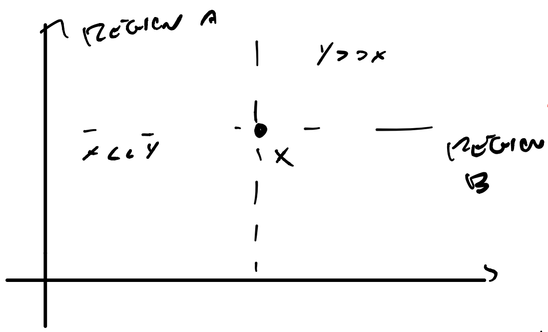
- UCS IMPLY ALL THE AREA AND THE LINE

- LCS AREA IN BASSO AND CONSUMPTION SET

$LCS \cap UCS = IND$

We saw properties of preference relation. Now we will see properties in indifference set (or curves)

Strong monotonicity



POSITIVE QUADRANT DIVIDED BY 4

ALL BUNDLES IN y>>x CONTAINS LARGER QT OF THIS GOODS. THEY ARE STRICTLY PREFE

SO IND CANNOT BE IN THIS REGION

IN x<<y IND CANNOT BE IN THIS AREA

THE ONLY AREA IS REGION A AND B,
WHICH MEANS CURVES ARE NEGATIVE SLOPES

We will have curve that decrease???

Convexity of preferences

A preference relation is convex if for every two bundle in consumption set such that
X weak pref y ==> ax + (1-a) y >= y —> like a weighted average a+(1-a) = 1

$a \in [0,1]$

$a = 1/2$ $\quad (x_1, x_2) \succsim (y_1, y_2)$

$$\left(\tfrac{1}{2}x_1, \tfrac{1}{2}x_2, \tfrac{1}{2}x_2, \tfrac{1}{2}y_2\right) \succ (y_1, y_2)$$

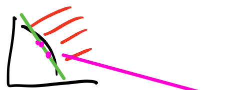
IF TRUE FOR ANY α

CONVEXITY OF PREFERENCE

TASTE FOR DIVERSIFY / YOUR ASSUMPTION

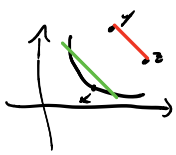
You can say another property of convexity with upper counter set (UCS).
So UCS(x) = {y app X: y>= x}

CONVEX -> ALL POINTS IN STRAIGHT LINE ARE IN THE SET

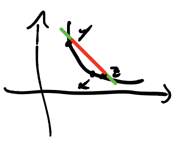


CONCAVE -> POINT NOT IN THE SET

Y in the UCS and if i have another bundle and if i have z also, then convex combination of the two good. So any bundle in this line is strictly pref to the original bundle. Not only weak but also strictly pref.



All bundle in the strictly line are preferred.



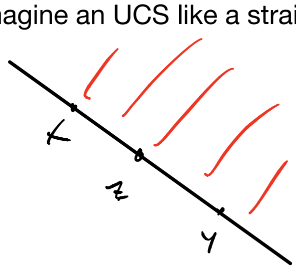
ALSO LIKE THIS - SO STRICTLY PREF OR IND

Convexity 1 we need just 2 bundles. For convexity 2 we need 3 bundles.

Strict convexity if you take x,y,z app X
If x weak pref z
If y weak pref to z
Then convex combination is strictly preferred.

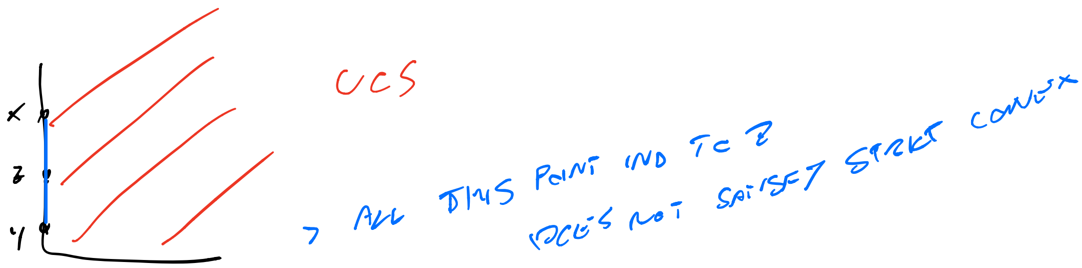
The only example strictly convex (have a shape like a curve)

Imagine an UCS like a straigh line



UCS

Taking two point x and y that are weak pref to z. Which mean z is in the indifference set.
Any points are indifferent to z and not strictly pref to z. So straight indifference curve (rette) represent preference that are not strictly convex but weakly convex. This correspond with linear utilty function which is the example of perfect substitutes goods.



In this case pref relation is not strictly convex. But in most of our example the curve will no have this shape.

Try to do example 1.7 as an exercise applying the definition that this u satisfy both convexity and strict convexity.

Interpretation of convexity

You consume a lot of good 1 and a small quantity of good2. The coordinate of y is high and the second is low. You don't like the bundles unbalance to the two good. We pref to consume a little bit of everything. Are weakly preferred.

Advanced Microeconomics (EPS)

Chapter 1: Utility functions, indifference sets, quasi-concavity

Utility Function

- A function $u: X \to \mathbb{R}$ is a ***utility function*** representing preference relations $\succsim$ if, for every pair of alternatives $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y)$$

Utility Function

- Two points:
  1) Only the ranking of alternatives matters.
     - That is, it does not matter if

$$u(x) = 14 \text{ or if } u(x) = 2000$$

$$u(y) = 10 \text{ or if } u(y) = 3$$

     - We do not care about *cardinality* (the number that the utility function associates with each alternative) but instead care about *ordinality* (ranking of utility values among alternatives).

Utility Function

2) If we apply any **strictly increasing function** $f(\cdot)$ on $u(x)$, i.e.,

$$f\colon \mathbb{R} \to \mathbb{R} \text{ such that } v(x) = f(u(x))$$

the new function keeps the ranking of alternatives intact and, therefore, the new function still represents the same preference relation.

– *Example*:

$$v(x) = 3u(x)$$
$$v(x) = 5u(x) + 8$$

Desirability

- We can express desirability in different ways.
  - Monotonicity
  - Strong monotonicity
  - Non-satiation
  - Local non-satiation
- In all the above definitions, consider that x is an n-dimensional bundle

$$x \in \mathbb{R}^n, \text{i.e., } x = (x_1, x_2, \dots, x_N)$$

where its k^{th} component represents the amount of good (or service) k, $x_k \in \mathbb{R}$.

Desirability

- ***Monotonicity***:
  - A preference relations satisfies monotonicity if, for all $x, y \in X$, where $x \neq y$,
    a) $x_k \geq y_k$ for every good k implies $x \succsim y$
    b) $x_k > y_k$ for every good k implies $x \succ y$
  - That is,
    - increasing the amounts of some commodities (without reducing the amount of any other commodity) **cannot hurt**, $x \succsim y$; and
    - increasing the amounts of all commodities is strictly preferred, $x \succ y$.

Desirability

- ***Strong Monotonicity***:
  - A preference relation satisfies strong monotonicity if, for all $x, y \in X$, where $x \neq y$,

    $x_k \geq y_k$ for every good k implies $x \succ y$

  - That is, even if we increase the amounts of only one of the commodities, we make the consumer strictly better off.

Desirability

- Relationship between **monotonicity** and utility function:
  - Monotonicity in preferences implies that the utility function **is weakly monotonic (weakly increasing) in its arguments**
    - That is, increasing some of its arguments weakly increases the value of the utility function, and increasing all its arguments strictly increases its value.
  - For any scalar $\alpha > 1$,

$$u(\alpha x_1, x_2) \geq u(x_1, x_2)$$
$$u(\alpha x_1, \alpha x_2) > u(x_1, x_2)$$

Desirability

- Relationship between **strong monotonicity** and utility function:
  - Strong monotonicity in preferences implies that the utility function **is strictly monotonic (strictly increasing) in all its arguments.**
    - That is, increasing some of its arguments strictly increases the value of the utility function.
  - For any scalar $\alpha > 1$,
$$u(\alpha x_1, x_2) > u(x_1, x_2)$$

Desirability

- ***Example 1.5***: $u(x_1, x_2) = \min\{x_1, x_2\}$
  - Monotone, since
    $$\min\{x_1 + \delta, x_2 + \delta\} > \min\{x_1, x_2\}$$
    for all $\delta > 0$.
  - Not strongly monotone, since
    $$\min\{x_1 + \delta, x_2\} \ngtr \min\{x_1, x_2\}$$
    if $\min\{x_1, x_2\} = x_2$.

Desirability

- ***Example 1.6***: $u(x_1, x_2) = x_1 + x_2$
  - Monotone, since
    $$(x_1 + \delta) + (x_2 + \delta) > x_1 + x_2$$
    for all $\delta > 0$.
  - Strongly monotone, since
    $$(x_1 + \delta) + x_2 > x_1 + x_2$$
- **Hence, strong monotonicity implies monotonicity, but the converse is not necessarily true.**

Desirability

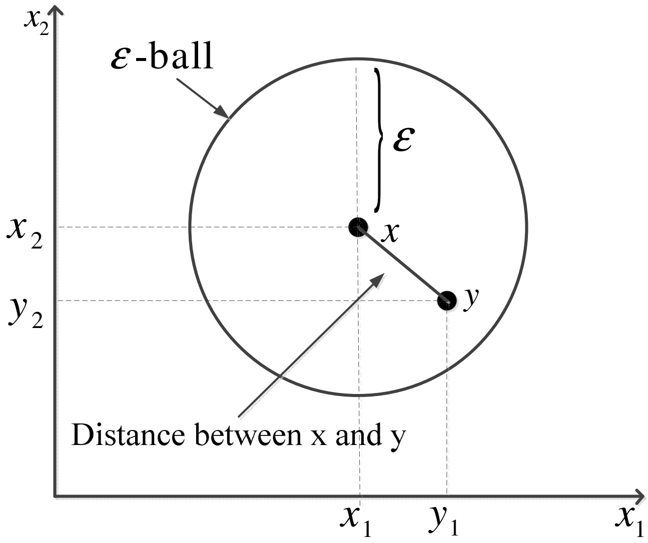
- ***Non-satiation*** (NS):
  - A preference relation satisfies NS if, for every $x \in X$, there is another bundle in set X, $y \in X$, which is strictly preferred to x, i.e., $y \succ x$.
    - NS is too general, since we could think about a bundle y containing extremely larger amounts of some goods than x.
    - How far away are y and x?

Desirability

- ***Local non-satiation*** (LNS):
  - A preference relation satisfies LNS if, for every bundle $x \in X$ and every $\varepsilon > 0$, there is another bundle $y \in X$ which is less than ε-away from x, $||y - x|| < \varepsilon$, and for which $y \succ x$.
    - $||y - x|| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$ is the Euclidean distance between x and y, where $x, y \in \mathbb{R}_+^2$.
    - In words, for every bundle x, and for ***every*** distance ε from x, we can find a more preferred bundle y.

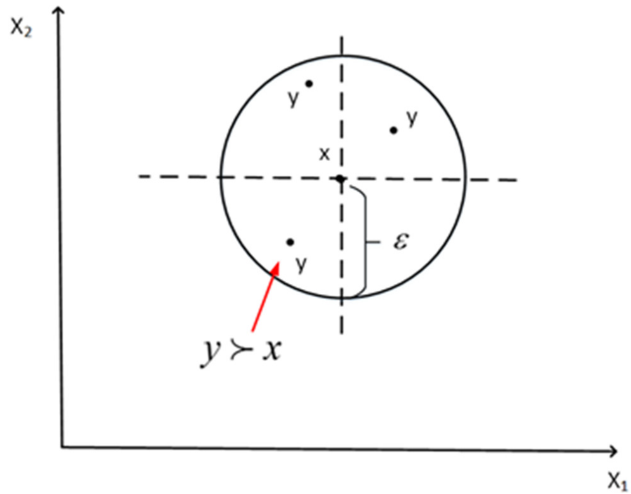
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of good 2 (but more of good 1) than bundle x.



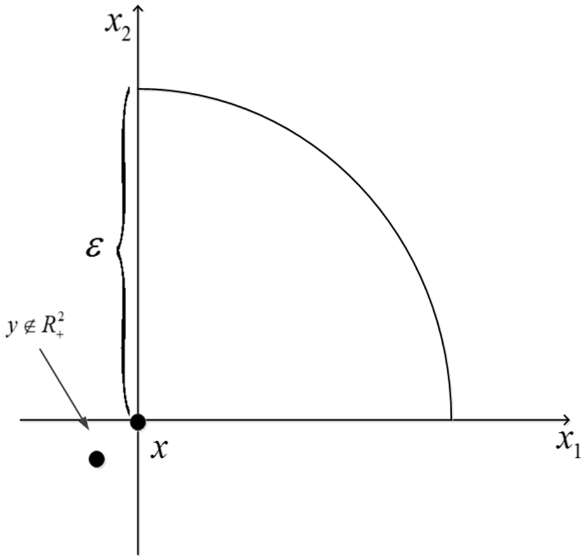
Desirability

– A preference relation satisfies $y \succ x$ even if bundle y contains less of *both* goods than bundle x.



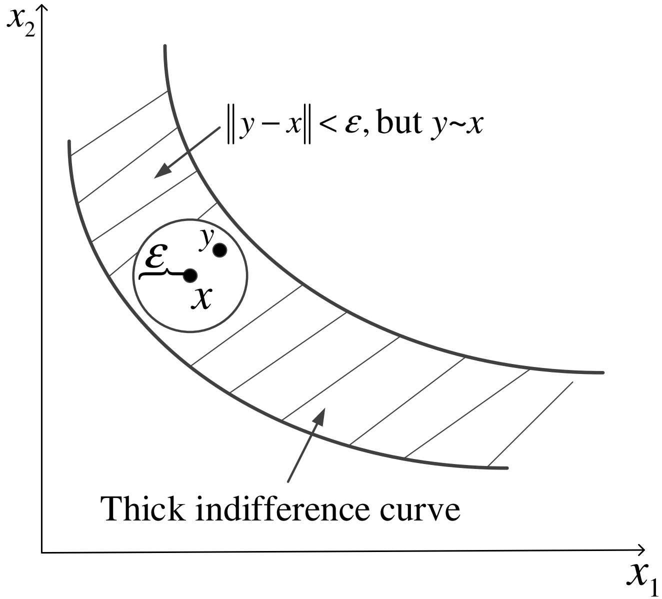
Desirability

- *Violation of LNS*
  - LNS rules out the case in which the decision-maker regards all goods as bads.
  - Although $y \succ x$, y is unfeasible given that it lies away from the consumption set, i.e., $y \notin \mathbb{R}^2_+$.



Desirability

- *Violation of LNS*
  - LNS also rules out "thick" indifference sets.
  - Bundles y and x lie on the same indifference curve.
  - Hence, decision maker is indifferent between x and y, i.e., $y \sim x$.



Desirability

- *Note*:
  - If a preference relation satisfies monotonicity, it must also satisfy LNS.
    - Given a bundle $x = (x_1, x_2)$, increasing all of its components yields a bundle $(x_1 + \delta, x_2 + \delta)$, which is strictly preferred to bundle (x_1, x_2) by monotonicity.
    - Hence, there is a bundle $y = (x_1 + \delta, x_2 + \delta)$ such that $y \succ x$ and $\|y - x\| < \varepsilon$.

Indifference sets

Indifference sets

- The **indifference set** of a bundle $x \in X$ is the set of all bundles $y \in X$, such that $y \sim x$.

$$IND(x) = \{y \in X: y \sim x\}$$

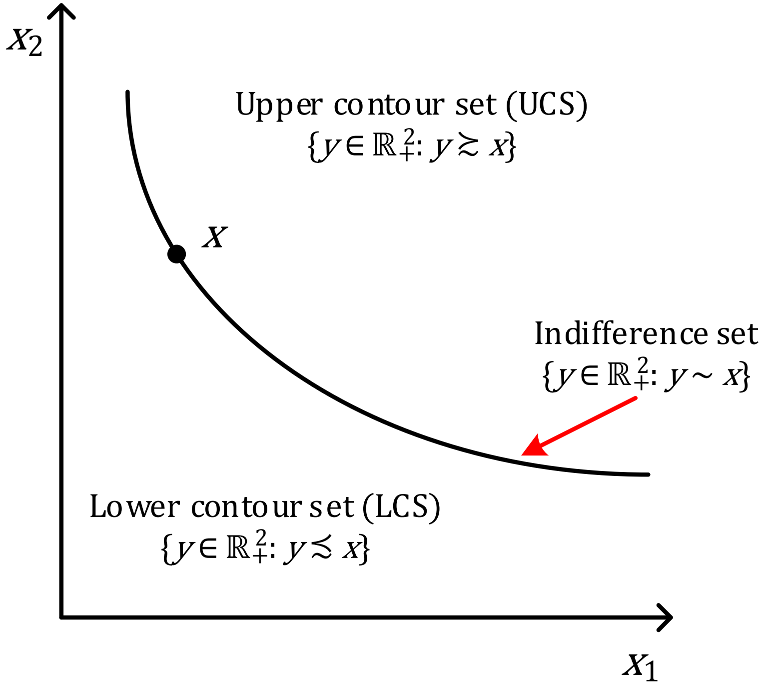
- The **upper-contour set** of bundle x is the set of all bundles $y \in X$, such that $y \succsim x$.

$$UCS(x) = \{y \in X: y \succsim x\}$$

- The **lower-contour set** of bundle x is the set of all bundles $y \in X$, such that $x \succsim y$.

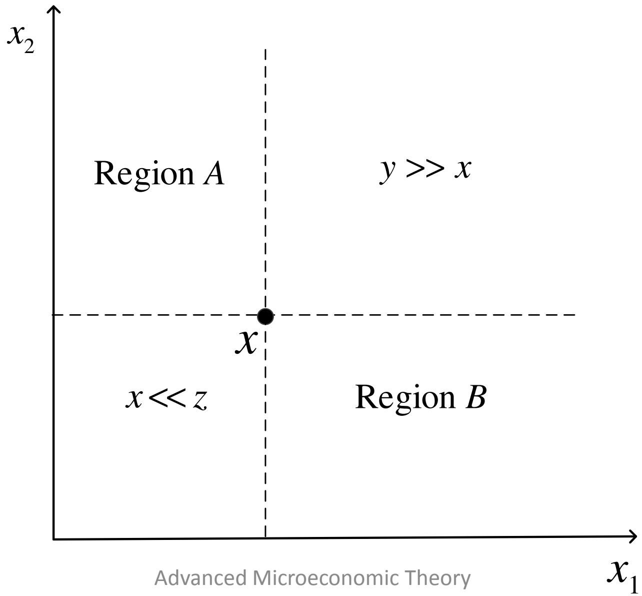
$$LCS(x) = \{y \in X: x \succsim y\}$$

Indifference sets



Indifference sets

- **Strong monotonicity** implies that indifference curves must be negatively sloped.



Indifference sets

- *Note*:
  - Strong monotonicity implies that indifference curves must be negatively sloped.
  - In contrast, if an individual preference relation satisfies LNS, indifference curves can be upward sloping.
    - This can happen if, for instance, the individual regards good 2 as desirable but good 1 as a bad.

Convexity of Preferences

Convexity of Preferences

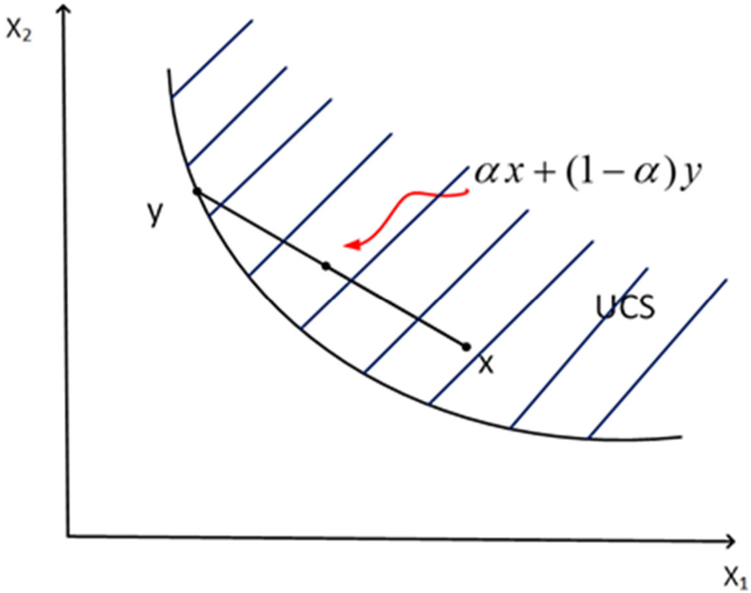
- ***Convexity 1***: A preference relation satisfies convexity if, for all $x, y \in X$,

$$x \succsim y \implies \alpha x + (1-\alpha)y \succsim y$$

for all $\alpha \in (0,1)$.

Convexity of Preferences

- Convexity 1



Convexity of Preferences

- ***Convexity 2***: A preference relation satisfies convexity if, for every bundle x, its upper contour set is convex.

$$UCS(x) = \{y \in X : y \succsim x\} \text{ is convex}$$

- That is, for every two bundles y and z,

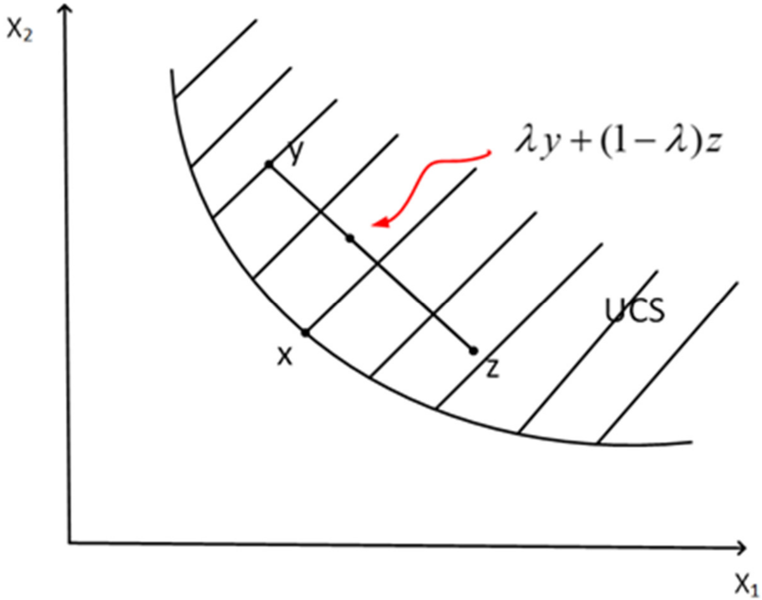
$$\begin{cases} y \succsim x \\ z \succsim x \end{cases} \implies \lambda y + (1-\lambda)z \succsim x$$

for any $\lambda \in [0,1]$.

- Hence, points y, z, and their convex combination belongs to the UCS of x.

Convexity of Preferences

- Convexity 2



Convexity of Preferences

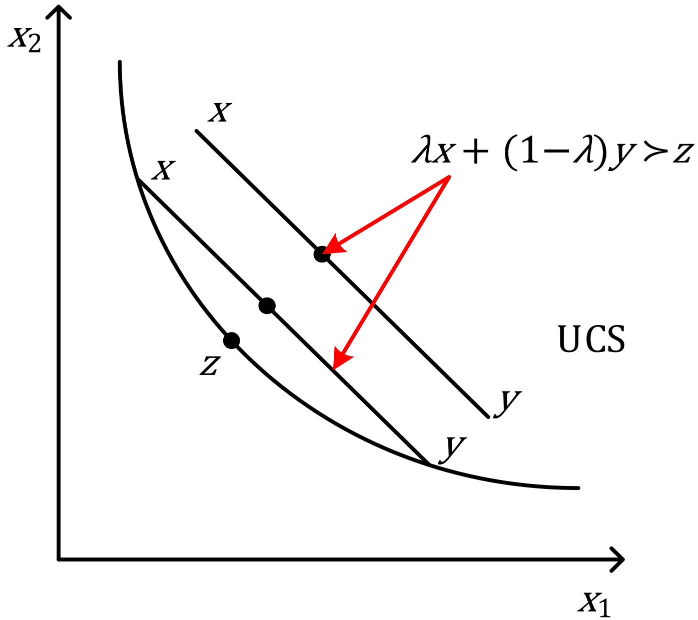
- ***Strict convexity***: A preference relation satisfies strict convexity if, for every $x, y \in X$ where $x \neq y$,

$$\begin{cases} x \succsim z \\ y \succsim z \end{cases} \Longrightarrow \lambda x + (1-\lambda)y \succ z$$

for all $\lambda \in [0,1]$.

Convexity of Preferences

- Strictly convex preferences

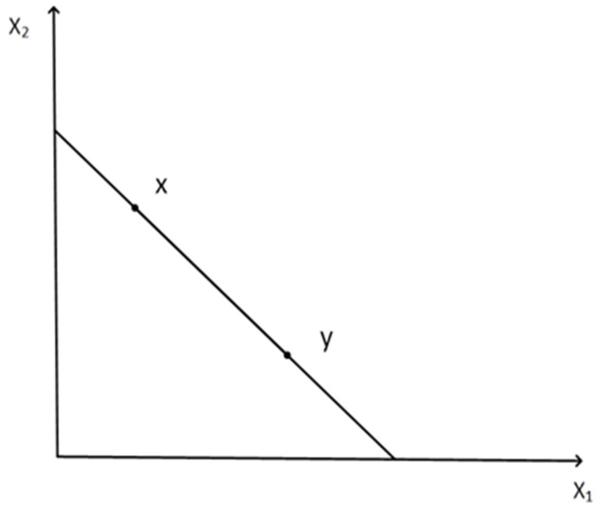


Convexity of Preferences

- **Convex but not strict convex preferences**
  - $\lambda x + (1 - \lambda)y \sim z$
  - This type of preference relation is represented by linear utility functions such as

    $$u(x_1, x_2) = ax_1 + bx_2$$

    where x_1 and x_2 are regarded as substitutes.

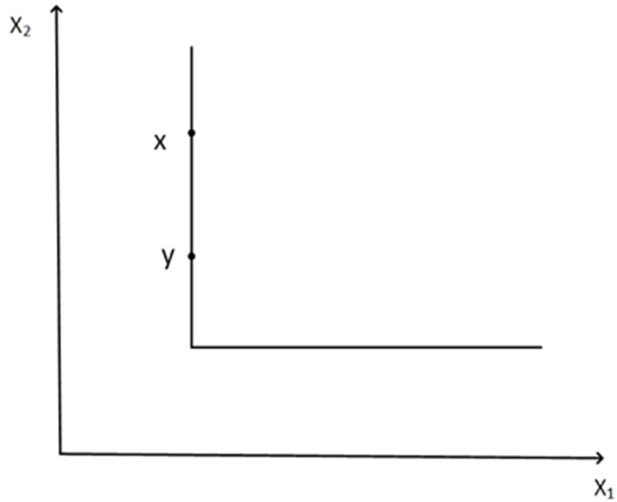


Convexity of Preferences

- **Convex but not strict convex preferences**
  - *Other example*: If a preference relation is represented by utility functions such as

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

where $a, b > 0$, then the pref. relation satisfies convexity, but not strict convexity.



Convexity of Preferences

- ***Example 1.7***

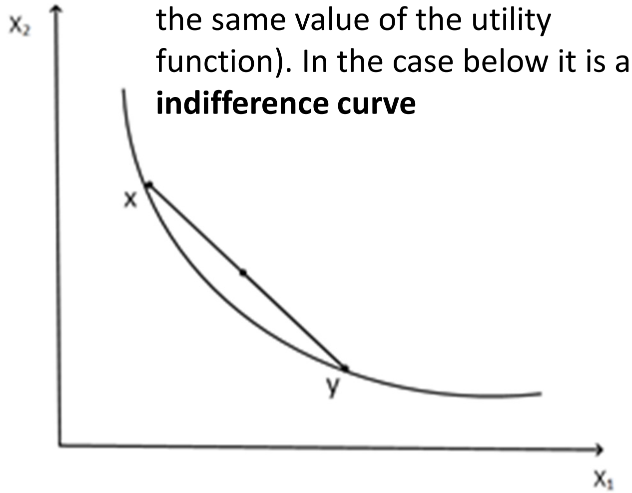
$u(x_1, x_2)$	Satisfies convexity	Satisfies strict convexity
$ax_1 + bx_2$	√	X
$\min\{ax_1, bx_2\}$	√	X
$ax_1^{\frac{1}{2}}bx_2^{\frac{1}{2}}$	√	√
$ax_1^2 + bx_2^2$	X	X

Do the last two for exercise

Convexity of Preferences

- ***Interpretation of convexity***
  - *1) Taste for diversification*:
    - An individual with convex preferences prefers the convex combination of bundles x and y, than either of those bundles alone.

Indifference sets can be interpreted as the bundles that give the same level of utility (i.e. the same value of the utility function). In the case below it is an **indifference curve**



MRS is the slope of this indifference curve. Now we will see how to compute the marginal rate of substitution.

SUPPOSE WE HAVE A UTILITY FUNCTION

$u(x_1, x_2, \dots x_n)$

$x_1 \dots x_n$ QUANTITY OF GOODS

MARGINAL UTILITY : $\frac{\partial u}{\partial x_1}$ → PARTIAL DER OF UTILITY WITH RESPECT TO x_1

INCR UTILITY GAIN OBTAINED BY A SMALL INCR OF x_1.

TOTAL DIFFERENTIAL → INCREASING THE VALUE IF WE INCREASE ALL ARGUMENTS OF THE FUNCTION

$$du = \frac{\partial u}{\partial x_1} \cdot dx_1 + \frac{\partial u}{\partial x_2} \cdot dx_2 + \dots + \frac{\partial u}{\partial x_n} dx_n$$

AMOUNT IN WHICH x_1 INCREASES

TOT VARIATION DEPENDS ON ... [MIN 8; 29]

WHAT IS THE MAIN PROPERTIES OF IND CURVE?

IF WE GO FROM POINT TO ANOTHER THEN VARIATION IS $\varnothing$

ALONG AN IND. CURVE $du = 0$

$0 =$

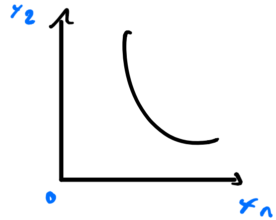
Marginal Rate of Substitution (MRS)

- *Remark*:
  - Let us show that the slope of the indifference curve is given by the MRS.
  - Consider a continuous and differentiable utility function $u(x_1, x_2, \dots, x_n)$.
  - Totally differentiating, we obtain

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \cdots + \frac{\partial u}{\partial x_n} dx_n$$

  - But since we move along the same indifference curve, $du = 0$. $\frac{\partial u}{\partial x_i}$ is called the **marginal utility** of x_i.

Convexity of Preferences



– Inserting $du = 0$, and taking any two goods

$$0 = \frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} dx_j$$

$u(x_i, x_j)$

or $$-\frac{\partial u}{\partial x_i} dx_i = \frac{\partial u}{\partial x_j} dx_j$$

SLOPE $\Rightarrow \boxed{\frac{dx_j}{dx_i}}$

– If we want to analyze the **rate at which the consumer substitutes units of good *i* for good *j***, we must solve for $\frac{dx_j}{dx_i}$, to obtain

$$\boxed{-\frac{dx_j}{dx_i} = \frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} \equiv MRS_{i,j}}$$

MARG UT IS POSITIVE
(IF I CONSUME MORE, IS BETTER)

For instance if $-\frac{dx_j}{dx_i}$ = 2/1 = 2 you have to replace 2 units of good *j* for one unit of good *i* to remain in the same indifference curve.

MU

$x_i = 2 \cdot x_j$ MU

BECAUSE RATION IS 2

$U(i) = 2U(j)$ IF I GIVE YOU x_1, YOU HAVE TO GIVE AT LEAST TWO x_2

MRS → RATIO OF MARG UTILITY OF GOODS

↳ SLOPE OF IND. CURVE!

Convexity of Preferences

- ***Interpretation of convexity***

  2) *Diminishing marginal rate of substitution:*

$$MRS_{1,2} \equiv -\frac{dx_2}{dx_1} = \frac{\partial u/\partial x_1}{\partial u/\partial x_2}$$

  – *MRS* describes the additional amount of good 2 that the consumer needs to receive in order to keep her utility level unaffected, when the amount of good 1 is reduced by one unit.

  – Hence, a *diminishing MRS* implies that the consumer needs to receive increasingly larger amounts of good 2 in order to accept further reductions of good 1.

One properties of MRS:
Since we are using convex Ind curve.

IND set or ind set is decreasing. So IND curve decreasing, slope is negative and then the slope is decreasing. What does it mean?

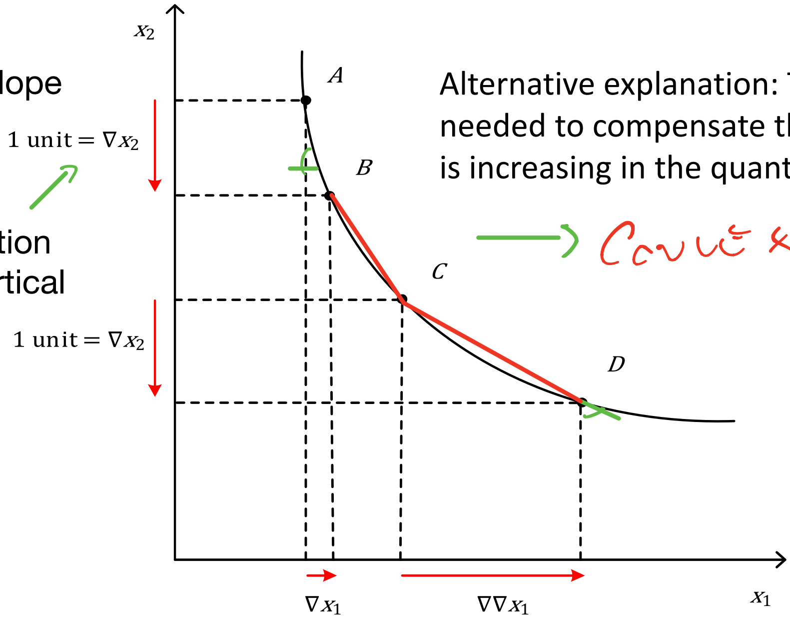
Slope of a curve in one point, if the slope of the angle in this point.

Convexity of Preferences

- Diminishing marginal rate of substitution

Small slope means slope is ..

Imagine having variation from x1 to x2 this vertical jump.



Alternative explanation: The increase in x1 needed to compensate the same decrease in x2, is increasing in the quantity consumed of x1

→ Convex → line above (strict) curve

Amount of x1 you need to mantain(mantenere) utilty invariance is larger.

Implication of marginal rate of substitution...
[25:]

Indifferent curve decreasing mean slope < 0 and the slope is decreasing. The slope is the Marginal rate of substitution.
So this are all thing we are using in the next lectures.

Quasiconcavity

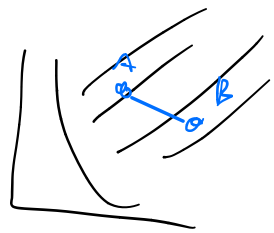
A utility function is concave if UCS is convex.

Quasiconcavity

- A utility function $u(\cdot)$ is ***quasiconcave*** if, for every bundle $y \in X$, the set of all bundles for which the consumer experiences a higher utility, i.e., the $UCS(x) = \{y \in X \mid u(y) \geq u(x)\}$ is convex.
- The following three properties are equivalent:

$$\text{Convexity of preferences} \Leftrightarrow UCS(x) \text{ is convex} \Leftrightarrow u(\cdot) \text{ is quasiconcave}$$

In the example before the UCS is convex. SO if we take two point in the set and link it with a straight line then they depends on the set.



Function convex, UCS convex ==> u(°) is quasiconcave.

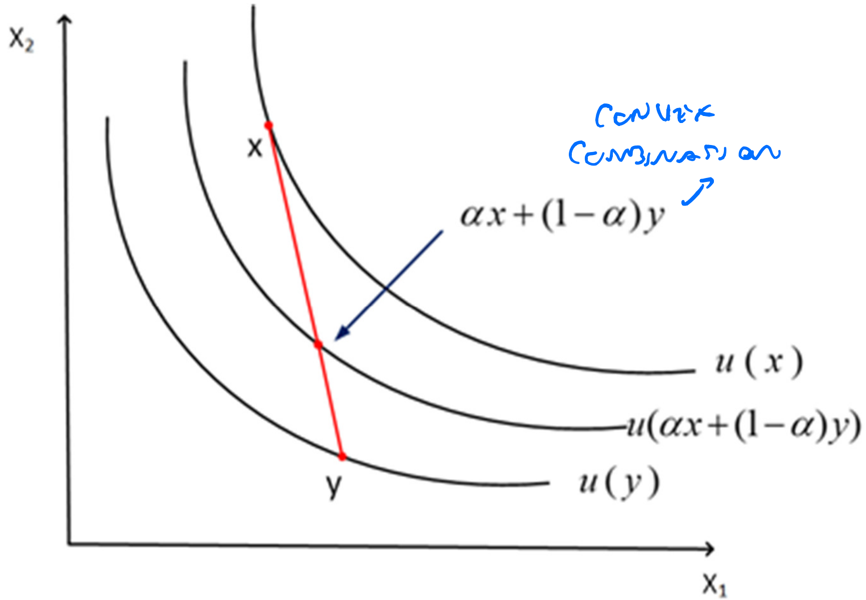
Quasiconcavity

- ***Alternative definition of quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *weakly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) \geq \min\{u(x), u(y)\}$$

Quasiconcavity

- Quasiconcavity (second definition)



IND CURV DIFFR BUNDLE

PREF REPRESENTED BY A MAP OF IND CURVE

IN THIS MAP COULD BE INFINITE IND CURVE

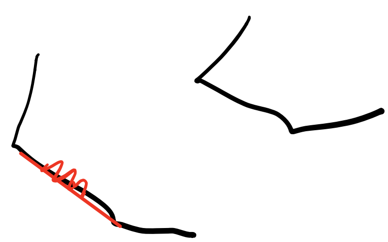
STRICT QUASICONCAVE IS THE SAME

BUT THE VALUE OF CONVEX COMBINATION

IS NO MORE $\geq$ BUT ONLY $>$ THAN

THE MIN UTILITY BETWEEN THE TWO

BUNDLES



QUASI

CONCAVE BUT

NOT STRICT

QUASICONC

EACH OF THIS POINT WILL

BE WEAKLY PREFERED

IN THIS CASE THEY ARE

STRICT

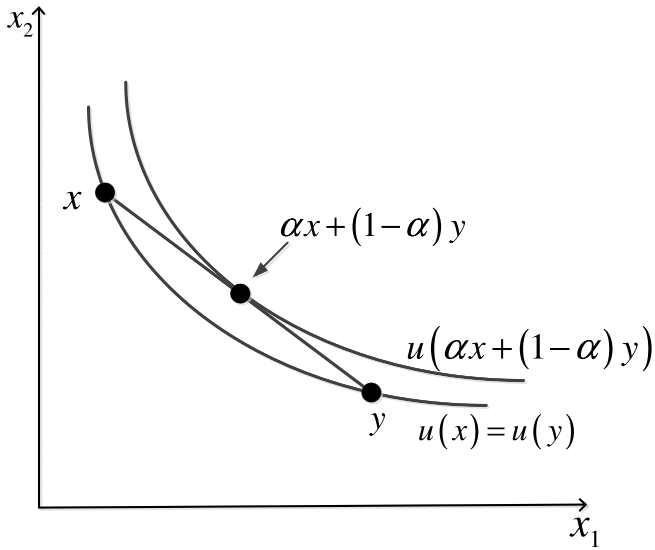
Quasiconcavity

- ***Strict quasiconcavity***:
  - A utility function $u(\cdot)$ satisfies *strict quasiconcavity* if, for every two bundles $x, y \in X$, the utility of consuming the convex combination of these two bundles, $u(\alpha x + (1-\alpha)y)$, is *strictly* higher than the minimal utility from consuming each bundle separately, $\min\{u(x), u(y)\}$:

$$u(\alpha x + (1-\alpha)y) > \min\{u(x), u(y)\}$$

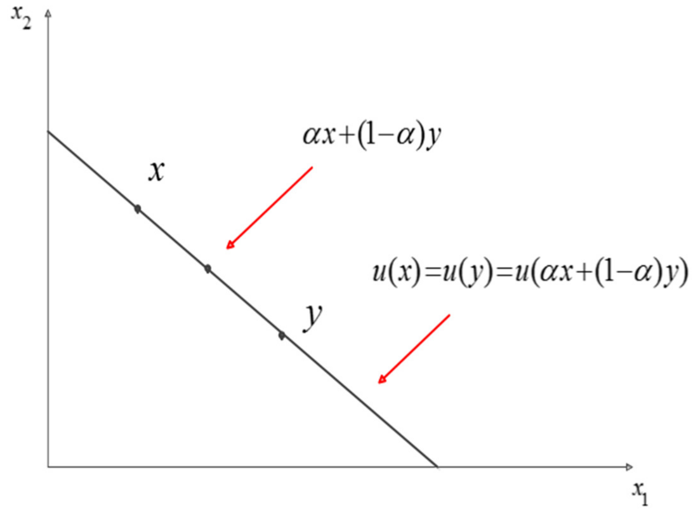
Quasiconcavity

- *What if bundles x and y lie on the same indifference curve?*
- Then, $u(x) = u(y)$.
- Since indifference curves are strictly convex, $u(\cdot)$ satisfies quasiconcavity.



Quasiconcavity

- *What if indifference curves are linear?*
- $u(\cdot)$ satisfies the definition of a quasiconcavity since $u(\alpha x+(1-\alpha)y) = \min\{u(x),u(y)\}$
- But $u(\cdot)$ does not satisfy *strict* quasiconcavity.

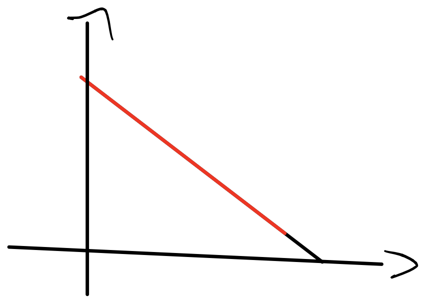


NOT STRICT!

POINT

IF I CONNECT

THIS POINT ARE NOT



STRICTLY CONCAVE
BUT IS THE SAME

Quasiconcavity

- ***Relationship between concavity and quasiconcavity***:

$$\text{Concavity} \underset{\not\Leftarrow}{\Longrightarrow} \text{Quasiconcavity}$$

  – If a function $f(\cdot)$ is *concave*, then for any two points $x, y \in X$,

$$f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$$
$$\geq \min\{f(x), f(y)\}$$

Since it is a weighted average of the two

  for all $\alpha \in (0,1)$.

    - The first inequality follows from the definition of concavity, while the second holds true for all concave functions.
    - Hence, **quasiconcavity is a weaker condition than concavity**.

IF FUNCTION IS CONCAVE IS QUASI CONCAVE BUT NOT THE OPPOSITE

WE DO NOT REQUIRE CONCAVITY IN OUR STUDY

YOU HAVE TO LOOK AT MATRIX

HESSIAN

FIRST DERIVATIVES $\dfrac{\partial u}{\partial x_1}$ $\dfrac{\partial u}{\partial x_2}$

$u(x_1, x_2)$

$$H = \begin{bmatrix} \underbrace{\dfrac{\partial^2 u}{\partial x_1 \partial x_1}}_{A_{11}} & \dfrac{\partial^2 u}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 u}{\partial x_2 \partial x_1} & \dfrac{\partial^2 u}{\partial x_2 \partial x_2} \end{bmatrix}$$

A_{22}

SYMMETRIC!

2,3 ARE EQUAL

HESSIAN MATRIX
(SECOND DERIVATIVE)

H IS POSITIVE DEFINITE $\Rightarrow u(\cdot)$ CONVEX

H IS NEG DEFINITE $\Rightarrow u(\cdot)$ IS CONCAVE

$|A_{11}| > 0$ $\quad |A_{22}| > 0$

$$\frac{\partial^2 u}{\partial x_1 \partial x_1} > 0 \qquad \frac{\partial^2 u}{\partial x_1 \partial x_1} \cdot \frac{\partial^2 u}{\partial x_2 \partial x_2} - \overbrace{\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2}^{\text{POSITIVE}} > 0$$

THIS IS VERIFIED BEFORE

THIS SHOULD BE POSITIVE THEN

PROVE MATRIX DEFINITE NEGATIVE

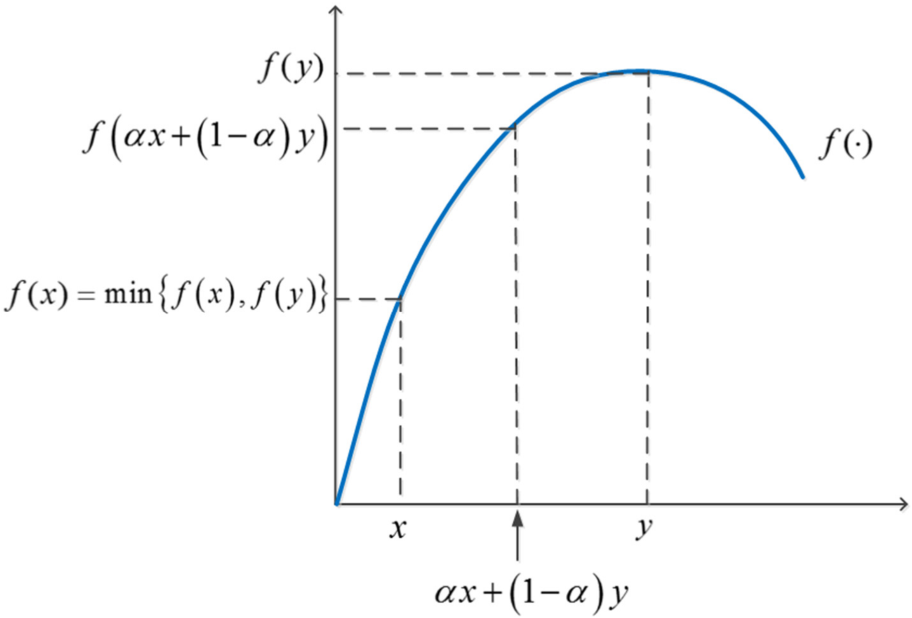
CONCAVE IF

$|A_{11}| < 0$ $|A_{22}| > 0$

SECOND DERIVATIVE IF FUNCTION CONCAVE

Quasiconcavity

- Concavity implies quasiconcavity



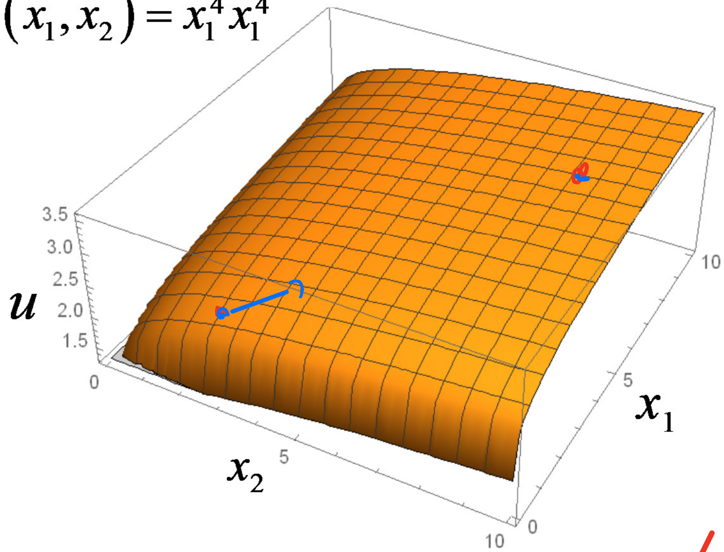
Quasiconcavity

- A concave $u(\cdot)$ exhibits diminishing marginal utility.
  – That is, **for an increase in the consumption bundle, the increase in utility is *smaller* as we move away from the origin**.
- The "jump" from one indifference curve to another requires:
  – a slight increase in the amount of x_1 and x_2 when we are close to the origin
  – a large increase in the amount of x_1 and x_2 as we get further away from the origin

Quasiconcavity

- Concave and quasiconcave utility function (3D)

$$u(x_1, x_2) = x_1^{\frac{1}{4}} x_1^{\frac{1}{4}}$$

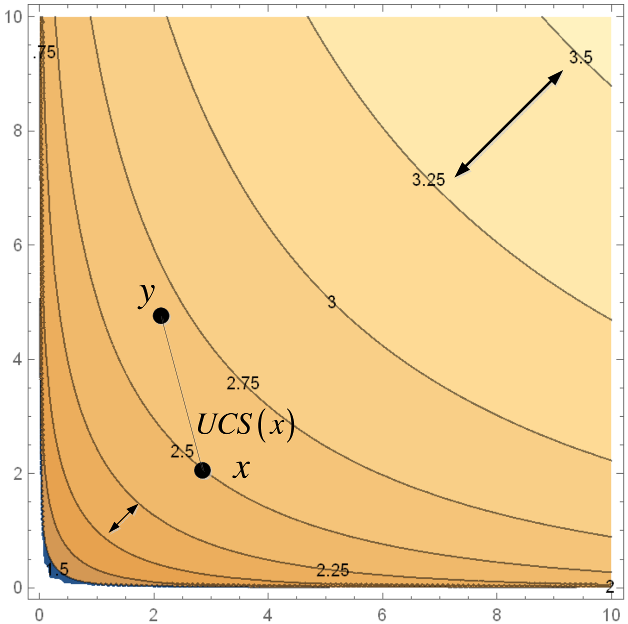


ALTITUDE?

CONTOUR MAP

Quasiconcavity

- Concave and quasiconcave utility function (2D)



PALE COLOUR
MEAN HIGHER
UTILITY

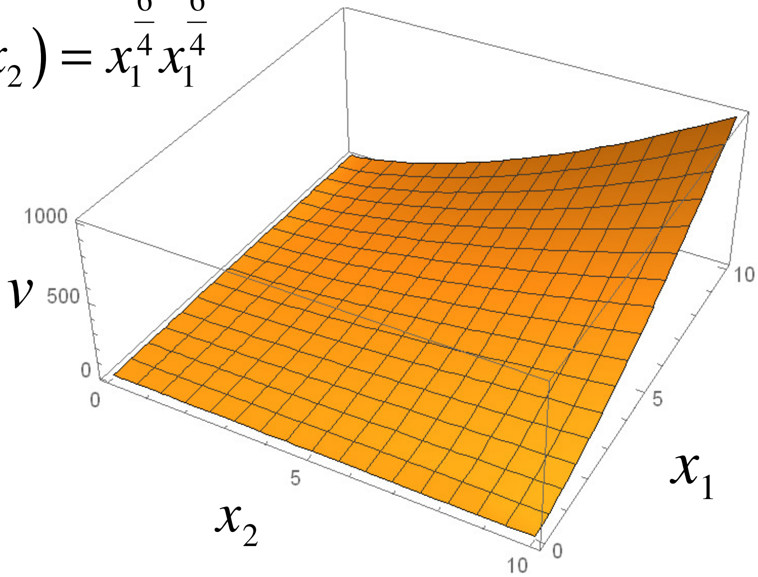
Quasiconcavity

- A convex $u(\cdot)$ exhibits increasing marginal utility.
  - That is, for an increase in the consumption bundle, the increase in utility is *larger* as we move away from the origin.
- The "jump" from one indifference curve to another requires:
  - a large increase in the amount of x_1 and x_2 when we are close to the origin, but…
  - a small increase in the amount of x_1 and x_2 as we get further away from the origin

Quasiconcavity

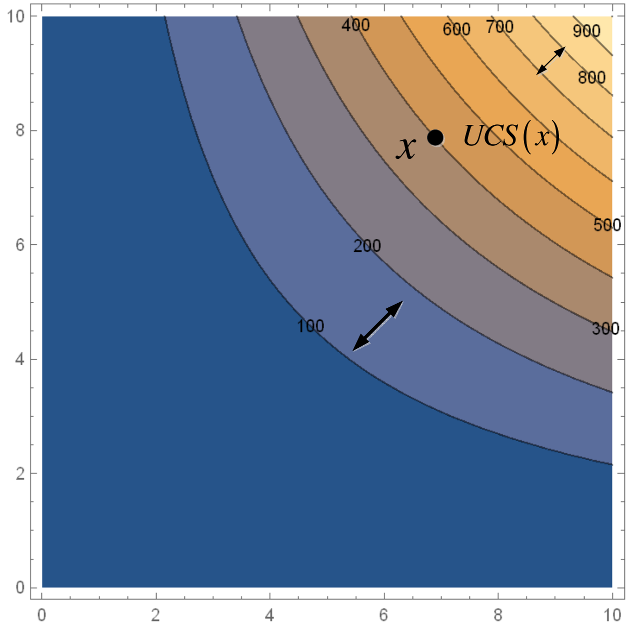
- Convex but quasiconcave utility function (3D)

$$v(x_1, x_2) = x_1^{\frac{6}{4}} x_1^{\frac{6}{4}}$$



Quasiconcavity

- Convex but quasiconcave utility function (2D)



Cobb-Douglas utility function

Quasiconcavity

- *Note*:
  - Utility function $v(x_1, x_2) = x_1^{\frac{6}{4}} x_2^{\frac{6}{4}}$ is a strictly monotonic transformation of $u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{1}{4}}$,
    - That is, $v(x_1, x_2) = f(u(x_1, x_2))$, where $f(u) = u^6$.
  - Therefore, utility functions $u(x_1, x_2)$ and $v(x_1, x_2)$ represent the same preference relation.
  - **Both utility functions are quasiconcave although one of them is concave and the other is convex**.
  - Hence, **we normally require utility functions to satisfy quasiconcavity alone**.

A1

Diapositiva 54

A1 Show quasi-concavity with the hessian?
Administrator; 04/01/2019

Quasiconcavity

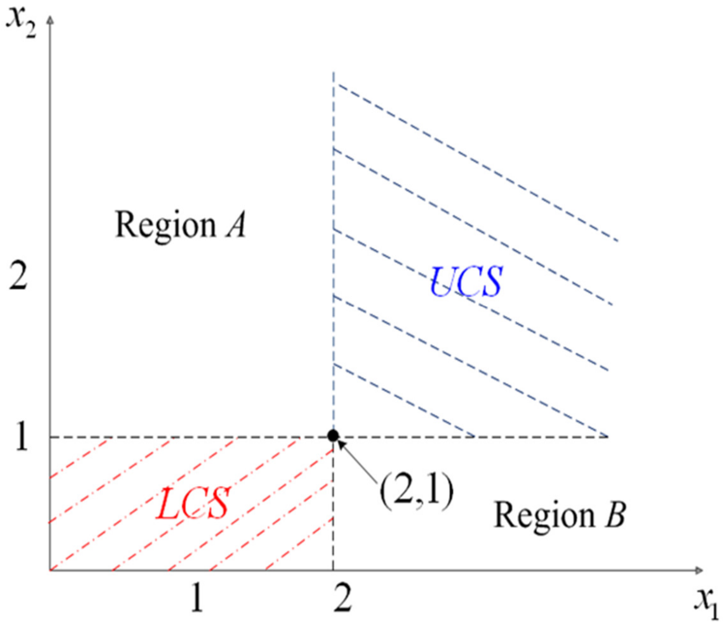
- ***Example 1.8*** (Testing properties of preference relations):
  - Consider an individual decision maker who consumes bundles in $\mathbb{R}_+^L$.
  - Informally, he "prefers more of everything"
  - Formally, for two bundles $x, y \in \mathbb{R}_+^L$, bundle x is weakly preferred to bundle y, $x \gtrsim y$, iff bundle x contains more units of every good than bundle y does, i.e., $x_k \geq y_k$ for every good k.
  - Let us check if this preference relation satisfies: (a) completeness, (b) transitivity, (c) strong monotonicity, (d) strict convexity, and (e) local non-satiation.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us consider the case of only two goods, $L = 2$.
  - Then, an individual prefers a bundle $x = (x_1, x_2)$ to another bundle $y = (y_1, y_2)$ iff x contains more units of both goods than bundle y, i.e., $x_1 \geq y_1$ and $x_2 \geq y_2$.
  - For illustration purposes, let us take bundle such as $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):



Quasiconcavity

- ***Example 1.8*** (continued):

  1) UCS:

  – The upper contour set of bundle $(2,1)$ contains bundles (x_1, x_2) with weakly more than 2 units of good 1 and/or weakly more than 1 unit of good 2:

  $$UCS(2,1) = \{(x_1, x_2) \succsim (2,1) \Leftrightarrow x_1 \geq 2, x_2 \geq 1\}$$

  – The frontiers of the UCS region also represent bundles preferred to $(2,1)$.

Quasiconcavity

- ***Example 1.8*** (continued):

  2) LCS:

  – The bundles in the lower contour set of bundle $(2,1)$ contain fewer units of both goods:

  $$LCS(2,1) = \{(2,1) \succsim (x_1, x_2) \ \Leftrightarrow x_1 \leq 2, x_2 \leq 1\}$$

  – The frontiers of the LCS region also represent bundles with fewer unis of either good 1 or good 2.

Quasiconcavity

- ***Example 1.8*** (continued):

  3) IND:

  – The indifference set comprising bundles (x_1, x_2) for which the consumer is indifferent between (x_1, x_2) and $(2,1)$ is empty:

  $$IND(2,1) = \{(2,1) \sim (x_1, x_2)\} = \emptyset$$

  – No region for which the upper contour set and the lower contour set overlap.

Quasiconcavity

- ***Example 1.8*** (continued):

  4) Regions A and B:

  – Region A contains bundles with more units of good 2 but fewer units of good 1 (the opposite argument applies to region B).

  – The consumer cannot compare bundles in either of these regions against bundle $(2,1)$.

  – For him to be able to rank one bundle against another, one of the bundles must contain the same or more units of all goods.

Quasiconcavity

- ***Example 1.8*** (continued):

  5) Preference relation is not complete:

  – Completeness requires for every pair x and y, either $x \succsim y$ or $y \succsim x$ (or both).
  – Consider two bundles $x, y \in \mathbb{R}_+^2$ with bundle x containing more units of good 1 than bundle y but fewer units of good 2, i.e., $x_1 > y_1$ and $x_2 < y_2$ (as in Region B)
  – Then, we have neither $x \succsim y$ (UCS) nor $y \succsim x$ (LCS).

Quasiconcavity

- ***Example 1.8*** (continued):

  6) Preference relation is transitive:

  – Transitivity requires that, for any three bundles x, y and z, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

  – Now $x \succsim y$ and $y \succsim z$ means $x_k \geq y_k$ and $y_k \geq z_k$ for all k goods.

  – Then, $x_k \geq z_k$ implies $x \succsim z$.

Quasiconcavity

- ***Example 1.8*** (continued):

  7) Preference relation is strongly monotone:

  – Strong monotonicity requires that if we increase one of the goods in a given bundle y, then the newly created bundle x must be strictly preferred to the original bundle.

  – Now $x \geq y$ and $x \neq y$ implies that $x_l \geq y_l$ for all good l and $x_k > y_k$ for at least one good k.

  – Thus, $x \geq y$ and $x \neq y$ implies $x \succsim y$ and not $y \succsim x$.

  – Thus, we can conclude that $x \succ y$.

Quasiconcavity

- ***Example 1.8*** (continued):

  8) Preference relation is strictly convex:

  – Strict convexity requires that if $x \succsim z$ and $y \succsim z$ and $x \neq y$, then $\alpha x + (1-\alpha)y \succ z$ for all $\alpha \in (0,1)$.

  – Now $x \succsim z$ and $y \succsim z$ implies that $x_l \geq y_l$ and $y_l \geq z_l$ for all good l.

  – $x \neq z$ implies, for some good k, we must have $x_k > z_k$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Hence, for any $\alpha \in (0,1)$, we must have that
    $$\alpha x_l + (1-\alpha)y_l \geq z_l \text{ for every good } l$$
    $$\alpha x_k + (1-\alpha)y_k > z_k \text{ for some } k$$
  - Thus, we have that $\alpha x + (1-\alpha)y \geq z$ and $\alpha x + (1-\alpha)y \neq z$, and so
    $$\alpha x + (1-\alpha)y \succsim z$$
    $$\text{and not } z \succsim \alpha x + (1-\alpha)y$$
  - Therefore, $\alpha x + (1-\alpha)y \succ z$.

Quasiconcavity

- ***Example 1.8*** (continued):
  9) ***Preference relation satisfies LNS***:
     - Take any bundle (x_1, x_2) and a scalar $\varepsilon > 0$.
     - Let us define a new bundle (y_1, y_2) where

$$(y_1, y_2) \equiv \left(x_1 + \frac{\varepsilon}{2}, x_2 + \frac{\varepsilon}{2}\right)$$

       so that $y_1 > x_1$ and $y_2 > x_2$.
     - Hence, $y \gtrsim x$ but not $x \gtrsim y$, which implies $y \succ x$.

Quasiconcavity

- ***Example 1.8*** (continued):
  - Let us know check if bundle y is within an ε-ball around x.
  - The Cartesian distance between x and y is

$$\|x - y\| = \sqrt{\left[x_1 - \left(x_1 + \frac{\varepsilon}{2}\right)\right]^2 + \left[x_1 - \left(x_1 + \frac{\varepsilon}{2}\right)\right]^2} = \frac{\varepsilon}{\sqrt{2}}$$

which is smaller than ε for all $\varepsilon > 0$.

Advanced Microeconomics (EPS)

Chapter 1: Common utility functions

Common Utility Functions

- ***Cobb-Douglas utility functions***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}$$

    where $A, \alpha, \beta > 0$.
  - Applying logs on both sides

$$\log u = \log A + \alpha \log x_1 + \beta \log x_2$$

  - Hence, the exponents in the original $u(\cdot)$ can be interpreted as *elasticities*:

$$\varepsilon_{u,x_1} = \frac{\partial u(x_1,x_2)}{\partial x_1} \cdot \frac{x_1}{u(x_1,x_2)} = \alpha A x_1^{\alpha-1} x_2^{\beta} \cdot \frac{x_1}{Ax_1^{\alpha}x_2^{\beta}} = \alpha$$

Compute marginal derivative.

$a x^2 b$

$$A x_1^a x_2^b \qquad \frac{\partial u}{\partial x_1} = a A x_2^\beta x_1^{a-1}$$

$K x_1^a$

$K x_1^{a-1}$

$$\frac{a A x_2^\beta x_1^{a-1}}{A b x_2^{b-1} x_1^a}$$

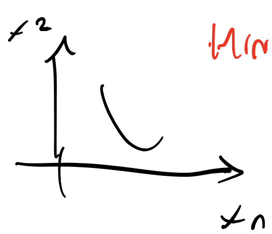
$$\frac{a x_2}{b x_1}$$

$$A x_1^a x_2^b \qquad \frac{\partial u}{\partial x_2} = A x_1^a b x_2^{b-1}$$

IF $A, a, \beta > 0$ SO MARG UTILITY IS POSITIVE!

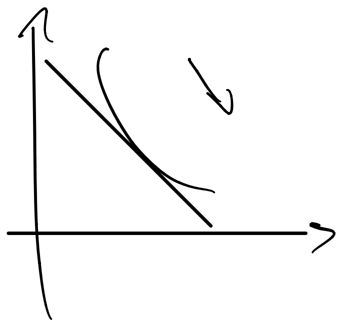
$$MRS = -\frac{dx_2}{dx_1} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}$$

SLOPE OF IND. CURVE



HINT TO REMEMBER

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{\cancel{A}\, \overset{x_1}{\cancel{x_2^{\beta}}}\, \overset{x_2}{\cancel{x_1^{\alpha-1}}}\, \alpha}{A x_1^{\alpha}\, \beta\, \underset{x_1}{\cancel{x_2^{\beta-1}}}\, \cancel{A}} = \frac{\alpha x_2}{\beta x_1}$$



IF $x_1 \uparrow$

MRS $\downarrow$

$\boxed{\text{ELASTICITY}}$ OF UTILITY IN THIS CASE

$$y = f(x) \Rightarrow \varepsilon_{y,x} = \frac{\partial f}{\partial x} \cdot \frac{x}{y}$$

$$\downarrow$$

$$\frac{\frac{\partial y}{\partial x}}{\frac{y}{x}}$$

% CHANGE IN y
PRODUCE BY CHANGING x

$$\frac{\frac{\partial y}{y}}{\frac{\partial x}{x}} \rightarrow \frac{\partial y}{y} \cdot \frac{x}{\partial x} \rightarrow \frac{\partial y}{\partial x} \cdot \frac{x}{y}$$

DER Y WITH RESPECT TO X MULTIPLY BY
X/Y

APPLY THIS TO UTILITY FUNCTION

$$\varepsilon_{u,x_1} = \frac{\partial u}{\partial x_1} \cdot \frac{x_1}{u}$$

ELASTICITY OF UTILITY WITH RESPECT TO X_1

MARGINAL UTILITY

INITIAL FUNCTION

$$\varepsilon_{u,x_1} = (\cancel{A} \cdot \cancel{x_2^{\beta}})\, \alpha \cancel{x_1^{\alpha-1}} \cdot \frac{\cancel{x_1}}{\cancel{A} \cancel{x_1^{\alpha}} \cancel{x_2^{\beta}}} = \boxed{\alpha}$$

SO ELASTICITY IS CONSTANT

If we have utility function and we apply

log of the product is the sum of the log of the product

$$\log x_1^{\alpha} = \alpha \log x_1$$

$$\log x_2^{\beta} = \beta \log x_2$$

SO IS JUST α AND IT'S SIMPLER

$$\varepsilon_{u,x_1} = \frac{d \log u}{d \log x_1} =$$

$(x_1^{\alpha} + x_2^{\beta})$ we cannot separate x_1, x_2 $\rightarrow$ we compute calculus did before

with log transformation we obtain something linear

Common Utility Functions

– Intuitively, a one-percent increase in the amount of good x_1 increases individual utility by α percent.

– Similarly, $\varepsilon_{u,x_2} = \beta$.

– Special cases:
  - $\alpha + \beta = 1$: $u(x_1, x_2) = Ax_1^{\alpha}x_2^{1-\alpha}$
  - $A = 1$: $u(x_1, x_2) = x_1^{\alpha}x_2^{\beta}$
  - $A = \alpha = \beta = 1$: $u(x_1, x_2) = x_1 x_2$

Common Utility Functions

– Marginal utilities:

$$\frac{\partial u}{\partial x_1} > 0 \text{ and } \frac{\partial u}{\partial x_2} > 0$$

– Diminishing MRS, since

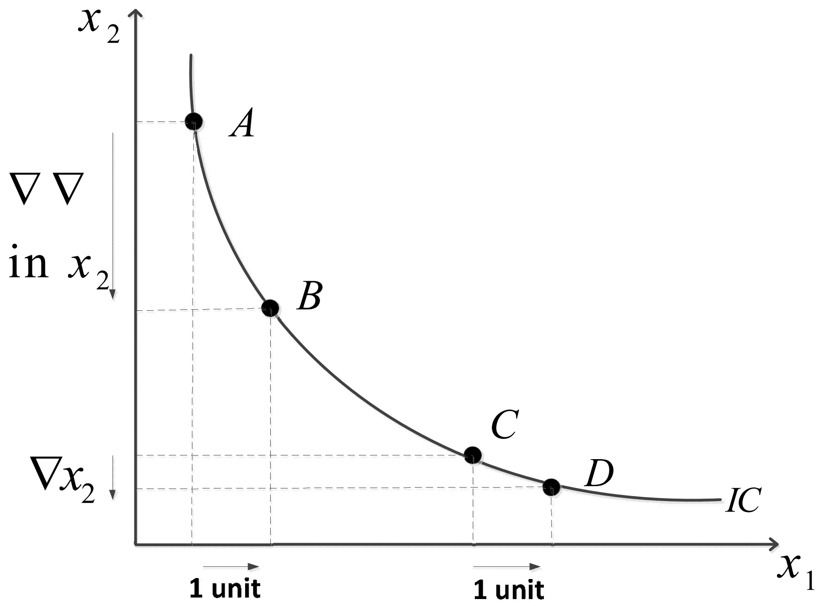
$$MRS_{x_1,x_2} = \frac{\alpha A x_1^{\alpha-1} x_2^{\beta}}{\beta A x_1^{\alpha} x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}$$

which is decreasing in x_1.

- Hence, indifference curves become flatter as x_1 increases.

Common Utility Functions

- Cobb-Douglas preference



Common Utility Functions

- ***Perfect substitutes***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = Ax_1 + Bx_2$$
  where $A, B > 0$.
  - Hence, the marginal utility of every good is constant:
  $$\frac{\partial u}{\partial x_1} = A \text{ and } \frac{\partial u}{\partial x_2} = B$$
  - MRS is also constant, i.e., $MRS_{x_1,x_2} = \frac{A}{B}$
    - Therefore, indifference curves are straight lines with a slope of $-\frac{A}{B}$.

Utility depends on x1 and x2 but they enter separately in the utility function. A and B must be greater than 0.

$A = 1$ $\Rightarrow$ $x_1 + x_2$
$B = 1$

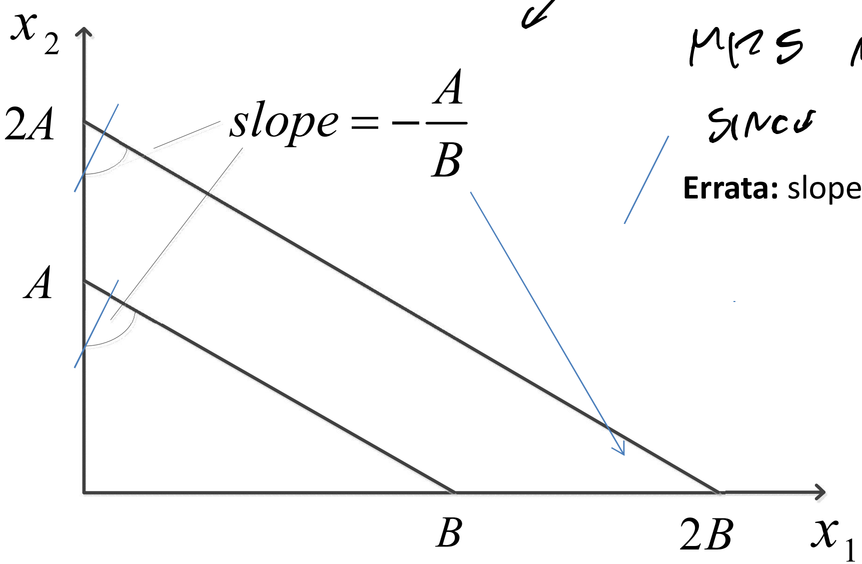
Marginal utility of this ?
Marginal of x1 is A and marginal of x2 is B.
In this case marginal utility is a constant and don’t depend on x1 and x2. In the linear utilty function, MU is constant. What does this imply for MRS (rateo of MU)? If the MU are constant then MRS is constant.

$$MRS = \frac{A}{B}$$

Common Utility Functions

- Perfect substitutes

CONSTANT SLOPE



MRS NOT DECREASE SINCE IS A CONSTANT!

Errata: slope is with x_1 axes

Common Utility Functions

– Intuitively, the individual is willing to give up $\frac{A}{B}$ units of x_2 to obtain one more unit of x_1 and keep his utility level unaffected.

– Unlike in the Cobb-Douglas case, such willingness is independent in the relative abundance of the two goods.

– *Examples*: butter and margarine, coffee and black tea, or two brands of unflavored mineral water

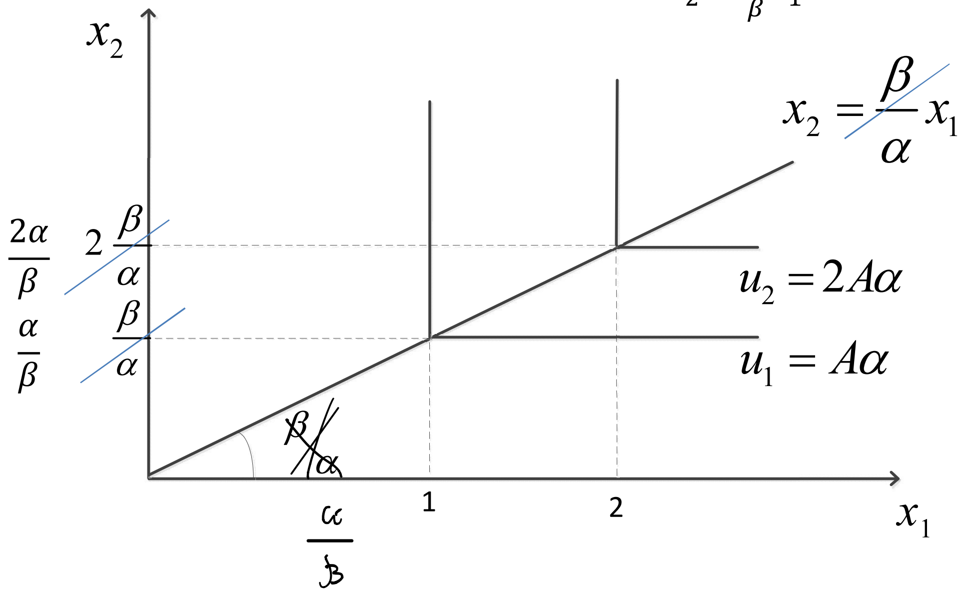
Common Utility Functions

- ***Perfect Complements***:
  - In the case of two goods, x_1 and x_2,
$$u(x_1, x_2) = A \cdot \min\{\alpha x_1, \beta x_2\}$$
where $A, \alpha, B > 0$.
  - Intuitively, increasing one of the goods without increasing the amount of the other good entails *no* increase in utility.
    - The amounts of *both* goods must increase for the utility to go up.
  - The indifference curve is right angle with a kink at $\alpha x_1 = \beta x_2$ that is $x_2 = (\alpha/\beta)\, x_1$

Common Utility Functions

- Perfect complements

Errata: $x_2 = \frac{\alpha}{\beta} x_1$



Common Utility Functions

– The slope of a ray $x_2 = \frac{\alpha}{\beta}x_1$, $\frac{\alpha}{\beta}$, indicates the rate at which goods x_1 and x_2 must be consumed in order to achieve utility gains.

– Special case: $\alpha = \beta$

$$\begin{aligned} u(x_1, x_2) &= A \cdot \min\{\alpha x_1, \alpha x_2\} \\ &= A\alpha \cdot \min\{x_1, x_2\} \\ &= B \cdot \min\{x_1, x_2\} \text{ if } B \equiv A\alpha \end{aligned}$$

– *Examples*: cars and gasoline, or peanut butter and jelly.

Common Utility Functions

- ***CES utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = \left[a{x_1}^{\frac{\sigma-1}{\sigma}} + b{x_2}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

    where σ measures the elasticity of substitution between goods x_1 and x_2.

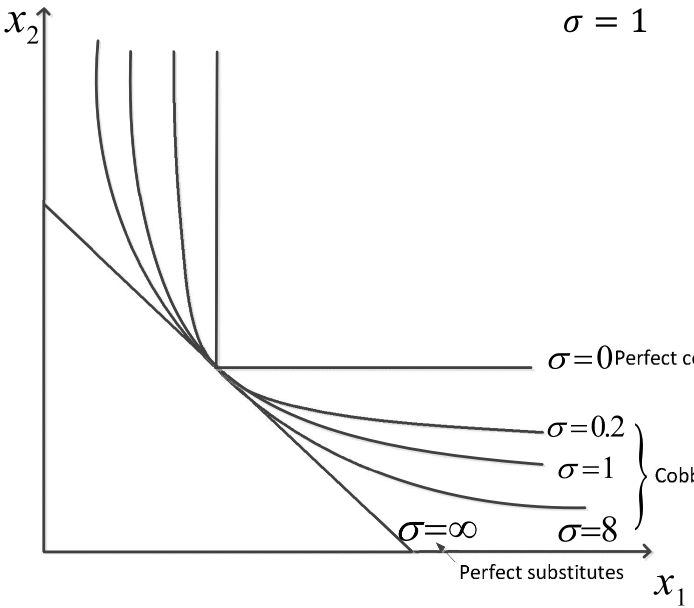
  - In particular,

$$\sigma = \frac{\partial\left(\frac{x_2}{x_1}\right)}{\partial MRS_{1,2}} \cdot \frac{MRS_{1,2}}{\frac{x_2}{x_1}}$$

Common Utility Functions

- CES preferences

Errata: Cobb Douglas if $\sigma = 1$



Common Utility Functions

– CES utility function is often presented as

$$u(x_1, x_2) = \left[ax_1^{\rho} + bx_2^{\rho}\right]^{\frac{1}{\rho}}$$

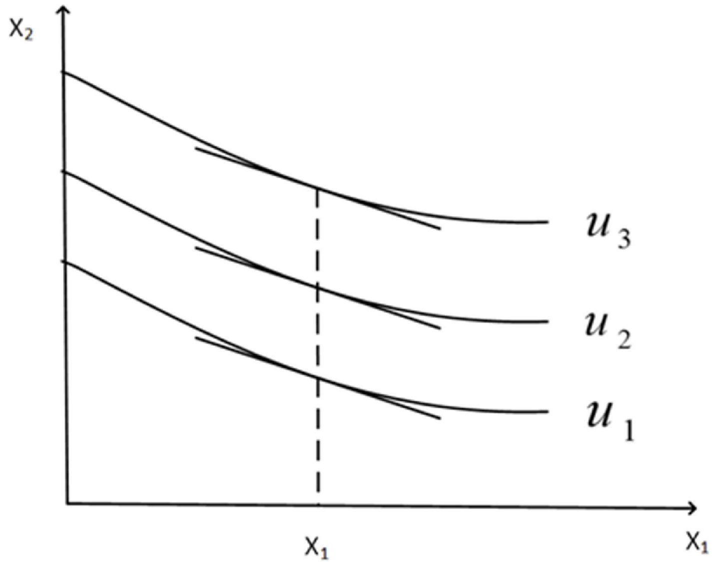
where $\rho \equiv \frac{\sigma - 1}{\sigma}$.

Common Utility Functions

- ***Quasilinear utility function***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = v(x_1) + bx_2$$
  where x_2 enters *linearly*, $b > 0$, and $v(x_1)$ is a *nonlinear* function of x_1.
    - For example, $v(x_1) = a \ln x_1$ or $v(x_1) = ax_1^{\alpha}$, where $a > 0$ and $\alpha \neq 1$.
  - The MRS is constant in the good that enters linearly in the utility function (x_2 in our case).

Common Utility Functions

- MRS of quasilinear preferences



Common Utility Functions

– For $u(x_1, x_2) = v(x_1) + bx_2$, the marginal utilities are

$$\frac{\partial u}{\partial x_2} = b \text{ and } \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$$

which implies

$$MRS_{x_1,x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

which is constant in the good entering linearly, x_2

– Quasilinear preferences are often used to represent the consumption of goods that are relatively insensitive to income.

– *Examples*: garlic, toothpaste, etc.

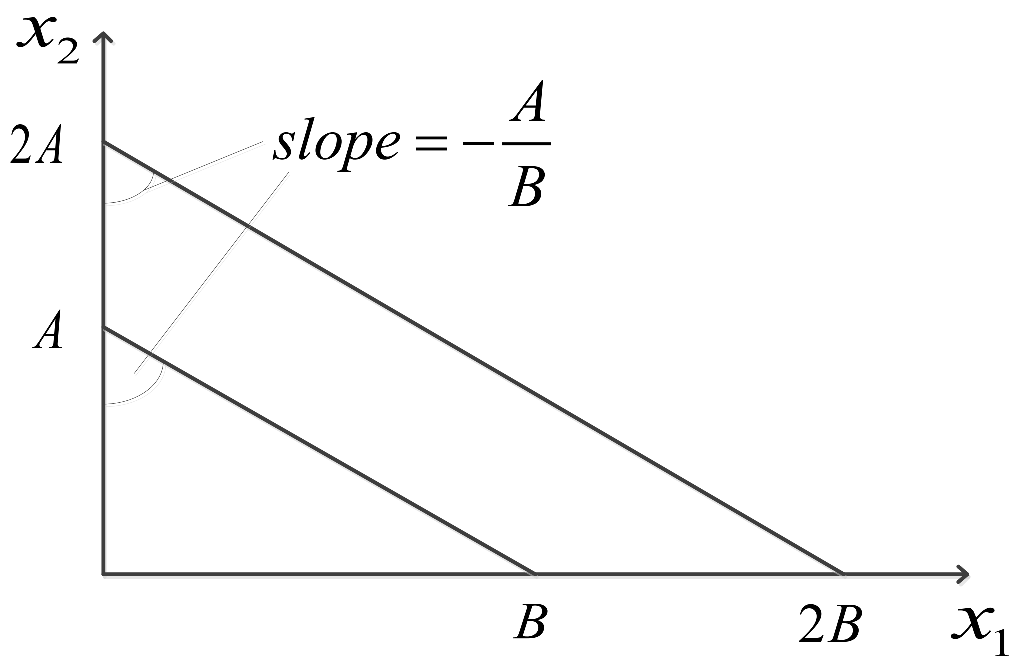
Summary
Perfect substitutes. A and B positive.
Last time introduce the concept of marginal utility = increase in the utility derived in infinitesima of x1.
MU of first good is der of u / der of x1 = A.
MRS is the slope of the indifference curve. In mathematics how do we compute? Ratio of the two MU.
If MU are constant also the ratio is constant. This mean that the slope is constant.

Common Utility Functions

- ***Perfect substitutes***:
  - In the case of two goods, x_1 and x_2,
    $$u(x_1, x_2) = Ax_1 + Bx_2$$
    where $A, B > 0$.
  - Hence, the marginal utility of every good is constant:
    $$\frac{\partial u}{\partial x_1} = A \text{ and } \frac{\partial u}{\partial x_2} = B$$
  - MRS is also constant, i.e., $MRS_{x_1,x_2} = \frac{A}{B}$
    - Therefore, indifference curves are straight lines with a slope of $-\frac{A}{B}$.

Common Utility Functions

- Perfect substitutes



How can you draw IC in a graph giving the utility function? (FOR PERFECT SUBSTITUTES)

$u(x_1, x_2) = x_1 + x_2$

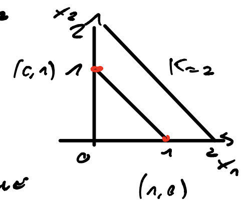
IS LINEAR? YES $A = 1$ $B = 1$

IS YOU WANT TO COMPUTE MU I HAVE TO COMPUTE DERIVATIVES

$\frac{\partial u}{\partial x_1} = 1$ $\frac{\partial u}{\partial x_2} = 1$

$MRS = -\frac{dx_2}{dx_1} = \frac{\frac{du}{dx_1}}{\frac{\partial u}{\partial x_2}} = 1$ HOW TO DRAW IT?

DEFINITION (SLOPE MRS IS)



∀ IND CURVE VALUE WILL GIVE SAME UTILITY

$K = x_1 + x_2$ → IND. CURVE WITH UTILITY LEVEL K

$K = 1$ → $1 = x_1 + x_2$ $x_2 = 1 - x_1$

(1,0) (0,1)

$K = 2$? DO THE SAME GET STRAIGHT LINE CROSSING 2 AND 2

u → LINEAR INC ARE STRAIGHT LINE

(MRS DO NOT DEP ON THE GOODS)

Common Utility Functions

– Intuitively, the individual is willing to give up $\frac{A}{B}$ units of x_2 to obtain one more unit of x_1 and keep his utility level unaffected.

– Unlike in the Cobb-Douglas case, such willingness is independent in the relative abundance of the two goods.

– *Examples*: butter and margarine, coffee and black tea, or two brands of unflavored mineral water

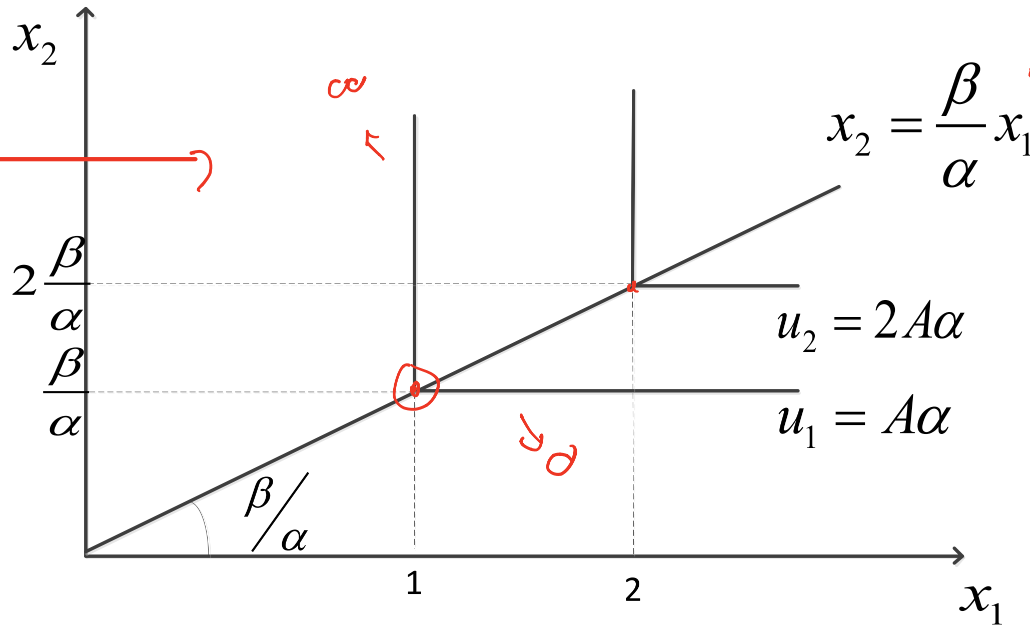
Common Utility Functions

- ***Perfect Complements***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = A \cdot \min\{\alpha x_1, \beta x_2\}$$
  where $A, \alpha, B > 0$.
  - Intuitively, increasing one of the goods without increasing the amount of the other good entails *no* increase in utility.
    - The amounts of *both* goods must increase for the utility to go up.
  - The indifference curve is right angle with a kink at $\alpha x_1 = \beta x_2$. $\longrightarrow \left(\frac{\alpha}{\beta}\right)x_1$

Common Utility Functions

- Perfect complements

All corners line on straight line



COT LINE OF CORNERS IS LIKE (??)

$\alpha x_1 = \beta x_2$

$x_2 = \frac{\alpha}{\beta} x_1$ SLOPE IS α / β

IMAGINE I WANT TO DRAW $A \cdot \min(\alpha x_1, \beta x_2)$

$A\alpha$

$2\alpha A$

$\alpha = 1$

In the case the slope is not decreasing. Slope is infinite in a vertical line, in orizzontal line slope is 0. In the point of corners the slope is not define.

Another function more complex that is called the constant elasticity of substitution.

Common Utility Functions

- The slope of a ray $x_2 = \frac{\beta}{\alpha} x_1$, $\frac{\beta}{\alpha}$, indicates the rate at which goods x_1 and x_2 must be consumed in order to achieve utility gains.
- Special case: $\alpha = \beta$

$$
\begin{aligned}
u(x_1, x_2) &= A \cdot \min\{\alpha x_1, \alpha x_2\} \\
&= A\alpha \cdot \min\{x_1, x_2\} \\
&= B \cdot \min\{x_1, x_2\} \text{ if } B \equiv A\alpha
\end{aligned}
$$

- *Examples*: cars and gasoline, or peanut butter and jelly.

Common Utility Functions

- ***CES utility function***:
  - In the case of two goods, x_1 and x_2,

$$u(x_1, x_2) = \left[a x_1^{\frac{\sigma-1}{\sigma}} + b x_2^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where σ measures the elasticity of substitution between goods x_1 and x_2.

  - In particular,

$$\boxed{\sigma = \frac{\partial\left(\frac{x_2}{x_1}\right)}{\partial MRS_{1,2}} \cdot \frac{MRS_{1,2}}{\frac{x_2}{x_1}}}$$

This form of the utility function that is called CES. A combination of cobddouglas function with only one good. We get a constant elasticity substitution.
This elasticity is define d in this way.

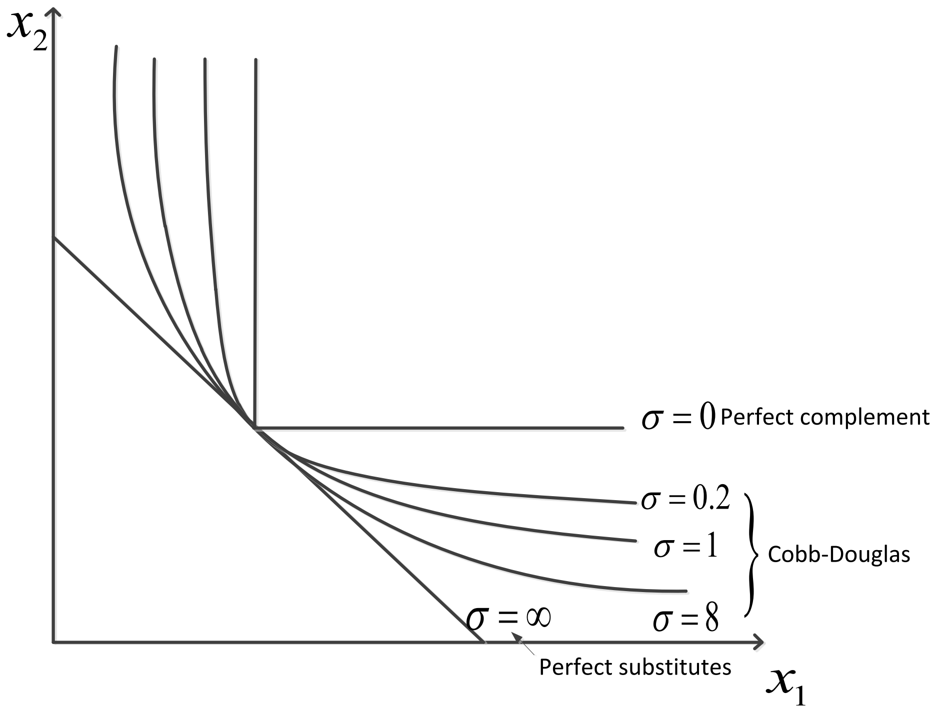
Elasticity is percentage change of one variable of the percentage change in the other
.

$$\frac{\dfrac{\partial\left(\frac{x_2}{x_1}\right)}{\frac{x_2}{x_1}}}{\dfrac{\partial MRS}{MRS}}$$

Graphical representation in the next slide.
Depending on value of sigma you can get all the utility functions that we have already introduce. If elasticity 1 we get cob Douglas, if 0 we obtain leonthief, if infinity you get a linear function.
Elasticity of substitution is infinity is that for me the two good are totally indifferent. So I don't care which of the two good consume.
On Ariel he will put the proof (not necessary).

Common Utility Functions

- CES preferences



Common Utility Functions

– CES utility function is often presented as

$$u(x_1, x_2) = \left[a x_1^{\rho} + b x_2^{\rho}\right]^{\frac{1}{\rho}}$$

where $\rho \equiv \frac{\sigma - 1}{\sigma}$.

Sometimes the CES is indicated in this way, using rho that is a function of sigma. Still remain constant.

Last utility function we consider is the quasi linear utilty function. This depend on the quantity consumed of two good. Quasilinear mean that one of the two good enter linearly in the utility function. X2 is linear.
Log function and cob double.

MRS of substitution (ratio of the two MU).

$$MU_{x_1} = \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1} \qquad \frac{\partial u}{\partial x_2} = b \qquad MRS_{x_1 x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

SO MRS DOES NOT DEPEND ON X_2!

IN CASE, LINEAR FUNCTION

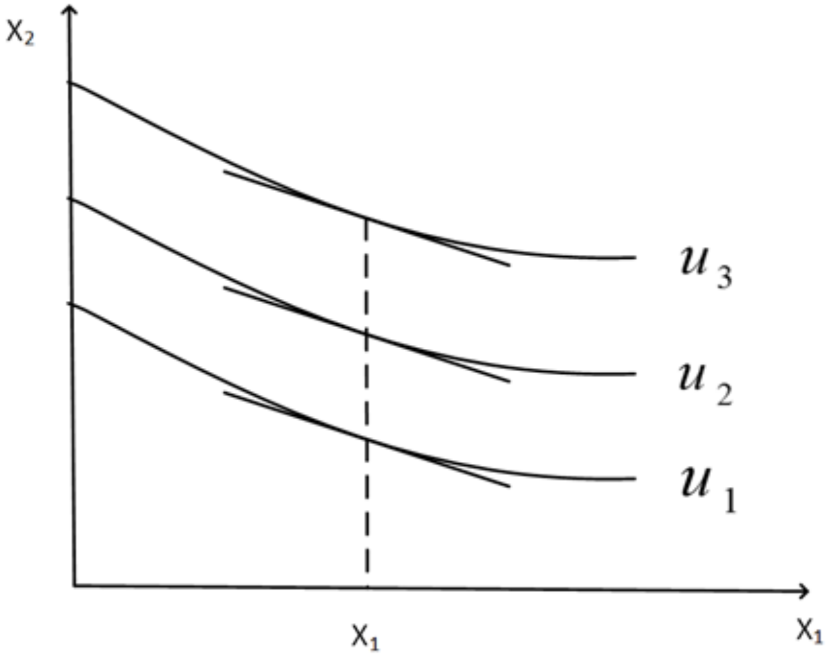
So this is the overview of all utility function we will consider.

Common Utility Functions

- ***Quasilinear utility function***:
  - In the case of two goods, x_1 and x_2,
  $$u(x_1, x_2) = v(x_1) + bx_2$$
  where x_2 enters *linearly*, $b > 0$, and $v(x_1)$ is a *nonlinear* function of x_1.
    - For example, $v(x_1) = a \ln x_1$ or $v(x_1) = ax_1^{\alpha}$, where $a > 0$ and $\alpha \neq 1$.
  - The MRS is constant in the good that enters linearly in the utility function (x_2 in our case).

Common Utility Functions

- MRS of quasilinear preferences



Common Utility Functions

– For $u(x_1, x_2) = v(x_1) + bx_2$, the marginal utilities are

$$\frac{\partial u}{\partial x_2} = b \text{ and } \frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_1}$$

which implies

$$MRS_{x_1,x_2} = \frac{\frac{\partial v}{\partial x_1}}{b}$$

which is constant in the good entering linearly, x_2

– Quasilinear preferences are often used to represent the consumption of goods that are relatively insensitive to income.

– *Examples*: garlic, toothpaste, etc.

Properties of Preference Relations

We go on with another section of the chapter and we will introduce other properties of preference relation.
Rational preference: completeness, transitivity.
Completeness: DM can define for any two bundle you are able to compare each goods in the bundle
Transitivity: 1° > 2° and 2°>3° then 1°> 3°

Now we define a bundle that is a combination of good in a given points.
We define other feature in the preference relation.

One is the definition of ***homogeneity***: utility function is homogeneous, if you take utility and multiply each argument by alpha then utility function is equal to utility multiply by a^k (with alpha > 0)
If 0<a<1 we are decreasing quantity of the original bundle.

If this happen we define the utility function as homogenous of degree k.

RATIO BETWEEN GOODS

$$\frac{a x_1}{x_1} = a$$

$$\frac{a x_2}{x_2} = a \qquad \frac{u(a x_1, a x_2)}{u(x_1, x_2)} = a^k \frac{(u(x_1, x_2)}{u(x_1, x_2)} = a^k$$

u CHANGES TO PROPORTION a TO POWER OF k

If a function if utilty of degree k, then the first derivative is homogeneous of degree k-1.
How to prove?

$$u(a x_1, a x_2) = a^k u(x_1, x_2)$$

$$\frac{\partial u(a x_1, a x_2)}{\partial x_1} = a^{k-1} \frac{\partial u(x_1, x_2)}{\partial x_1}$$

TO PROVE!

HINT → DERIVE EACH SIDE

$$\frac{\partial u(a x_1, a x_2)}{\partial x_1} a = a^k \frac{\partial u(x_1, x_2)}{\partial x_1}$$

$$u'(a x_1, a x_2) = \frac{a^k}{a} u'(x_1, x_2) \qquad u'(a x_1, a x_2) = a^{k-1} u(x_1, x_2)$$

Properties of Preference Relations

- ***Homogeneity***:
  - A utility function is *homogeneous of degree k* if varying the amounts of all goods by a common factor $\alpha > 0$ produces an increase in the utility level by α^k.
  - That is, for the case of two goods,
    $$u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$$
    where $\alpha > 0$. This allows for:
    - $\alpha > 1$ in the case of a common increase
    - $0 < \alpha < 1$ in the case of a common decrease

Properties of Preference Relations

– Three properties:

1) *The first-order derivative of a function $u(x_1, x_2)$ which is **homogeneous of degree k** is homogeneous of degree $k - 1$.*

- Given $u(\alpha x_1, \alpha x_2) = \alpha^k u(x_1, x_2)$, we can take derivatives of both sides with respect to x_i that is

$$\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial(\alpha x_i)} \cdot \alpha = \alpha^k \cdot \frac{\partial u(x_1, x_2)}{\partial x_i}$$

and re-arranging

$$u_i'(\alpha x_1, \alpha x_2) = \alpha^{k-1} u_i'(x_1, x_2)$$

Where u_i' denotes partial derivative w.r.t. *i* argument.

1. If function is homogeneous the IC has a specific shape. In particular is radial expansion of one another. If we increase with the same quantity the two bundle then they lie on the same indifference curve. Radial expaction because with increase the value by the same proportion of alpha.
2. If we compute the MRS along radial expansion the slope of first IC is equal to the slope of the second IC. So marginal rate of substitution is constant then IC are parallel curve.

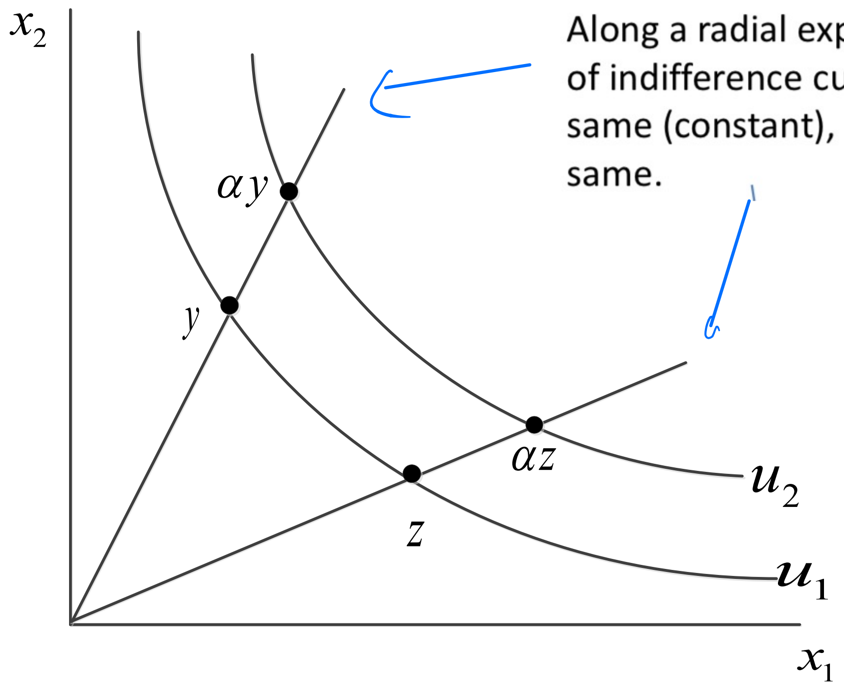
Properties of Preference Relations

2) The indifference curves of homogeneous functions are radial expansions of one another.

- That is, if two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$.

Properties of Preference Relations

- Homogenous preference



Along a radial expansion slope of indifference curve is the same (constant), i.e. MRS is the same.

Properties of Preference Relations

3) *The MRS of a homogeneous function is constant for all points along each ray from the origin.*

- That is, the slope of the indifference curve at point y coincides with the slope at a "scaled-up version" of point y, αy, where $\alpha > 1$. TRY PROVE IT.
- The MRS at bundle $x = (x_1, x_2)$ is

$$MRS_{1,2}(x_1, x_2) = -\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

- The MRS at $(\alpha x_1, \alpha x_2)$ is

Since is homogeneous the derivative is equal to u time a^k-1 degree

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = -\frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

$$= -\frac{\alpha^{k-1}\frac{\partial u(x_1, x_2)}{\partial x_1}}{\alpha^{k-1}\frac{\partial u(x_1, x_2)}{\partial x_2}} = -\frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}}$$

where the second equality uses the first property.

- Hence, the MRS is unaffected along all the points crossed by a ray from the origin.

Along radial expansion we prove MRS is the same since has degree k

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
  - Proof: if $u(y) = u(z) \Longrightarrow g\big(v(y)\big) = g\big(v(z)\big)$ and being $g(.)$ monotonic then $v(y) = v(z)$ (two arguments cannot have the same value of the function. From homogeneity of degree k of v (.) we know that

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k\, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k\, v(z))$$

Hence, $\alpha^k\, v(y)$= $\alpha^k\, v(z)$ and $u(\alpha y)$= $u(\alpha z)$.

Increasing transformation of (similar to say monotonic transformation) this new utilty function is called homothetic.

Monotonic preserve the ordering of the arguments.

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
  - Proof: if $u(y) = u(z) \Longrightarrow g\big(v(y)\big) = g\big(v(z)\big)$ and being g(.) monotonic then $v(y) \ = v(z)$ (two arguments cannot have the same value of the function. From homogeneity of degree k of v (.) we know that

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k\, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k\, v(z))$$

Hence, $\alpha^k\, v(y)$= $\alpha^k\, v(z)$ and $u(\alpha y)$= $u(\alpha z)$.

$$u(ax_1, ax_2) \cdot g(v(ax_1, ax_2))$$

$$u(az_1, az_2) \cdot g(v(az_1, az_2))$$

GARCIA

Properties of Preference Relations

– The MRS of a homothetic function is homogeneous of degree zero.

– In particular,

Slope of IC will be equal. So along expaction, MRS is equal.

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\cancel{\frac{\partial g}{\partial u}} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\cancel{\frac{\partial g}{\partial u}} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

$\cancel{a}$ $\cancel{a}$

$\partial(\alpha x_1)$

$\partial(\alpha x_2)$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

– Canceling the $\frac{\partial g}{\partial u}$ terms yields (V is homogeneous of degree k)

Since v hom. of degree k

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\cancel{\alpha^{k-1}} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\cancel{\alpha^{k-1}} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

– **The MRS of a homothetic function is homogeneous of degree zero.**

Proof.

$$|MRS_{1,2}(\alpha x_1, \alpha x_2)| = \frac{\frac{\partial u(\alpha x_1,\alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1,\alpha x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial v}\cdot\frac{\partial v(\alpha x_1,\alpha x_2)}{\partial(\alpha x_1)}\alpha}{\frac{\partial g}{\partial v}\cdot\frac{\partial v(\alpha x_1,\alpha x_2)}{\partial(\alpha x_2)}\alpha}$$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

– Canceling the $\frac{\partial g}{\partial v}\alpha$ terms yields (v is homogeneous of degree k)

$$\frac{\frac{\partial v(\alpha x_1,\alpha x_2)}{\partial(\alpha x_1)}}{\frac{\partial v(\alpha x_1,\alpha x_2)}{\partial(\alpha x_2)}} = \frac{\alpha^{k-1}\cdot\frac{\partial v(x_1,x_2)}{\partial x_1}}{\alpha^{k-1}\cdot\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

Vèra

GARCIA

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \Bigg| \qquad \Bigg| =$$

PROVE THIS!

$$= \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = MRS_{1,2}(x_1, x_2)$$

$$\frac{\frac{\cancel{\alpha^k}\, \delta u(x_1, x_2)}{\delta x_1}}{\frac{\alpha^k\, \delta u(x_1, x_2)}{\delta x_2}}$$

Graph 1. If we increase by 2 both arguments also the value of the function doblued. So this IC will correspond with twice the level of utilty.
In homothetic actually the level of utilty does not doble in some case. So all the thing i notice graphically are summarised in the slide (homogeneous function are homothetic.. but homothetic function are not necessary homogeneous).

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(\alpha x_1,\alpha x_2)}{\partial(\alpha x_1)}}{\frac{\partial v(\alpha x_1,\alpha x_2)}{\partial(\alpha x_2)}} = \frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$|MRS_{1,2}(\alpha x_1, \alpha x_2)| = \frac{\frac{\partial u(\alpha x_1,\alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1,\alpha x_2)}{\partial x_2}} = \frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

$= \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} \equiv |MRS_{1,2}(x_1, x_2)| = \frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$ (do the proof in this line for exercise, proof in the following slide)

VERA

Properties of Preference Relations

- But we also have

$$|MRS_{1,2}(x_1, x_2)| = \frac{\frac{\partial u(x_1,x_2)}{\partial x_1}}{\frac{\partial u(x_1,x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial u} \cdot \frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial g}{\partial u} \cdot \frac{\partial v(x_1,x_2)}{\partial x_2}} = \frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}.$$

Hence $|MRS_{1,2}(\alpha x_1, \alpha x_2)| = |MRS_{1,2}(x_1, x_2)|$.

i.e. MRS is the same along radial expansions.

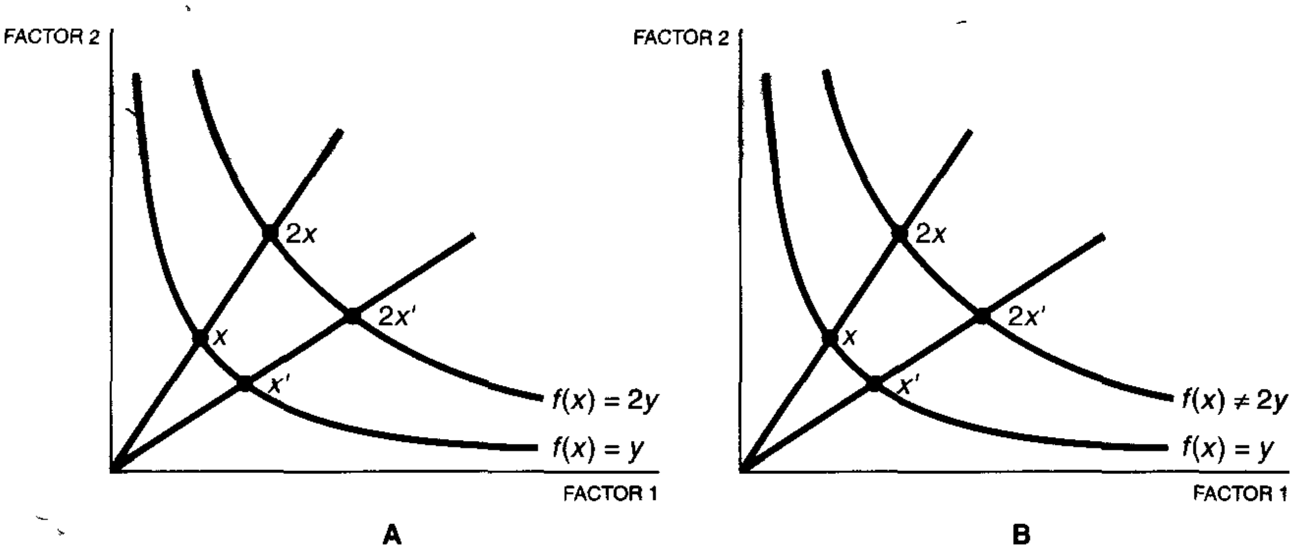
Properties of Preference Relations

- ***Homotheticity (graphical interpretation)***
  - A preference relation on $X = \mathbb{R}^L_+$ is homothetic if all indifference sets are related to proportional expansions along the rays.
  - That is, if the consumer is indifferent between bundles x and y, i.e., $x \sim y$, he must also be indifferent between a common scaling in these two bundles, i.e., $\alpha x \sim \alpha y$, for every scalar $\alpha \geq 0$.

Properties of Preference Relations

- For a given ray from the origin, the slope of the indifference curves (i.e., the MRS) that the ray crosses coincides.
  - The ratio between the two goods x_1/x_2 remains constant along all points in the ray.
- Intuitively, the rate at which a consumer is willing to substitute one good for another (his MRS) only depends on:
  - the rate at which he consumes the two goods, i.e., x_1/x_2, but does not depend on the utility level he obtains.
- But it is independent in the volume of goods he consumes, and in the utility he achieves.

Properties of Preference Relations



Homogeneous of degree k=1

Homothethic

Properties of Preference Relations

- ***Homogeneity and homotheticity***:
  - Homogeneous functions are homothetic.
    - We only need to apply a monotonic transformation $g(\cdot)$ on $v(x_1, x_2)$, i.e., $u(x_1, x_2) = g(v(x_1, x_2))$.
  - But homothetic functions are not necessarily homogeneous.
    - Take a homogeneous (of degree one) function $v(x_1, x_2) = x_1 x_2$.
    - Apply a monotonic transformation $g(y) = y + a$, where $a > 0$, to obtain homothetic function
$$u(x_1, x_2) = x_1 x_2 + a$$

$u(x_1, x_2) = x_1 x_2$ ORIGINAL UTILITY FUNCTION

APPLY DEFINITION (INC BOTH QUANTITY BY α)

$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) = \alpha^2 (x_1 x_2) = \underline{\alpha^2 \cdot u(x_1, x_2)}$

$\alpha > 1$

THIS IS HOMOGENEOUS OF DEGREE 2

STRICTLY INCR TRANSFORMATION EXAMPLE

PRESERVE ORDERING

$u(x_1, x_2) = x_1 \cdot x_2 + c$

SO THIS IS HOMOTHETIC → CHECK HOMOGENEOUS

$u(\alpha x_1, \alpha x_2) = \alpha^2 (x_1 \cdot x_2) + c$ NOT HOMOGENEOUS

NO WAY DEFINE VALUE ... [1 : A8]

$\neq \alpha^K u(x_1, x_2)$??

Homogeneous with strictly incr transformation we get homothetic function.
If we get hothetic function is not implied that we also get it homogeneous.

Properties of Preference Relations

- This function is not homogeneous, since increasing all arguments by α yields

$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) + a$$
$$= \alpha^2 v(x_1, x_2) + a$$
$$\neq \alpha^K u(x_1, x_2)$$

- Other monotonic transformations yielding non-homogeneous utility functions are

$$g(y) = (ay^\gamma) + by, \quad \text{where } a, b, \gamma > 0, \quad \text{or}$$
$$g(y) = a(\ln y), \quad \text{where } a > 0$$

Do as an exercise: prove that the two function are homogeneous.

NOT HOMOGENEOUS -> ORIGINAL FUNCTION V WAS

$$g(y) = a(y^3) + by$$

$$g(ay) = a^2 y^3 + aby \neq a^k u(x_1, x_2)$$

$$g(y) = a \ln y \qquad g(ay) = a^2 \ln y \neq a^k u(y)$$

Properties of Preference Relations

$u(\alpha x_1, \alpha x_2) = a(\alpha x_1) + b(\alpha x_1) = [a x_1 + b x_2]\alpha$

$u(x_1, x_2)$

- Utility functions that satisfy homotheticity:
  - Linear utility function $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$

    $u(tx_1, tx_2) = a(tx_1) + b(tx_2)$

    derivative wrt $x_1 \to at$ der. wrt $x_2 \to bt$

    - Goods x_1 and x_2 are perfect substitutes
    - $MRS(x_1, x_2) = \frac{a}{b}$ and $MRS(tx_1, tx_2) = \frac{at}{bt} = \frac{a}{b}$

    $\frac{at}{bt} = \frac{a}{b}$

    MRS is same of the original bundles and drawn, it's homothetic
  - The Leontief utility function $u(x_1, x_2) = A \cdot \min\{ax_1, bx_2\}$, where $A > 0$

    $a x_1 = \beta x_2$

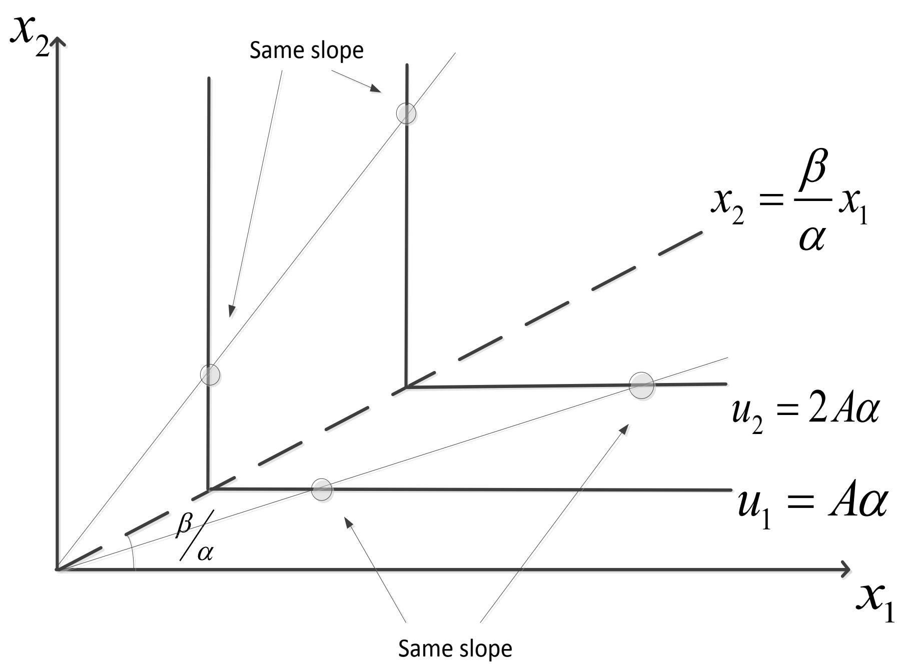
    $x_2 = \frac{a}{b} x_2$

    Manca roba qua!!

    - Goods x_1 and x_2 are perfect complements
    - We cannot define the MRS along all the points of the indifference curves
    - However, the slope of the indifference curves coincide for those points where these curves are crossed by a ray from the origin.

Properties of Preference Relations

- Perfect complements and homotheticity



Properties of Preference Relations

- ***Homotheticity***:
  - A utility function $u(x)$ is homothetic if it is a monotonic transformation of a homogeneous function.
  - That is, $u(x) = g(v(x))$, where
    - $g: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, and
    - $v: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k.

Properties of Preference Relations

- Properties:
  - If $u(x)$ is homothetic, and two bundles y and z lie on the same indifference curve, i.e., $u(y) = u(z)$, bundles αy and αz also lie on the same indifference curve, i.e., $u(\alpha y) = u(\alpha z)$ for all $\alpha > 0$.
    - In particular,

$$u(\alpha y) = g(v(\alpha y)) = g(\alpha^k \, v(y))$$
$$u(\alpha z) = g(v(\alpha z)) = g(\alpha^k \, v(z))$$

Properties of Preference Relations

– The MRS of a homothetic function is homogeneous of degree zero.

– In particular,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\frac{\partial g}{\partial u} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial g}{\partial u} \cdot \frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}}$$

where $u(x_1, x_2) \equiv g(v(x_1, x_2))$.

– Canceling the $\frac{\partial g}{\partial u}$ terms yields

$$\frac{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial v(\alpha x_1, \alpha x_2)}{\partial x_2}} = \frac{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_1}}{\alpha^{k-1} \cdot \frac{\partial v(x_1, x_2)}{\partial x_2}}$$

Properties of Preference Relations

– Canceling the α^{k-1} terms yields

$$\frac{\frac{\partial v(x_1,x_2)}{\partial x_1}}{\frac{\partial v(x_1,x_2)}{\partial x_2}}$$

– In summary,

$$MRS_{1,2}(\alpha x_1, \alpha x_2) = \frac{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_1}}{\frac{\partial u(\alpha x_1, \alpha x_2)}{\partial x_2}} =$$

$$= \frac{\frac{\partial u(x_1, x_2)}{\partial x_1}}{\frac{\partial u(x_1, x_2)}{\partial x_2}} = MRS_{1,2}(x_1, x_2)$$

Properties of Preference Relations

- ***Homotheticity (graphical interpretation)***
  - A preference relation on $X = \mathbb{R}_+^L$ is homothetic if all indifference sets are related to proportional expansions along the rays.
  - That is, if the consumer is indifferent between bundles x and y, i.e., $x \sim y$, he must also be indifferent between a common scaling in these two bundles, i.e., $\alpha x \sim \alpha y$, for every scalar $\alpha \geq 0$.

Properties of Preference Relations

- For a given ray from the origin, the slope of the indifference curves (i.e., the MRS) that the ray crosses coincides.
  - The ratio between the two goods x_1/x_2 remains constant along all points in the ray.
- Intuitively, the rate at which a consumer is willing to substitute one good for another (his MRS) only depends on:
  - the rate at which he consumes the two goods, i.e., x_1/x_2, but does not depend on the utility level he obtains.
- But it is independent in the volume of goods he consumes, and in the utility he achieves.

Properties of Preference Relations

- ***Homogeneity and homotheticity***:
  - Homogeneous functions are homothetic.
    - We only need to apply a monotonic transformation $g(\cdot)$ on $v(x_1, x_2)$, i.e., $u(x_1, x_2) = g(v(x_1, x_2))$.
  - But homothetic functions are not necessarily homogeneous.
    - Take a homogeneous (of degree one) function $v(x_1, x_2) = x_1 x_2$.
    - Apply a monotonic transformation $g(y) = y + a$, where $a > 0$, to obtain homothetic function
$$u(x_1, x_2) = x_1 x_2 + a$$

Properties of Preference Relations

- This function is not homogeneous, since increasing all arguments by α yields
$$u(\alpha x_1, \alpha x_2) = (\alpha x_1)(\alpha x_2) + a$$
$$= \alpha^2 v(x_1, x_2) + a$$

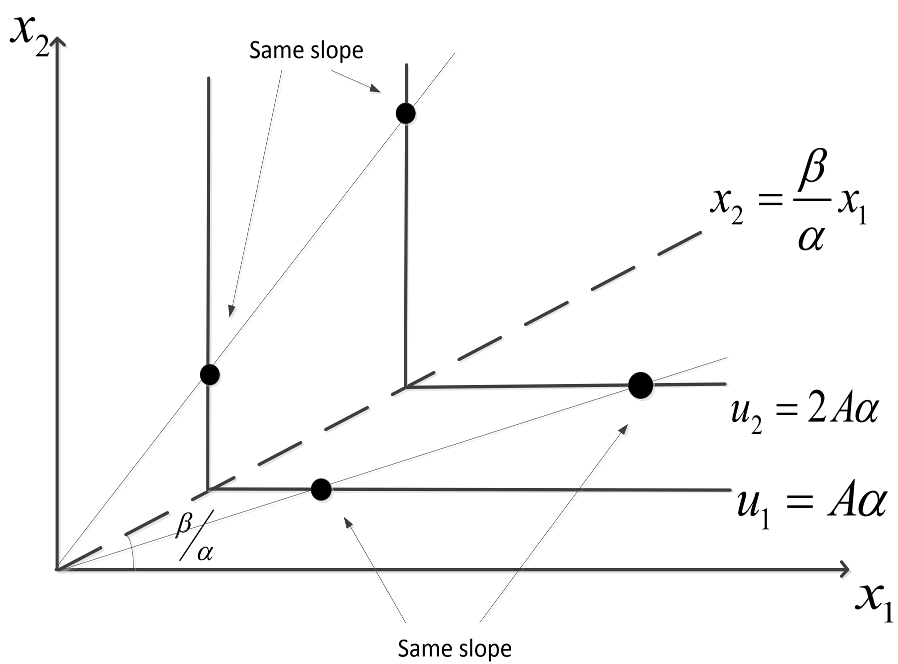
- Other monotonic transformations yielding non-homogeneous utility functions are
$$g(y) = ay^\gamma + by, \text{ where } a, b, \gamma > 0, \quad \text{or}$$
$$g(y) = a \ln y, \text{ where } a > 0$$

Properties of Preference Relations

- Utility functions that satisfy homotheticity:
  - Linear utility function $u(x_1, x_2) = ax_1 + bx_2$, where $a, b > 0$
    - Goods x_1 and x_2 are perfect substitutes
    - $MRS(x_1, x_2) = \frac{a}{b}$ and $MRS(tx_1, tx_2) = \frac{at}{bt} = \frac{a}{b}$
  - The Leontief utility function $u(x_1, x_2) = A \cdot \min\{ax_1, bx_2\}$, where $A > 0$
    - Goods x_1 and x_2 are perfect complements
    - We cannot define the MRS along all the points of the indifference curves
    - However, the slope of the indifference curves coincide for those points where these curves are crossed by a ray from the origin.

Properties of Preference Relations

- Perfect complements and homotheticity



Properties of Preference Relations

- ***Example 1.9*** (Testing for quasiconcavity and homotheticity):
  - Let us determine if $u(x_1, x_2) = \ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave, homothetic, both or neither.
  - *Quasiconcavity*:
    - Note that $\ln(x_1^{0.3} x_2^{0.6})$ is a monotonic transformation of the Cobb-Douglas function $x_1^{0.3} x_2^{0.6}$.
    - Since $x_1^{0.3} x_2^{0.6}$ is a Cobb-Douglas function, where $\alpha + \beta = 0.3 + 0.6 < 1$, it must be a concave function.
    - Hence, $x_1^{0.3} x_2^{0.6}$ is also quasiconcave, which implies $\ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave (as quasiconcavity is preserved through a monotonic transformation).

Properties of Preference Relations

- ***Example 1.9*** (continued):
  - *Homogeneity*:
    - Increasing all arguments by a common factor α,

$$(\alpha x_1)^{0.3}(\alpha x_2)^{0.6} = \alpha^{0.3} x_1^{0.3} \alpha^{0.6} x_2^{0.6} = \alpha^{0.9} x_1^{0.3} x_2^{0.6}$$

    - Hence, $x_1^{0.3} x_2^{0.6}$ is homogeneous of degree 0.9

  - *Homotheticity:*
    - Therefore, $x_1^{0.3} x_2^{0.6}$ is also homothetic.
    - As a consequence, its transformation, $\ln(x_1^{0.3} x_2^{0.6})$, is also homothetic (as homotheticity is preserved through a monotonic transformation).

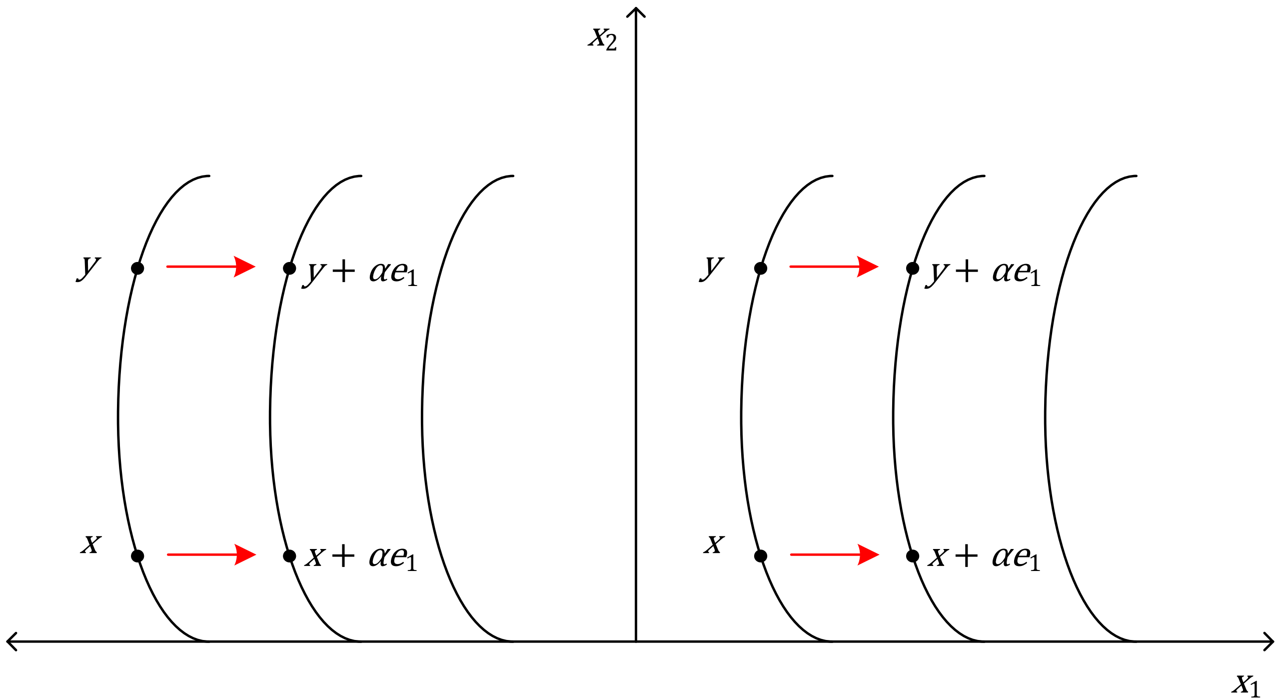
Properties of Preference Relations

- ***Quasilinear preference relations***:
  – The preference relation on $X = (-\infty, \infty)$ $x \in \mathbb{R}_+^{L-1}$ is *quasilinear* with respect to good 1 if:

    1) All indifference sets are parallel displacements of each other along the axis of good 1.
       - That is, if $x \sim y$, then $(x + \alpha e_1) \sim (y + \alpha e_1)$, where $e_1 = (1,0, \dots, 0)$.
    2) Good 1 is desirable.
       - That is, $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Properties of Preference Relations

- Quasilinear preference-I

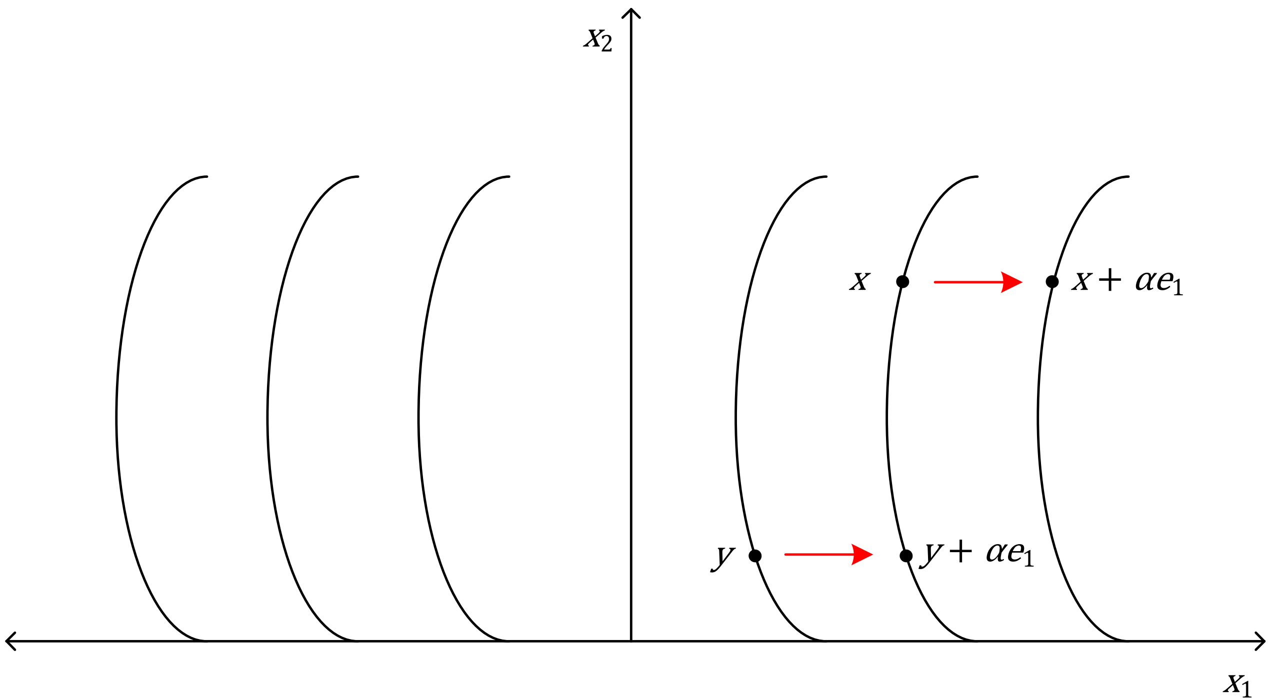


Properties of Preference Relations

- *Notes*:
  - No lower bound on the consumption of good 1, i.e., $x_1 \in (-\infty, \infty)$.
  - If $x \succ y$, then $(x + \alpha e_1) \succ (y + \alpha e_1)$.

Properties of Preference Relations

- Quasilinear preference-II



Properties of Preference Relations

- ***Example 1.9*** (Testing for quasiconcavity and homotheticity):
  - Let us determine if $u(x_1, x_2) = \ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave, homothetic, both or neither.
  - *Quasiconcavity*:
    - Note that $\ln(x_1^{0.3} x_2^{0.6})$ is a monotonic transformation of the Cobb-Douglas function $x_1^{0.3} x_2^{0.6}$.
    - Since $x_1^{0.3} x_2^{0.6}$ is a Cobb-Douglas function, where $\alpha + \beta = 0.3 + 0.6 < 1$, it must be a concave function.
    - Hence, $x_1^{0.3} x_2^{0.6}$ is also quasiconcave, which implies $\ln(x_1^{0.3} x_2^{0.6})$ is quasiconcave (as quasiconcavity is preserved through a monotonic transformation).

EXAMPLE 1.9

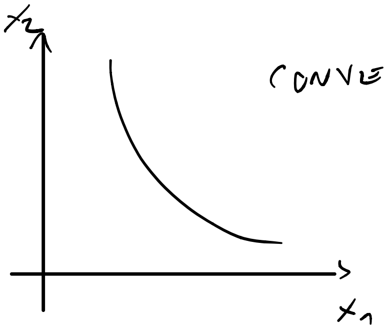
$$u(x_1, x_2) = \ln\left(x_1^{0.3} x_2^{0.6}\right)$$

IS QUASI CONCAVE, HOMOTETIC, BOTH, NEITHER

QUASI CONCAVITY

IND. CURVE ARE CONVEX $\Rightarrow$ UTILITY FUNCTION IS QUASI CONCAVE

$$\ln\left(x_1^{0.3}, x_2^{0.6}\right) = k$$



CONVEX ?

1. ISOLATE VARIABLE

$$\left(x_1^{0.3} x_2^{0.6}\right) = \underbrace{\exp(k)}_{a}$$

$$x_2^{0.6} = \exp \frac{a}{x_1^{0.3}}$$

$$x_2 = \left(\frac{a}{x_1^{0.3}}\right)^{\frac{1}{0.6}} > 1 \qquad \frac{1}{x_1} = x_1^{-1}$$

$$a^{1/0.6} \cdot x_1^{-0.3 \cdot \frac{1}{0.6}} = a^{1/0.6} \cdot x_1^{-1/2}$$

PROVE IT'S CONVEX $\longrightarrow$ 2° DERIVATIVE !

$$\frac{\partial x_2}{\partial x_1} = a^{1/0.6}\left(-\frac{1}{2}\right) x_1^{-3/2}$$

$$\frac{\partial^2 x_2}{\partial^2 x_1} = \left(-\frac{1}{2} a^{1/0.6}\right) -\frac{3}{2} \cdot x_1^{-\frac{5}{2}}$$

SIGN POSITIVE TO BE CONVEX

$$\left(-\frac{1}{2} a^{1/0.6}\right) -\frac{3}{2} \cdot x_1^{-\frac{5}{2}} > 0$$ SO IT'S CONVEX

CONVEX $\Rightarrow$ QUASI CONCAVE $u(x_1, x_2)$

HOMOTHETICITY

STRICTLY INCREASING ~~MONOTONIC~~ TRANSFORMATION OF AN HOMOGENEOUS FUNCTION (OF DEGREE K)

MONOTONIC

$\ln(x_1^{0.3}, x_2^{0.6})$ COB-DOUGLAS WITH LOG TRANSFORMATION

LINEAR HOMOGENEITY

COB-DOUGLAS $\rightarrow x_1^a x_2^b$

$$u(a x_1, a x_2) = a^k u(x_1, x_2) \Big\}$$

$a > 1$

IF WE CAN WRITE THIS, THEN IT'S HOMOGENEOUS OF DEGREE K

t BECAUSE α AS EXPONENT

$$w(t x_1, t x_2) = t^k w(x_1, x_2)$$

$t > 1$

$$w(t x_1, t x_2) = (t x_1)^\alpha (t x_2)^\beta = t^{\alpha+\beta}(x_1^\alpha x_2^\beta)$$

$$= t^{\alpha+\beta} u(x_1, x_2)$$

LN IS INCREASING

u IS HOMOTHETIC BECAUSE IS A LOG TRANSFORMATION OF A COB-DOUGLAS WHICH IS HOMOGENEOUS

OF DEGREE 0.5

1. 2.

CONVEXITY OF PREF $\neq$ CONVEXITY UTILITY

1. PREF CONVEX IF YOU TAKE UCS CONVEX

2. LINEAR QUASI CONCAVE

Social preferences

u(x1, x2) is utility function of an individual. Is not indexed by individual i.

Social and Reference-Dependent Preferences

Social Preferences

- We now examine social, as opposed to individual, preferences.
- Consider additively separable utility functions of the form
$$u_i(x_i, x) = f(x_i) + g_i(x)$$
where
  - $f(x_i)$ captures individual i's utility from the monetary amount that he receives, x_i;
  - $g_i(x)$ measures the utility/disutility he derives from the distribution of payoffs $x = (x_1, x_2, \ldots, x_N)$ among all N individuals.

This is a case in which is indexed by individual. Utility of individual is define by his consumption Xi but also the consumption of all other people. So f(xi) is the egoistic part, and gi(x) is the consumption of all other people. Gi mean that can be some sort of altruism.

In this example we don't take average consumption. In x we have all bundle of consumption of all individual (kindy absurd to have all consumption so we have average). X is a vector of consumption of all the other individual. Xi could be a vector and also x2, x3 ... could be a vector.
Usually we will take much simpler utility function.

Social Preferences

- **Fehr and Schmidt (1999)**:
  - For the case of two players,

ENVY ALTRUISM

$$u_i(x_i, x_j) = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\}$$

where x_i is player i's payoff and $j \neq i$.

  - Parameter α_i represents player i's disutility from envy
    - When $x_i < x_j$, $\max\{x_j - x_i, 0\} = x_j - x_i > 0$ but $\max\{x_i - x_j, 0\} = 0$.
    - Hence, $u_i(x_i, x_j) = x_i - \alpha_i(x_j - x_i)$.

Fehr and Schmidt we assume we have only two individuals, so we have only two consumption of the two individual. We also have consumption of j.
Xi is your consumption and the from this level of utility we subtract something: ai max(xk-xi, 0) if i consume less than xj i get a max of 0. Else if you consuming more than the other guy you take in the utility function Bi (xi-xj). So which between the two are altruistic consort. If you consume more Than the other guys you are not happy. a is for envy.

In this model we assume that player envy is stronger than their guilt. So alpha >= bi. You don't want to be the poor one.

Social Preferences

– Parameter $\beta_i \geq 0$ captures player i's disutility from guilt
  - When $x_i > x_j$, $\max\{x_i - x_j, 0\} = x_i - x_j > 0$ but $\max\{x_j - x_i, 0\} = 0$.
  - Hence, $u_i(x_i, x_j) = x_i - \beta_i(x_i - x_j)$.

– Players' envy is stronger than their guilt, i.e., $\alpha_i \geq \beta_i$ for $0 \leq \beta_i < 1$.
  - Intuitively, players (weakly) suffer more from inequality directed at them than inequality directed at others.

Social Preferences

– Thus players exhibit “concerns for fairness” (or “social preferences”) in the distribution of payoffs.

– If $\alpha_i = \beta_i = 0$ for every player i, individuals only care about their material payoff $u_i(x_i, x_j) = x_i$.

  - Preferences coincide with the individual preferences.

Social Preferences

- Let's depict the indifference curves of this utility function.
- Fix the utility level at $u = \overline{u}$. Solving for x_j yields

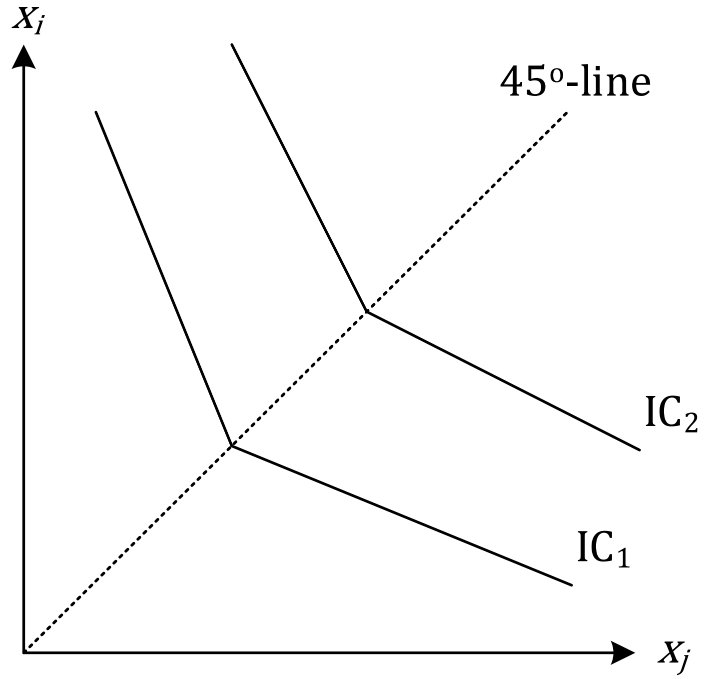
$$x_j = \frac{\overline{u}}{\beta} - \frac{1-\beta}{\beta} x_i \text{ if } x_i > x_j$$

$$x_j = \frac{\overline{u}}{\alpha} - \frac{1-\alpha}{\alpha} x_i \text{if } x_i < x_j$$

- Hence each indifference curve has two segments:
  - one with slope $\frac{1-\beta}{\beta}$above the 45-degree line
  - another with slope $\frac{1-\alpha}{\alpha}$below 45-degree line
- Note that (x_i, x_j)-pairs to the northeast yield larger utility levels for individual i.

Social Preferences

- Fehr and Schmidt's (1999) preferences



Advanced Microeconomic Theory

Chapter 2: Demand Theory

Consumption Sets

Consumption Sets

- ***Consumption set***: a subset of the commodity space $\mathbb{R}^L$, denoted by $x \subset \mathbb{R}^L$, whose elements are the consumption bundles that the individual can conceivably consume, given the physical constraints imposed by his environment.
- Let us denote a commodity bundle x as a vector of L components.

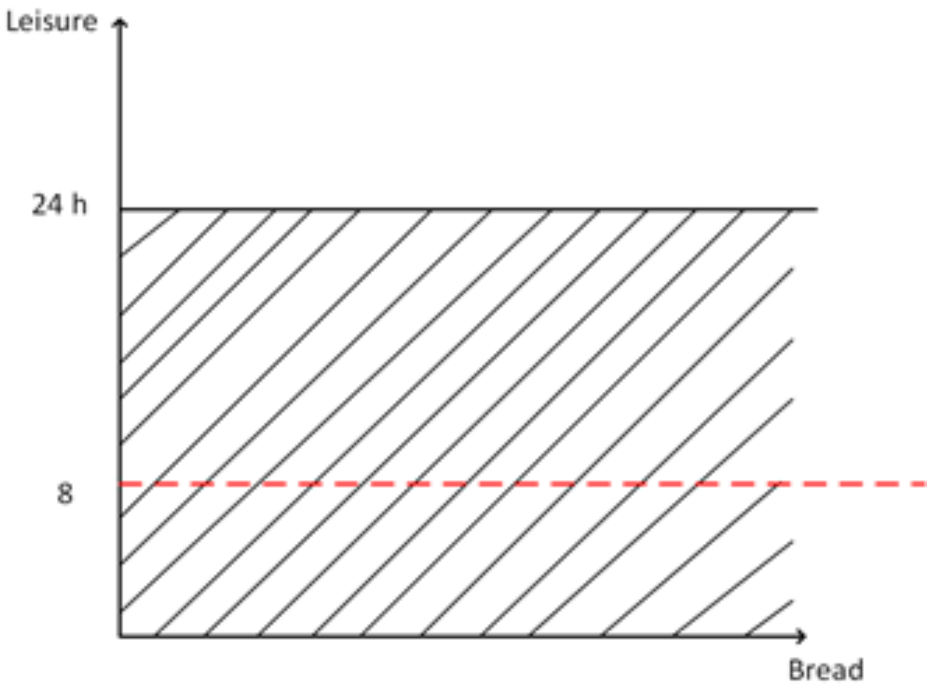
Consumption Set

Set of all possible alternatives (which are bundles)

sometime some bundle are not feasible, so we cannot consume it because there are constrained imposed by his environment.
A bundle is a vector of L components.

Consumption Sets

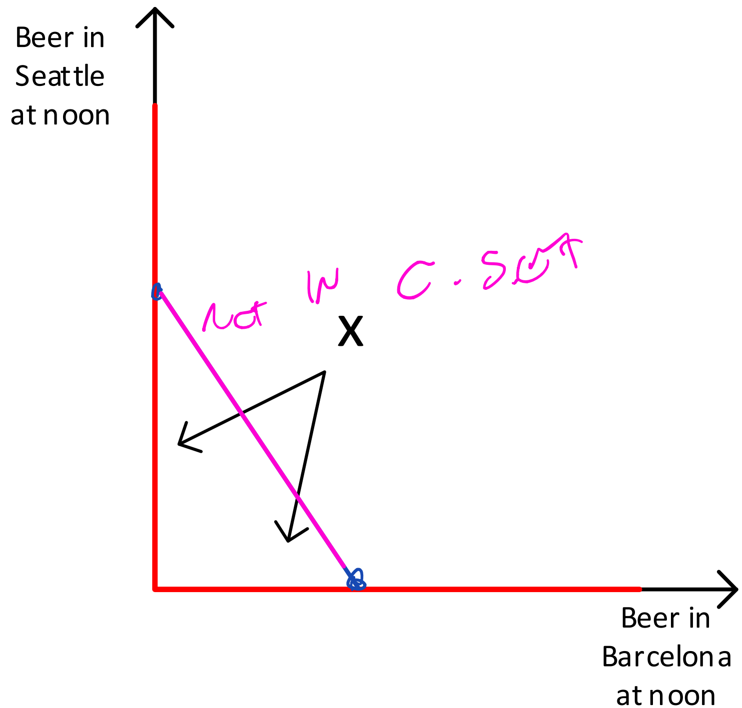
- Physical constraint on the labor market



How people decide labour supply (so how many hours they work). Leisure can also be called as house work. If we consider leisure as a good and bread. People don't want to work all day but you want to have some leisure. You have to sleep, what are the main free activity. Studying working and having fun. Even if you don't sleep any hour you do not consume any bread, the maximum amount of leisure is 24h. It's physical constraint on the environment.
If you have pleasure you don't work, if you work you have more income and more pleasure.

Consumption Sets

- Consumption at two different locations



Imagine this two goods are beer in Seattle and Barcellona at the same day. So there’s a physical constraint. The consumption set is in the axes. Since Barcellona is 0 if I’m in Seattle and vice versa. Not convex, if i take point in a straight line they are not in the consumption set.

Consumption Sets

- ***Convex consumption sets***:
  - A consumption set X is convex if, for two consumption bundles $x, x' \in X$, the bundle
  $$x'' = \alpha x + (1 - \alpha)x'$$
  is also an element of X for any $\alpha \in (0,1)$.
  - Intuitively, a consumption set is convex if, for any two bundles that belong to the set, we can construct a straight line connecting them that lies completely within the set.

Consumption Sets: Economic Constraints

- Assumptions on the price vector in $\mathbb{R}^L$:
  1) All commodities can be traded in a market, at prices that are publicly observable.
     - This is the principle of completeness of markets
     - It discards the possibility that some goods cannot be traded, such as pollution.
  2) Prices are strictly positive for all L goods, i.e., $p \gg 0$ for every good k.
     - Some prices could be negative, such as pollution.

Economic constraint —> we do some additional assumption that characterise perfect competition. All commodities can be traded in a marker and all good has a price in the market. This is called a market completeness.
For instance, we do not consider pollution because cannot be traded. Even though expert create market with pollution.
[Let's say 100 firm, each one 100 and then sell certificate and trade the right to pollute. The reason to create a market is that if you have a cost to pollute. You sell the right to pollute.]

Also price is positive. If something is free i can ask for infinite amount of the good??

Consumption Sets: Economic Constraints

3) Price taking assumption: a consumer's demand for all L goods represents a small fraction of the total demand for the good.

Consumer cannot affect the price.
In some situation consumer can affect the price. Big enterprise in the retail distribution and you supply all shop and then you go to people working on agriculture if price is this, then i get it else i will go to another one.

Consumption Sets: Economic Constraints

- Bundle $x \in \mathbb{R}_+^L$ is affordable if
$$p_1 x_1 + p_2 x_2 + \cdots + p_L x_L \leq w$$
or, in vector notation, $p \cdot x \leq w$.

- Note that $p \cdot x$ is the total cost of buying bundle $x = (x_1, x_2, \ldots, x_L)$ at market prices $p = (p_1, p_2, \ldots, p_L)$, and w is the total wealth of the consumer.

- When $x \in \mathbb{R}_+^L$ then the set of feasible consumption bundles consists of the elements of the set:
$$B_{p,w} = \{x \in \mathbb{R}_+^L : p \cdot x \leq w\}$$

Consumer have some wealth and cannot spend more on this wealth. So consumer cannot borrow money to his consumption (???) [56.00]
Amount of goods that are consumed and income is endogenous (variables explained in the model).
Endogenous decision are about x1, x2 so the amount of consumed.
The budget inequality is saying that you expenditure must be less or equal than your income.
So this define the so called budget set.
B is a set qand then the budget set depend on price and wealth in which components are positive for which the product of price vector moltiply by good vector is less or equal of w (amount of wealth that you have, it's a scalar! Not a vector like p and x).

How is budget set represented? In the following way.

Consumption Sets: Economic Constraints

- ***Example for two goods***:

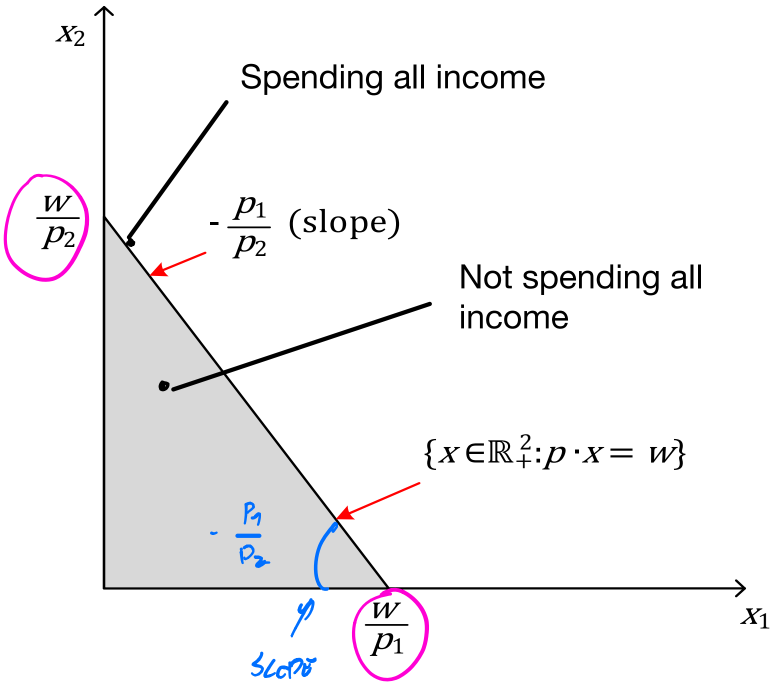
$$B_{p,w} = \{x \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \le w\}$$

The budget line is

$p_1 x_1 + p_2 x_2 = w$

Hence, solving for the good on the vertical axis, x_2, we obtain

$$x_2 = \frac{w}{p_2} - \frac{p_1}{p_2} x_1$$



Two components. How do you represent graphically a budget set?
You see you have inequality and you can take this inequality as equality and define the graph of the function of p1x1 +p2x2 = w.

$$\left.p_1x_1 + p_2x_2 = w \rightarrow x_2 = \frac{w}{p_2} - \frac{p_1}{p_2}x_1\right\} \text{BUDGET LINE}$$

IF $x_1 = 0$ YOU CAN CONSUME x_2 MAX AMOUNT OF $\frac{w}{p_2}$

ALSO $x_2 = 0$ x_1 MAX AMOUNT IS $\frac{w}{p_1}$

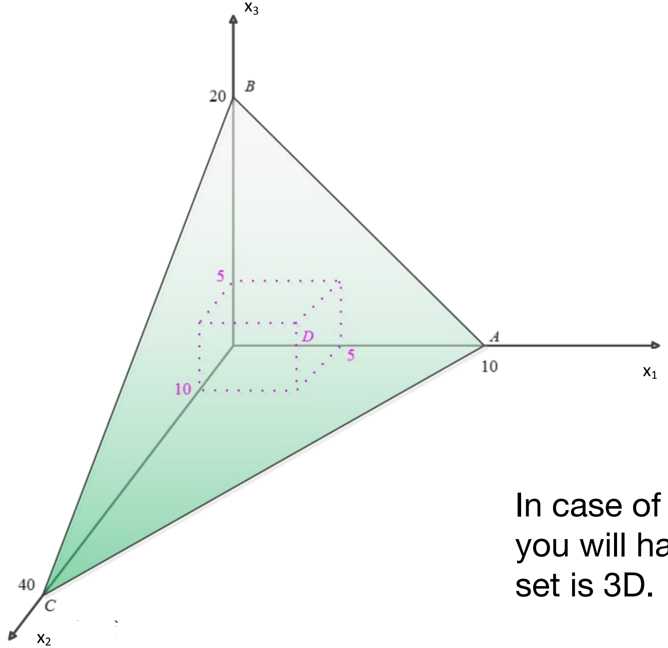
Set of all feasible bundle depending on your income and the price.

Consumption Sets: Economic Constraints

- ***Example for three goods***:

$$B_{p,w} = \{x \in \mathbb{R}^3_+ : p_1x_1 + p_2x_2 + p_3x_3 \leq w\}$$

  – The surface $p_1x_1 + p_2x_2 + p_3x_3 = w$ is referred to as the "Budget hyperplane"



In case of three goods you will have the budget set is 3D.

Consumption Sets: Economic Constraints

- *Price vector p is orthogonal (perpendicular) to the budget line $B_{p,w}$.*
  - Note that $p \cdot x = w$ holds for any bundle x on the budget line.
  - Take any other bundle x' which also lies on $B_{p,w}$. Hence, $p \cdot x' = w$.
  - Then,

$$p \cdot x' = p \cdot x = w$$

$$p \cdot (x' - x) = 0 \text{ or } p \cdot \Delta x = 0$$

Consumption Sets: Economic Constraints

– Since this is valid for any two bundles on the budget line, then p must be perpendicular to Δx on $B_{p,w}$.

– This implies that the price vector is perpendicular (orthogonal) to $B_{p,w}$.

Price vector is orthogonal to the budget line Bp,w.

$(p_1, p_2)(x_1, x_2) = p_1 x_1 + p_2 x_2$

Consumption Sets: Economic Constraints

- *The budget set $B_{p,w}$ is convex*.
  - We need that, for any two bundles $x, x' \in B_{p,w}$, their convex combination

$$x'' = \alpha x + (1-\alpha)x'$$

    also belongs to the $B_{p,w}$, where $\alpha \in (0,1)$.

CHECK

  - Since $p \cdot x \leq w$ and $p \cdot x' \leq w$, then

$$p \cdot x'' = p\alpha x + p(1-\alpha)x'$$
$$= \alpha p x + (1-\alpha)px' \leq w$$

why is $\leq w$?

$$a\,[\,px + (1-a)\,px'\,] \le w$$

<1 $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $<w$

EXERCISES 1.7

THIS IS ABOUT PREFERENCES

$$u(x_1, x_2) = a\,x_1^{1/2}\,b\,x_2^{1/2}$$

PREFERENCES CONVEX? UCS IS CONVEX?

1. DRAW FUNCTION OF I.C. $\rightarrow$ TAKE FUNCTION AND PUT IT = TO A K VALUE

$$a\,x_1^{1/2}\,b\,x_2^{1/2} = K \qquad x_2 = \left(\frac{K}{ab\,x_1^{1/2}}\right)^2$$

$$x_2 = \left(\frac{K}{ab}\right)^2 x_1^{-1} \quad \boxed{= A}$$

FUNCTION OF IC

TO CHECK IF IT IS CONVEX?

$$\frac{\partial x_2}{\partial x_1} = A \cdot -x_1^{-2}$$

$$\frac{\partial^2 x_2}{\partial x_1\,\partial x_1} = A\,(2\,x_1^{-3}) > 0 \quad \text{IF } x_1 > 0$$

I.C. CONVEX $\Rightarrow$ UTILITY IS QUASI CONCAVE

PREF CURVE ARE CONVEX

$ax_1^2 + b\, x_2^2$

$$u(x_1, x_2) = a x_1^2 + a\, x_2^2$$

CONVEX PREFERENCES ?

$$a x_1^2 + b x_2^2 = K \qquad x_2 = \left[\frac{(K - a x_1^2)}{b}\right]^{1/2} = A^{1/2}$$

$$\frac{\partial x_2}{\partial x_1} = \frac{1}{2} A^{-1/2} \cdot -\left(\frac{a}{b}\right) \cdot 2x_1 = -\frac{a}{b} A^{-1/2} \cdot x_1$$

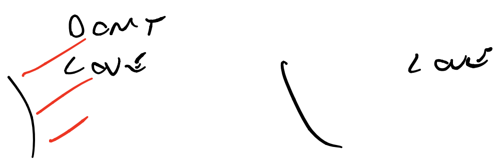
DUE TO PRODUCT!

$$\frac{\partial^2 x_2}{\partial x_1 \partial x_2} = -\frac{a}{b} \cdot \left[-\frac{1}{2} A^{-3/2}\left(-\frac{a}{b}\right) 2x_1 \cdot x_1 + A^{-1/2}\right] < 0$$

IC. CONCAVE! $\Rightarrow$ PREF ARE NOT CONVEX

$\Downarrow$ UTILITY IS NOT QUASI CONCAVE

$$d(xy) = dx \cdot y - dy \cdot x$$



CHECKING PROPERTIES OF PREFERENCE RELATIONS

a) $(x_1, x_2) \succsim (y_1, y_2) \iff x_1 \geq y_1 - 1$

$\boxed{\text{COMPLETENESS}}$ $\rightarrow$ $\forall x, y \in X$ either $x \succsim y$

or $y \succsim x$

or BOTH $x \succsim y \wedge y \succsim x \Rightarrow x \sim y$

$x_1, x_2, y_1, y_2 \in \mathbb{R}$

COMPLETENESS IS RESPECTED

$\rightarrow x_1 \geq y_1 - 1 \Rightarrow x \succsim y$

$\rightarrow x_1 < y_1 - 1 \Rightarrow y_1 \geq x_1 - 1 \Rightarrow y \succsim x$

$\rightarrow x_1 = y_1 - 1 \Rightarrow x \succsim y \wedge y \succsim x \Rightarrow x \sim y$

$\boxed{\text{TRANSITIVITY}}$ $\forall x, y \in X;\ x \succsim y \wedge y \succsim z \Rightarrow x \succsim z$

SUBSTITUTE OUR CONDITION ($x_1 \geq y_1 - 1$) IN THIS PROPERTIES

$x_1 \geq y_1 - 1$ $\quad$ $y_1 \geq z_1 - 1$ $\quad$ $x_1 \geq z_1 - 1$

$\downarrow$ $\quad$ $\downarrow$ $\quad$ $\downarrow$

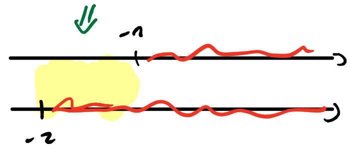
$\underline{x_1 - y_1 \geq -1}$ $\quad$ $\underline{y_1 - z_1 \geq -1}$ $\quad$ $|\, x_1 - z_1 \geq -1 \ ?\, |$

SUMMING THIS TWO

$|\, x_1 - \cancel{y_1} + \cancel{y_1} - z_1 \geq -2 \,|$

THIS ARE DIFFERENT!

IN THIS REGION, TRANSITIVITY DOES NOT HOLD



$(-2, -1) \rightarrow$ THIS PROPERTY DOES NOT HOLD

MONOTONICITY

INCREASE OF ONE GOOD BY α AMOUNT DOES NOT HURT OUR PREFERENCE

$x \succeq y \iff x_1 \geq y_1 - 1 \longrightarrow x_1 + \alpha \geq x_1 - 1$ TRUE FOR $\alpha \geq -1$

GET VALUE OF $x - 1$ TO GET VALUE OF y

BY ASSUMPTION α HAS TO BE GREATER THAN $\alpha > 0$

CONVEXITY

IF $x \succeq y \Rightarrow \alpha x + (1-\alpha) y \succeq y$

WE HAVE TO SUBSTITUTE x AND y BY INITIAL ASSUMPTION

$(x_1 \geq y_1 - 1)$

$\alpha x_1 + (1 - \alpha) y_1 \geq y_1 - 1$

I WANT THIS TO BE TRUE

$\alpha(x_1 - y_1) + \cancel{y_1} \geq \cancel{y_1} - 1 \Rightarrow x_1 - y_1 \geq -\frac{1}{\alpha}$

I USE α TO GET A CONVEX COMBINATION (SO $0 < \alpha < 1$)

SINCE $\alpha \in (0,1) \Rightarrow -\frac{1}{\alpha} < -1$

$x_1 - y_1 \geq -1$

THEREFORE $(x_1 - y_1) + 1 \geq 0$ SO IT'S A SUFFICIENT CONDITION FOR

$\alpha(x_1 - y_1) + 1 \geq 0$

$\Downarrow$

THIS PREF. RELATION IS CONVEX

ALWAYS TRUE THANKS TO ASSUMPTION

$x_1 - y_1 \geq -\frac{1}{\alpha} \Longleftarrow x_1 - y_1 \geq -1$

$\boxed{\text{CH 1 - EX 5}}$

$u(x)$ WHERE $x \in \mathbb{R}^n_+$ WITH N COMPONENTS

$v(x) = f(u(x))$ WHERE $f(\cdot)$ IS STRICTLY INCREASING AND CONCAVE

$v(x)$ MUST TO BE CONVEX

PROPERTY OF CONVEXITY

CONVEX PREFERENCES (DEF.)

$$x_1 \succsim y \wedge x_2 \succsim y \Rightarrow \bar{x} = a x_1 + (1-a) x_2 \succsim y$$

I CAN USE UTILITY FUNCTION

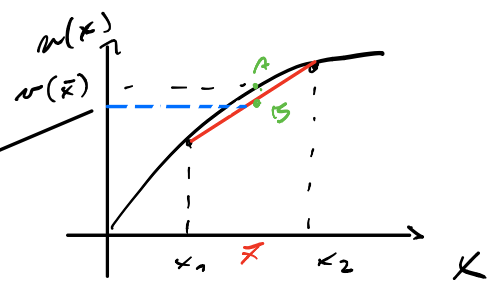
$$u(x_1) \geq u(y) \wedge u(x_2) \geq u(y) \Rightarrow u(\bar{x}) \geq u(y)$$

PROOF

IF $v(\cdot)$ IS CONCAVE (FROM INITIAL ASSUMPTION) $\Rightarrow$

$$v(\bar{x}) \geq a\, v(x_1) + (1-a)\, v(x_2)$$

UTILITY OF $\bar{x}$ $\longrightarrow$ $v(\bar{x})$



$a\, v(x_1) + (1-a)\, v(x_2)$

USING $v(x) = f(u(x))$ WE CAN WRITE

$$f(u(\bar{x})) \geq a\, f(u(x_1)) + (1-a)\, f(u(x_2))$$

$a\, f(u(x_1)) \geq f(u(y))$ $\qquad$ $a\, f(u(x_2)) \geq f(u(y))$

$$f(u(\bar{x})) \geq \cancel{\alpha f(u(y))} + (1-\cancel{\alpha})\, f(u(y))$$

$$f(u(\bar{x})) \geq f(u(y))$$

$f \quad u(x) \geq u(y) \Rightarrow \bar{x} \geq y$ CONVEXITY!

$\boxed{\text{EX 8 – MONOTONIC TRANSFORMATION}}$

WE WANT TO SEE IF THIS TRANSFORMATION PRESERVE THE SLOPE OF THE FUNCTION

$u(x) \geq 0 \quad \forall x \in \mathbb{R}^2_+$

(a) $f(x) = a\, u(x) + b\,[u(x)]^2$ WHERE $a, b > 0$

$u(x) = K$ TO MAKE DERIVATIVE EASIER

$$f(K) = aK + bK^2 \qquad \frac{\delta f(K)}{\delta K} = a + 2bK$$

UTILITY ≥ 0 BY ASSUMPTION

SO INCREASING TRANSFORMATION

THEREFORE $f(x)$ REPRESENT THE PREFERENCES OF THE ORIGINAL TRANSFORMATION $u(x)$

b) $f(x) = a \cdot u(x) - b\,[u(x)]^2$ WITH $a, b \geq 0$

$$f(z) = az - bz^2$$

$$\underline{\delta f(z)} = a - 2bz \geq 0 \rightarrow z \leq \frac{a}{2b} \Rightarrow u(x) \leq \frac{a}{2b}$$

∂z

$f(x)$ NOT! MONOTONIC TRANSFORMATION OF $u(x)$

NOT REPRESENT SAME PREFERENCES AS ORIGINAL FUNCTION
UTILITY $u(x)$

c) $f(x) = u(x) + \sum_{i=1}^{2} x_i$ $\quad u(x) \geq u(y) \Rightarrow f(x) < f(y)$

SO NOT MONOTONIC TRANSF.

ASSUME $x \geq y \iff x_1 \geq y_1$ $\quad (x_1, x_2) \geq (y_1, y_2)$

$(1, 2) \geq (0, 5)$

$u(x_1) \geq u(y_1) \Rightarrow u(1) > u(0)$

IF YOU TRY TO OBTAIN UTILITY OF TRANSFORMATION

$f(1,2) = \underbrace{1}_{u(x)=x} + 1 + 2 = 4 \quad f(0, 5) = 0 + 0 + 5 = 5$

$f(1,2) < f(0,5) \Rightarrow$ THIS TRANSFORMATION IS NOT MONOTONIC SINCE DOES NOT REPRESENT SAME PREFERENCE AS $u(x)$

d) $f(x) = [u(x)]^2 + b\,u(x) + c$ WITH $b, c > 0$

$u(x) = z$

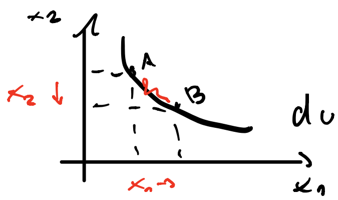
$\frac{\partial f(z)}{\partial z} = 2z + b \Rightarrow$ ALWAYS > 0 IF $z > 0 \rightarrow u(x) \geq 0 \in \mathbb{R}^+$

EX 1C – ADDITIVELY SEPARABLE UTILITY

(b) $u(x_1, x_2) : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}$ WHERE $u(x_1, x_2) = u_1(x_1) + u_2(x_2)$

$u(x_1)$ AND $u(x_2)$ STRICTLY INCREASING, STRICTLY CONCAVE, DIFFERENTIABLE

SHOW THAT IC ARE CONVEX WHERE MU_i FOR $i = 1, 2$ IS DECREASING



$du = 0$

A, B GENERATE SAME UTILITY → A ~ B

ALONG AND IC → BY DEFINITION

PROOF

$$du = \underbrace{\frac{\partial u_1(x_1)}{\partial x_1}}_{u_1'(x_1)} dx_1 + \underbrace{\frac{\partial u_2(x_2)}{\partial x_2}}_{u_2'(x_2)} dx_2 = 0$$

$$u_1'(x_1)\,dx_1 + u_2'(x_2)\,dx_2 = 0$$

$$-\frac{dx_2}{dx_1} = \frac{u_1'(x_1)}{u_2'(x_2)} \equiv |MRS|$$ → THIS IS EXPRESSION IS MRS

∂ MRS AND FIND DECR THEN WE CAN SAY WE HAVE CONVEX CURVE

$$\frac{\partial |MRS|}{\partial x_1} = \frac{\overbrace{u_1''(x_1)}^{\text{NEGATIVE}}}{\underbrace{u_2'(x_2)}_{\text{POSITIVE}}} < 0$$

SO IT'S NEGATIVE AND MRS IS DECREASING

$\Downarrow$

WE HAVE A CONVEX CURVE

ALSO YOU CAN USE HESSIAN MATRIX

$\boxed{\text{EX 14 - CHECKING PREFERENCE OF COB - DOUGLAS FUNCTION}}$

$$u(x) = \prod_{i=1}^{n} x_i^{\alpha_i} \qquad x \in \mathbb{R}^n_+ \quad \text{AND} \quad 1 > \alpha_i > 0$$

ADDITIVITY, HOM OF DEGREE K AND, HOMOTHETIC

ADDITIVITY ⟶ MARGINAL UTILITY OF GOOD x_i ONLY DEPENDS ON GOOD x_i

1. TAKE DERIVATIVE $u(x)$ WITH RESPECT TO A GOOD K

IF INCR UTILITY AND DEP ONLY ON GOOD K THEN UTILITY ADDITIVITY HOLDS

$$\frac{\partial u(x)}{\partial x_k} = \underbrace{\frac{\alpha_k}{x_k}}_{\text{POSITIVE}} \cdot \underbrace{\prod_{i=1}^{m} x_i^{\alpha_i}}_{\substack{u(x) \\ \text{POSITIVE}}} > 0$$

NOT ONLY CONSUMPTION OF x_k BUT ALSO FOR ALL OTHER GOODS

ONLY TWO GOODS

$$i = 1,2 \Rightarrow u(x_1, x_2) = x_1^{\alpha_1} \cdot x_2^{\alpha_2}$$

$$\frac{\partial u}{\partial x_1} = \alpha_1 \cdot x_1^{\alpha_1 - 1} \cdot x_2^{\alpha_2} = \frac{\alpha_1}{x_1} \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} = \frac{\alpha_i}{x_i} u(x_1, x_2)$$

ADDITIVITY MEANS CONSUMPTION x_k DEP ONLY ON x_k IN THIS CASE DEPENDS ON CONSUMPTION OF ALL THE OTHER GOOD.

$$u(x) = x_1^2 + 2x_2 \qquad \frac{\partial u_1(x)}{\partial x_1} = 2x_1$$

HOMOGENEITY

$$u(tx) = \prod_{i=1}^{n} (t x_i)^{a_i} = \prod_{i=1}^{n} t^{a_i} \cdot x^{a_i} = t^{\sum_{i=1}^{n} a_i} \cdot \prod_{i=1}^{n} x_i^{a_i} =$$

$$= t^{\sum a_i} \cdot u(x)$$

HOMOGENEITY HOLDS $\to$ DEGREE $\sum a_i$

$$i = 1,2 \to \sum_{i=1}^{n} a_i = a_1 + a_2 \begin{cases} > 1 & \text{MORE THAN PROPORTIONAL} \\ = 1 & \text{PROPORTIONAL} \\ < 1 & \text{LESS THAN PROPORTIONAL} \end{cases}$$

HOMOTHETIC $\to$ IS ALWAYS IMPLIED IN HOMOGENEITY

to CHECK IT BY USING MRS

$$|MRS| = \frac{\frac{\partial u(x)}{\partial x_\ell}}{\frac{\partial u(x)}{\partial x_k}} = \frac{\frac{a_i}{x_k} \cdot \prod x_i^{a_i}}{\frac{a_k}{x_k} \cdot \prod x_i^{a_i}} = \frac{a_\ell}{a_k} \cdot \frac{x_k}{x_\ell}$$

$$|MRS'| = \frac{t^{\sum a_i}}{t^{\sum a_\ell}} \cdot \frac{a_\ell}{a_k} \cdot \frac{x_k}{x_\ell} = |MRS|$$

Advanced Microeconomic Theory

Chapter 2: Utility Maximization Problem (UMP), Walrasian demand, indirect utility function

Outline

- Utility maximization problem (UMP)
- Walrasian demand and indirect utility function
- WARP and Walrasian demand (no, skip)
- Income and substitution effects (Slutsky equation)
- Duality between UMP and expenditure minimization problem (EMP)
- Hicksian demand and expenditure function
- Connections

Utility Maximization Problem

Utility Maximization Problem

- Consumer maximizes his utility level by selecting a bundle x (where x can be a vector) subject to his budget constraint:

$$\max_{x \geq 0} u(x)$$

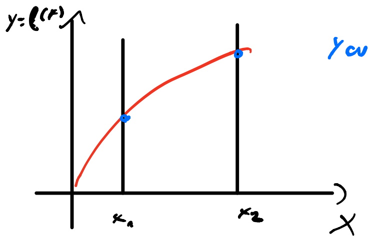
$$\text{s.t.} \quad p \cdot x \leq w$$

BUDGET CONSTRAINT

- ***Weierstrass Theorem***: for optimization problems defined on the reals, if the objective function is continuous and constraints define a closed and bounded set, then the solution to such optimization problem exists.

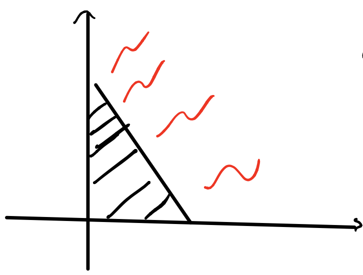
Vector is the quantity of goods. Max u(x) is a vector. Quantity must be positive. This is a constraint that we see last time.

P1x1 +p2 x2 ... is what you spend for good one and w is the total wealth.



You can always find the maximum

this area contains the boundary

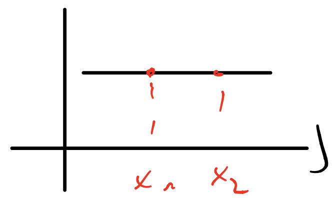


Closed & bounded set

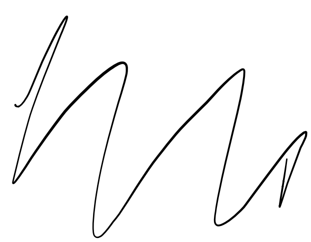
Utility Maximization Problem

- ***Existence***: if $p \gg 0$ and $w > 0$ (i.e., if $B_{p,w}$ is closed and bounded), and if $u(\cdot)$ is continuous, then there exists at least one solution to the UMP.
  - If, in addition, preferences are strictly convex, then the solution to the UMP is <u>unique</u>. only one x maximize the function
- We denote the solution of the UMP as the **argmax** of the UMP (the argument, x, that solves the optimization problem), and we denote it as $x(p, w)$. $x^* = \{x_1^*, x_2^*, \ldots\}$
  - $x(p, w)$ is the ***Walrasian demand*** correspondence, which specifies a demand of every good in $\mathbb{R}_+^L$ for every possible price vector, p, and every possible wealth level, w.

We can show that solution is unique if preferences are strictly convex and u(°) continuous.

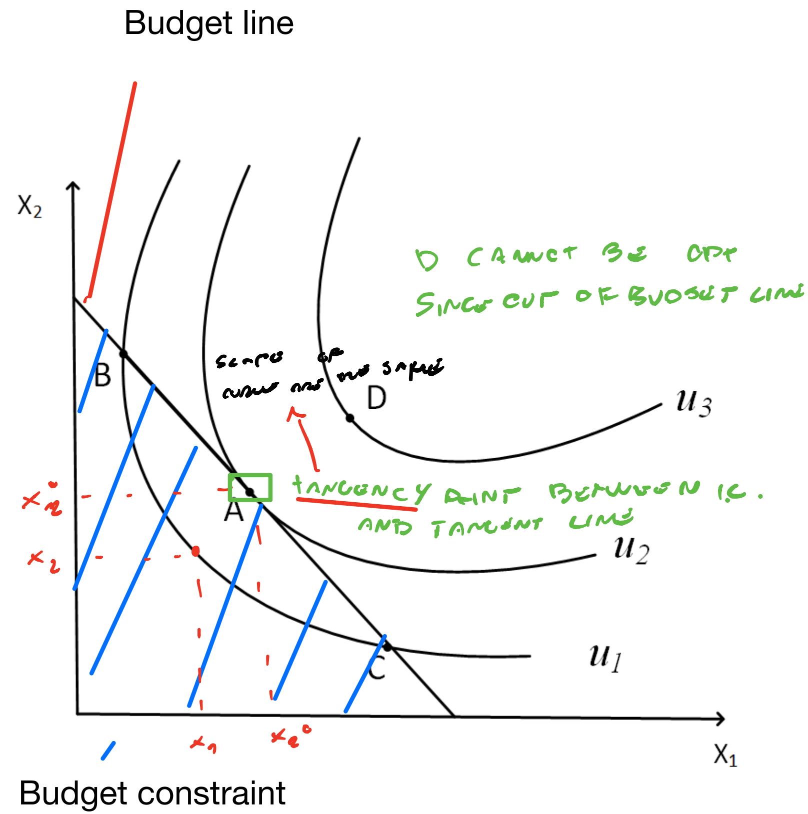


Depends on prices and wealth!
So it’s why opt solution depend on w and p.



Utility Maximization Problem

- Walrasian demand $x(p, w)$ at bundle A is optimal, as the consumer reaches a utility level of u_2 by exhausting all his wealth.
- Bundles B and C are not optimal, despite exhausting the consumer's wealth. They yield a lower utility level u_1, where $u_1 < u_2$.
- Bundle D is unaffordable and, hence, it cannot be the argmax of the UMP given a wealth level of w.



$x^* = (x_1^*, x_2^*)$

ARG MAX

x° PREFERABLE TO x

MONOTONE MORE IS BETTER!

IN A, WE WILL HAVE MRS $= -\frac{p_1}{p_2}$

ARGMAX IS A

Properties of Walrasian Demand

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}^L_+$, then the Walrasian demand $x(p, w)$ satisfies:

1) Homogeneity of degree zero:

$$x(p, w) = x(\alpha p, \alpha w) \text{ for all } p, w, \text{ and for all } \alpha > \cancel{0}\ 1$$

That is, the budget set is unchanged!

$$\{x \in \mathbb{R}^L_+ \colon\ p \cdot x \leq w\} = \{x \in \mathbb{R}^L_+ \colon\ \alpha p \cdot x \leq \alpha w\}$$

Note that we don't need any assumption on the preference relation to show this. We only rely on the budget set being affected.

We will assume this properties for any problem of utility maximisation problem.

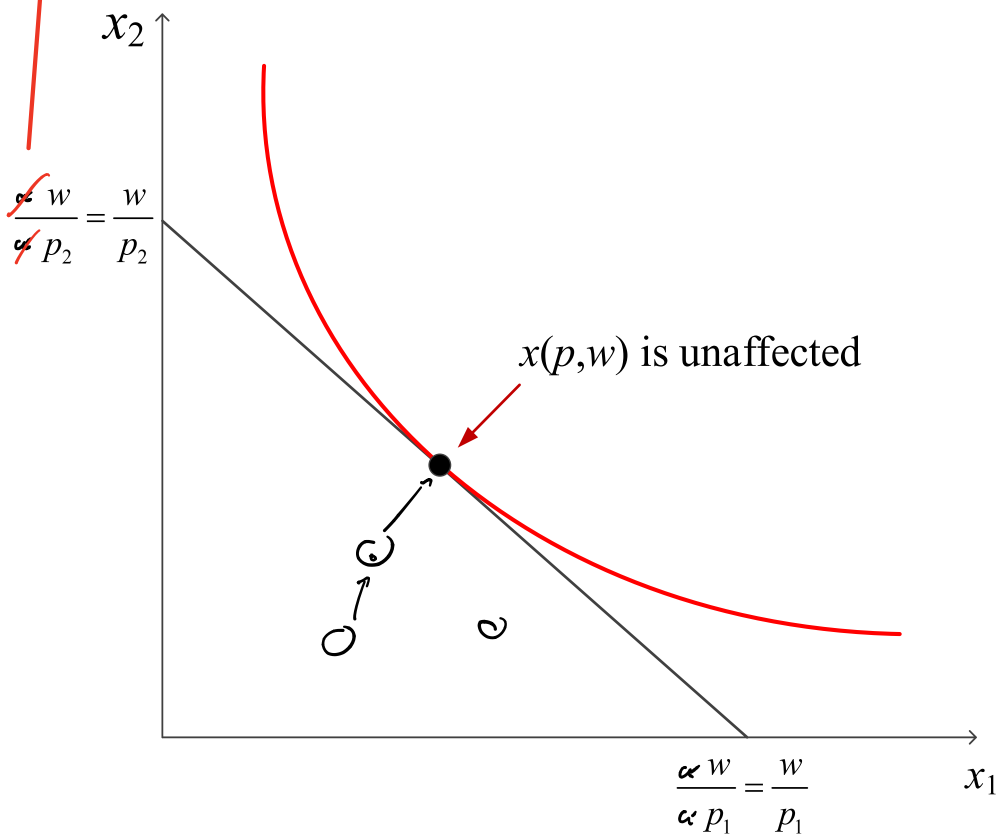
1. Homogeneity —> moltiply by alpha doesn't change the value of the function.

Why increasing prices and wealth by same alpha we obtain a solution that is the same also for the MUP? Is easy to demonstrate with the graphical solution before.

If we increase everything by alpha.

If i multiply for alpha i obtain the same solution.

Properties of Walrasian Demand



– Note that the preference relation can be linear, and homog(0) would still hold.

x^* is arg max

$u(x^*)$ MAX VALUE OF UTILITY

(COMPUTED UTILITY IN ARG MAX)

Properties of Walrasian Demand

2) ***Walras' Law:***

$$p \cdot x = w \quad \text{for all } x = x(p, w)$$

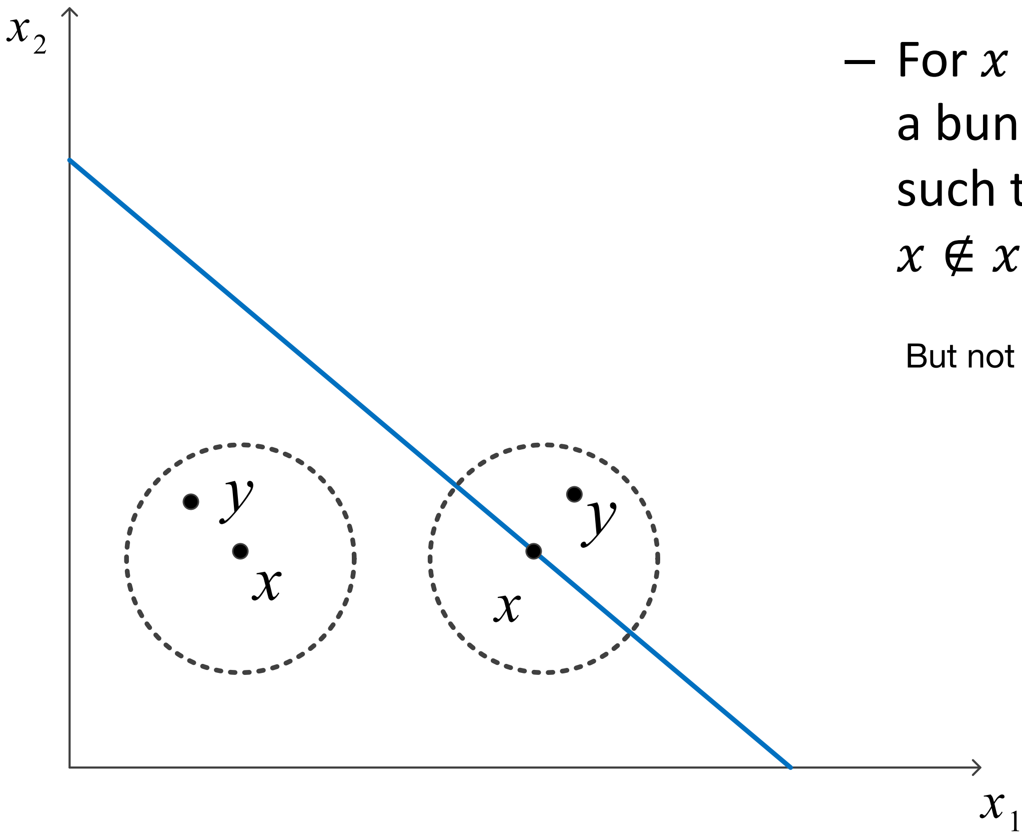
It follows from LNS: if the consumer selects a Walrasian demand $x \in x(p, w)$, where $p \cdot x < w$, then it means we can still find other bundle y, which is ε–close to x, where consumer can improve his utility level.

If the bundle the consumer chooses lies on the budget line, i.e., $p \cdot x' = w$, we could then identify bundles that are *strictly* preferred to x', but these bundles would be unaffordable to the consumer.

Walras' law. In the opt solution the consumer spends all income. Consume cannot remain with income not spent. It's irrational. In graphical term is intuitive because we must be in the budget line. In the opt solution you are in the tangency point and this define the walras law. In opt you don't have any unspent income. This depend on the fact that the utilty function satisfy LNS: you can find very close point that give you the same utility.

a) If Preferences are weakly convex then walrasian demand correspondence deifines a convex set.
b) if preference are strictly convex, then walrasian demand correspondence cointain a single element.

Properties of Walrasian Demand



– For $x \in x(p, w)$, there is a bundle y, ε–close to x, such that $y \succ x$. Then, $x \notin x(p, w)$.

But not affordable

Properties of Walrasian Demand

3) Convexity/Uniqueness:

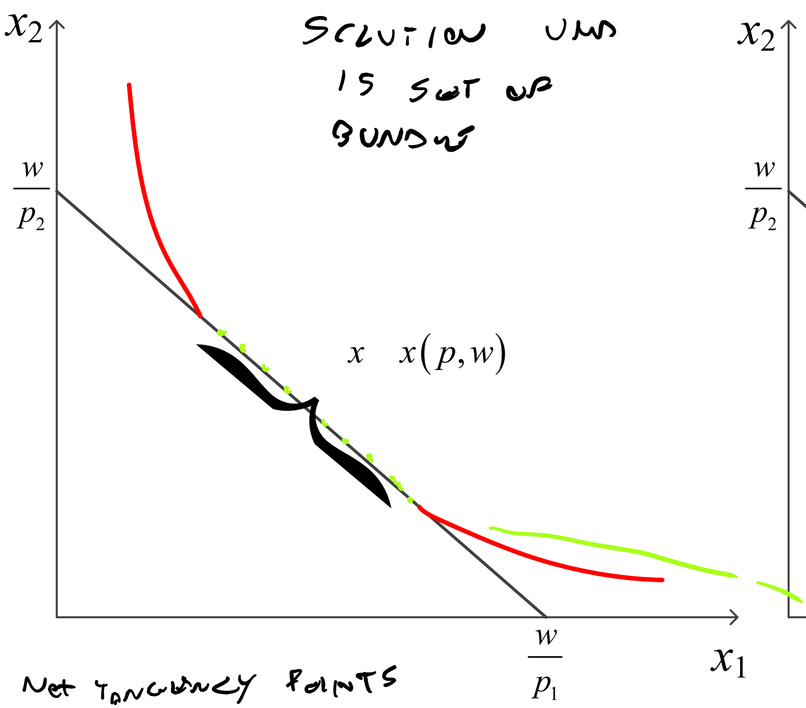
a) If the preferences are convex, then the Walrasian demand correspondence $x(p, w)$ defines a convex set, i.e., a continuum of bundles are utility maximizing. (For a given p and a given w)

b) If the preferences are strictly convex, then the Walrasian demand correspondence $x(p, w)$ contains a single element. (For a given p and a given w)

Properties of Walrasian Demand

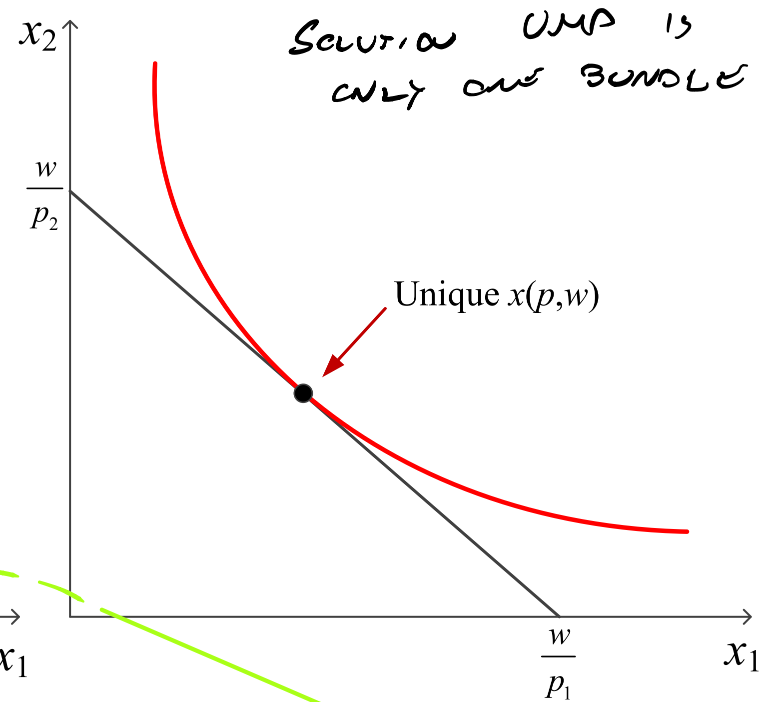
weak

Convex preferences



Not tangency points

Strictly convex preferences



I DON'T HAVE ONLY ONE BUNDLE BUT A SET OF BUNDLES

IN STRICTLY THIS HAVE A UNIQUE SOLUTION.

WE WILL SEE BOTH OF THE CASES.

NOW GO FOR ANALITICAL DEMONSTRATION → TANGENT

UMP: Necessary Condition

SOLUTION NON NEGATIVE

$$\max_{x \geq 0} u(x) \quad \text{s.t. } p \cdot x \leq w$$

- We solve it using Kuhn-Tucker conditions over the Lagrangian $L = u(x) + \lambda(w - p \cdot x)$,

KKT

COND

$$\frac{\partial L}{\partial x_k} = \frac{\partial u(x^*)}{\partial x_k} - \lambda p_k \leq 0 \text{ for all } k, \ = 0 \text{ if } x_k^* > 0$$

$u(x_1, x_2) + \lambda(w - p_1 x_1 - p_2 x_2) \geq 0$

$$\frac{\partial L}{\partial \lambda} = w - p \cdot x^* = 0$$

∂L

COMPLEMENTARY SLACKNESS

- That is, in a *interior* optimum, $\frac{\partial u(x^*)}{\partial x_k} = \lambda p_k$ for every good k, which implies

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k} \Leftrightarrow MRS_{l,k} = \frac{p_l}{p_k} \Leftrightarrow \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

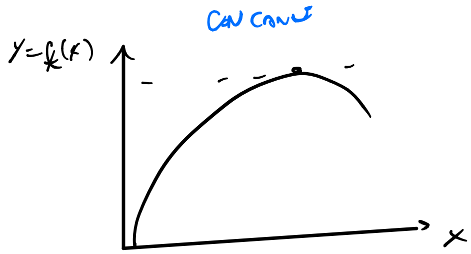
USE LAGRANGIAN INSTEAD MAX u(x) WE MAXIMISE LAGRANGIAN FUNCTION TO GET ANOTHER FUNCTION TO MAX

$$L = u(x) + \lambda(w - p \cdot x)$$ → LAGRANGIAN

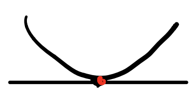
λ → LAGRANGIAN MULTIPLIER

ONE VARIABLE CASE

$L(x, \lambda)$



1. SLOPE CURVE MUST BE 0 (STATIONARY POINT
2. DER. 2 NEGATIVE IN STATIONARY POINT



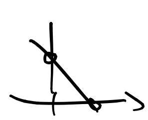
SLOPE IS MINIMUM

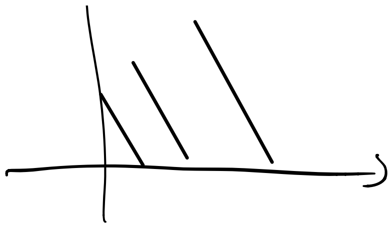
$$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$$ FOR INTERNAL SOLUTION

WANTS LOW → OPT CONSUMER CONSUME MORE INCOME

$$\frac{\partial L}{\partial \lambda} = 0 \qquad w - p \cdot x = 0 \rightarrow w = px$$

CORNER SOLUTION → INTERNAL SOLUTION OR CORNERS





IN CASE OF PERF SUB SOLUTION IN CORNERS

INSTEAD USE $=0$ WE USE KKT CONDITION

SEE ABOVE ↑

COMPLEMENTARY SLACKNESS

IF $x_k^* > 0 \Rightarrow \frac{\partial L}{\partial x_k} = 0$ ← WE WILL CONSIDER THIS CASE

IF $x_k^* = 0 \rightarrow \frac{\partial L}{\partial x_k} < 0$

1) $\frac{\partial u}{\partial x_1} - \lambda p_1 = 0$

2) $\frac{\partial u}{\partial x_2} - \lambda p_2 = 0$

MOVE TO RATIO! → $\frac{p_1}{p_2} = \frac{\cancel{\lambda}\frac{\partial u}{\partial x_1}}{\cancel{\lambda}\frac{\partial u}{\partial x_2}} = MRS$

LAMBDA GOES

MARGINAL UTILITIES

3) $w - px = 0$

LHS → MRS (SLOPE I.C.)

$\frac{p_1}{p_2}$ → SLOPE OF BUDGET CONSTRAINT

(IN REALITY IS THE OPPOSITE

I.C AND BUD CONSTRAIN NEGATIVELY SLOPED BUT YOU CAN MULTIPLY BY -1.

$-\frac{p_1}{p_2} \cdot -1 = \frac{p_1}{p_2}$

$$\frac{\frac{\partial u}{\partial x_1}}{p_1} = \frac{\frac{\partial u}{\partial x_2}}{p_2} \Rightarrow$$

INCOME 1€ HOW MANY UNIT OF GOOD 1?

$$\underline{\frac{1€}{p_1} \cdot \frac{\partial u}{\partial x_1}} = \frac{1€}{p_2} \cdot \frac{\partial u}{\partial x_2} \quad \text{FOR GOOD 2}$$

MU MUST BE THE SAME SPEND FOR THE SECOND GOOD

MOVING INCOME FROM 1 TO ANOTHER GOOD DOES NOT GET MORE OPTIMAL

UMP: Sufficient Condition

- When are Kuhn-Tucker (necessary) conditions, also sufficient?
  - That is, when can we guarantee that $x(p, w)$ is the $\max$ of the UMP and not the $\min$?

UMP: Sufficient Condition

- Interpretation of $\frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$

The marginal utility of the last dollar ("marginal" euro) spent in good l must produce the same utility of the last euro spent in good k. [Hint. With one dollar you buy $1/p_l$units of good l and $1/p_l$units of good k)

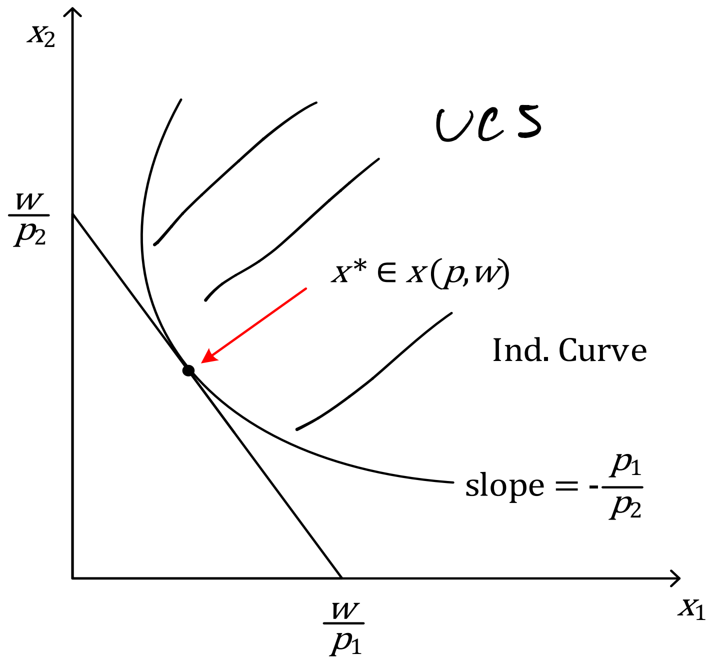
- When are Kuhn-Tucker (necessary) conditions, also sufficient?
  – That is, when can we guarantee that $x(p, w)$ is the max of the UMP and not the min?

UMP: Sufficient Condition

SECOND ORDER CONDITION

- Kuhn-Tucker conditions are sufficient for a max if:

  1) $u(x)$ is quasiconcave, i.e., convex upper contour set (UCS).
  2) $u(x)$ is monotone.
  3) $\nabla u(x) \neq 0$ for $x \in \mathbb{R}_{+}^{L}$.
     – If $\nabla u(x) = 0$ for some x, then we would be at the "top of the mountain" (i.e., blissing point), which violates both LNS and monotonicity.



UMP: Violations of Sufficient Condition

VIOLATION OF MONOTONICITY → BLISSING POINT

1) $u(\cdot)$ is non-monotone:

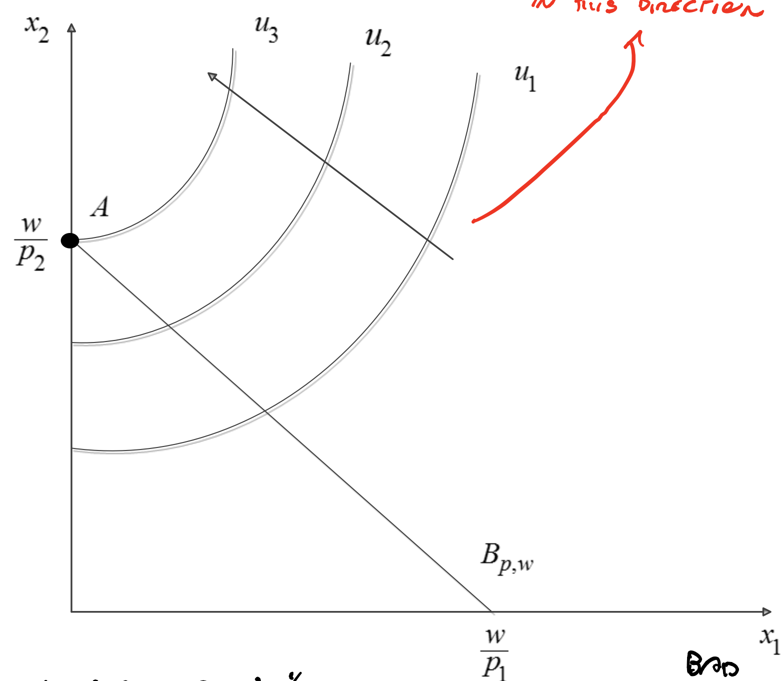
- The consumer chooses bundle *A* (at a corner) since it yields the highest utility level given his budget constraint.

CORNER SOLUTION

- At point *A*, however, the tangency condition $MRS_{1,2} = \frac{p_1}{p_2}$ does not hold.

GOOD

UTILITY INCR IN THIS DIRECTION



BAD

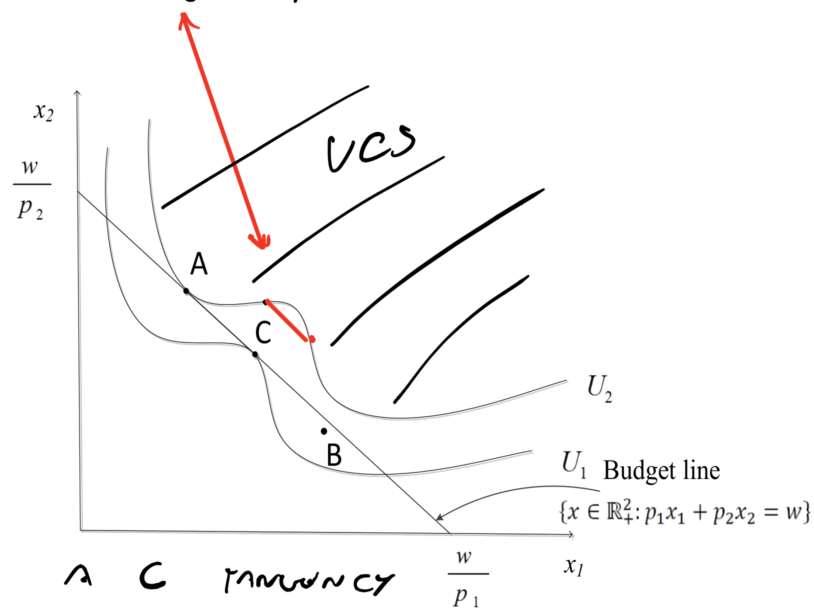
WE WILL ONLY CONSIDER CORNER SOLUTION OF PERFECT SUBSTITUTES

UMP: Violations of Sufficient Condition

2) $u(\cdot)$ is not quasiconcave:

- The upper contour sets (UCS) are not convex.
- $MRS_{1,2} = \frac{p_1}{p_2}$ is not a sufficient condition for a max.
- A point of tangency (*C*) gives a lower utility level than a point of non-tangency (*B*).
- True maximum is at point *A*.

UCS NOT CONVEX



A C TANGENCY
BUT ONLY IS MAX AND THE OTHER
NO. VIOLATION OF SUFF CONDITION

UMP: Corner Solution

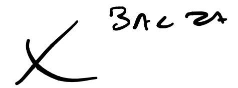
- Analyzing differential changes in x_l and x_l, that keep individual's utility unchanged, $du = 0$,

$$\frac{du(x)}{dx_l} dx_l + \frac{du(x)}{dx_k} dx_k = 0 \text{ (total diff.)}$$

- Rearranging,

$$\frac{dx_k}{dx_l} = -\frac{\frac{du(x)}{dx_l}}{\frac{du(x)}{dx_k}} = -MRS_{l,k}$$

- ***Corner Solution***: $MRS_{l,k} > \frac{p_l}{p_k}$, or alternatively, $\frac{\frac{du(x^*)}{dx_l}}{p_l} > \frac{\frac{du(x^*)}{dx_k}}{p_k}$, i.e., the consumer prefers to consume more of good l.



UMP: Corner Solution

- In the FOCs, this implies:
  a) $\frac{\partial u(x^*)}{\partial x_k} \leq \lambda p_k$ for the goods whose consumption is zero, $x_k^* = 0$, and
  b) $\frac{\partial u(x^*)}{\partial x_l} = \lambda p_l$ for the good whose consumption is positive, $x_l^* > 0$.

- *Intuition*: the marginal utility per dollar spent on good l is still larger than that on good k.

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} = \lambda \geq \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

UMP: Corner Solution

CASES OF CORNER SOLUTION

WE CANNOT FIND TANGENCY CONDITION

(NO TANGENCY POINT → SLOPE IS SAME)

- Consumer seeks to consume good 1 alone.
- At the corner solution, the indifference curve is steeper than the budget line, i.e.,

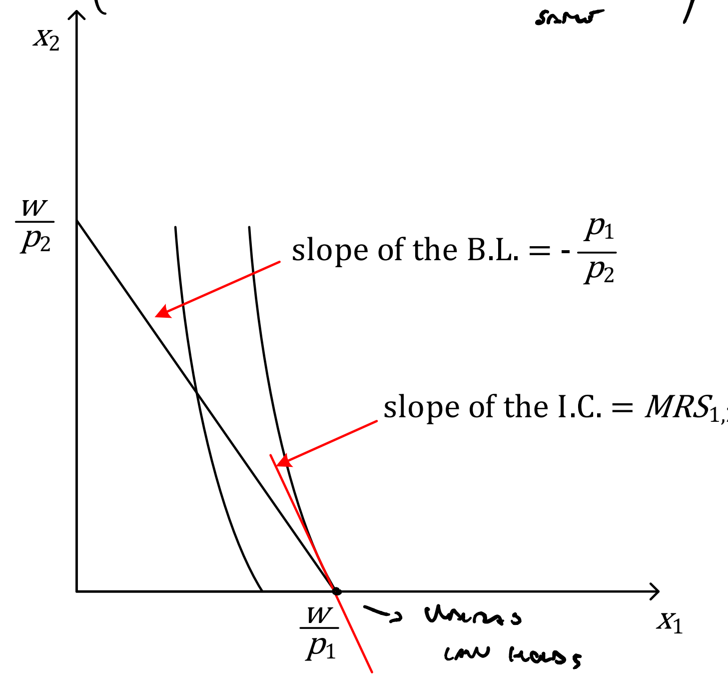
$$MRS_{1,2} > \frac{p_1}{p_2} \text{ or } \frac{MU_1}{p_1} > \frac{MU_2}{p_2}$$

- Intuitively, the consumer would like to consume more of good 1, even after spending his entire wealth on good 1 alone. , BUDGET

SLOPE IC AND BC?

SLOPE IS GREATER IN IC THAN B.C.

$|MRS| > p_1/p_2$



→ WALRAS LAW HOLDS

$x_2 = 0 \quad x_1 = \frac{w}{p_1}$

$$\frac{\frac{\partial u}{\partial x_1}}{p_1} > \frac{\frac{\partial u}{\partial x_2}}{p_2}$$

YOU CONSUME ALL INCOME IN x_1 SO YOU CAN'T MOVE FROM ONE GOOD TO ANOTHER

IN CORNER SOLUTION $MRS > \frac{p_1}{p_2}$ IS NOT EQUALITY

ERRORS IN EXERCISES!

$$\frac{\frac{\delta u}{\delta x_1}}{\frac{\delta u}{\delta x_2}} = MRS$$

DEFINITION

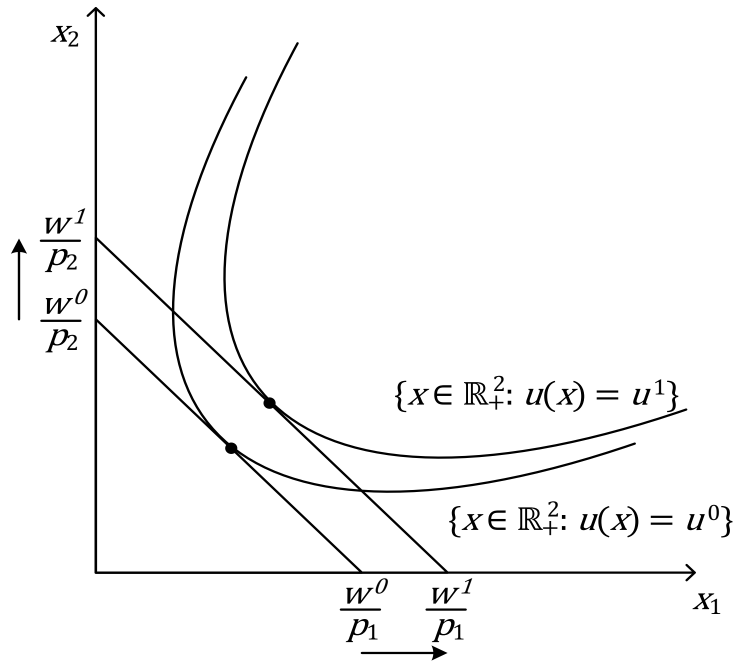
$$= \frac{p_1}{p_2}$$

THIS ONLY HOLDS FOR THE MAXIMUM!!

UMP: Lagrange Multiplier → λ

- λ is referred to as the "marginal values of relaxing the constraint" in the UMP (a.k.a. "***shadow price of wealth***").

- If we provide more wealth to the consumer, he is capable of reaching a higher indifference curve and, as a consequence, obtaining a higher utility level.
  - We want to measure the change in utility resulting from a marginal increase in wealth.



NEXT LESSON

UMP: Lagrange Multiplier

- Let us take $u(x(p,w))$, and analyze the change in utility from change in wealth. Using the chain rule yields,

$$\nabla u(x(p,w)) \cdot D_w x(p,w)$$

- Substituting $\nabla u(x(p,w)) = \lambda p$ (in interior solutions),

$$\lambda p \cdot D_w x(p,w)$$

NB. ∇ means differential with respect to a vector, $x = (x_1, x_2, \dots . x_n)$ the result is a vector next lesson

UMP: Lagrange Multiplier

- From Walras' Law, $p \cdot x(p,w) = w$, the change in expenditure from an increase in wealth is given by
$$p \cdot D_w x(p,w) = D_w[p \cdot x(p,w)] = D_w(w) = 1$$

- Hence,
$$\nabla u(x(p,w)) \cdot D_w x(p,w) = \lambda \underbrace{p \cdot D_w x(p,w)}_{1} = \lambda$$

- *Intuition*: If $\lambda = 5$, then a \$1 increase in wealth implies an increase in 5 units of utility. At the maximum this must be the same for all goods, otherwise we are not at the maximum

Walrasian Demand: Wealth Effects

PARTIAL DER. IN RESPECT OF WEALTH IF > NORM
IF < INF

- Normal vs. Inferior goods

$$\frac{\partial x(p,w)}{\partial w} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{normal} \\ \text{inferior} \end{Bmatrix}$$

INCREASING OF INCOME DOESN'T MAKE YOU TAKE MORE CRAPPY WINE BUT YOU GET BETTER WINE

- Examples of inferior goods:
  - Two-buck chuck (a really cheap wine)
  - Walmart during the economic crisis
  - POTATOES
  - A CRAP OF (ANY CHEAP) WINE

Walrasian Demand: Wealth Effects

Prices constant → slope same

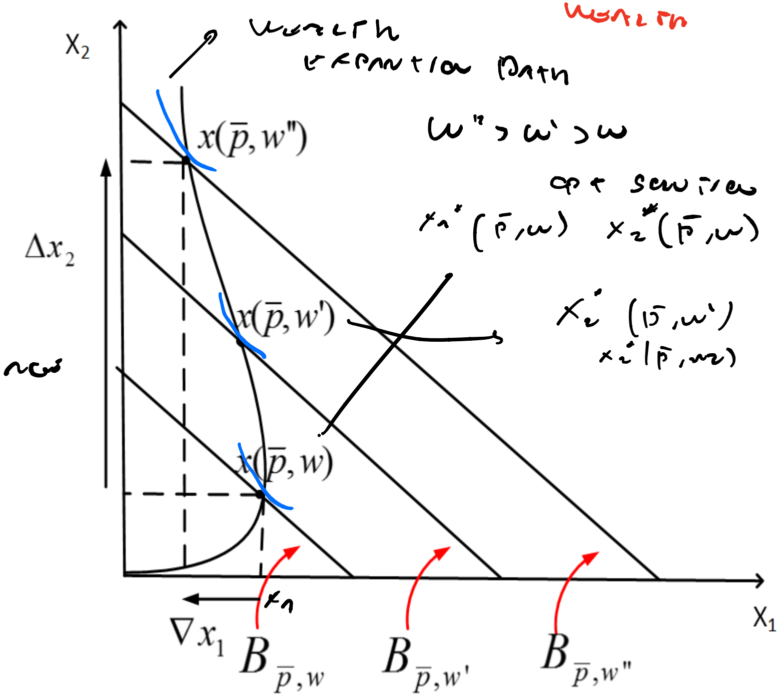
Intercept changes → ie incr. wealth

- An increase in the wealth level produces an outward shift in the budget line.

- x_2 is normal as $\frac{\partial x_2(p,w)}{\partial w} > 0$, while x_1 is inferior as $\frac{\partial x(p,w)}{\partial w} < 0$.

How demand changes ie wealth changes

- ***Wealth expansion path:***
  - connects the optimal consumption bundle for different levels of wealth
  - indicates how the consumption of a good changes as a consequence of changes in the wealth level



Wealth expansion path

$w'' > w' > w$

opt solution

$x_1^*(\bar{p},w)$ $x_2^*(\bar{p},w)$

$x_2^*(\bar{p},w')$

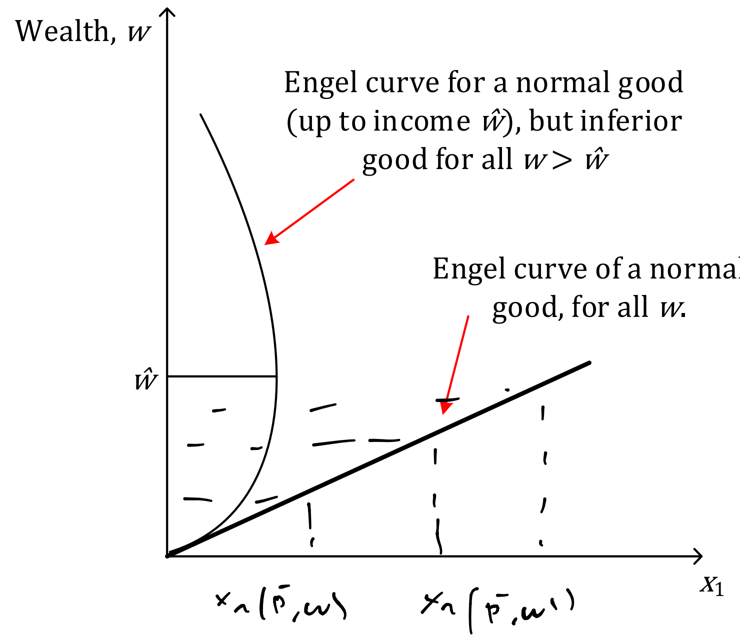
$x_1^*(\bar{p},w')$

x_1

Inc increases, you consume more x_2 (normal) and less x_1 (inferior)

Walrasian Demand: Wealth Effects

- ***Engel curve*** depicts the consumption of a particular good in the horizontal axis and wealth on the vertical axis.
- The slope of the Engel curve is:
  - positive if the good is normal
  - negative if the good is inferior
- Engel curve can be positively slopped for low wealth levels and become negatively slopped afterwards.



DRAW POINT AND I OBTAIN THE ENGEL CURVE

If price for sugar increase, then you demand less coffe. If prime derivative is positive

Walrasian Demand: Price Effects

- Own price effect:

$$\frac{\partial x_k(p,w)}{\partial p_k} \begin{Bmatrix} < \\ > \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Usual} \\ \text{Giffen} \end{Bmatrix}$$

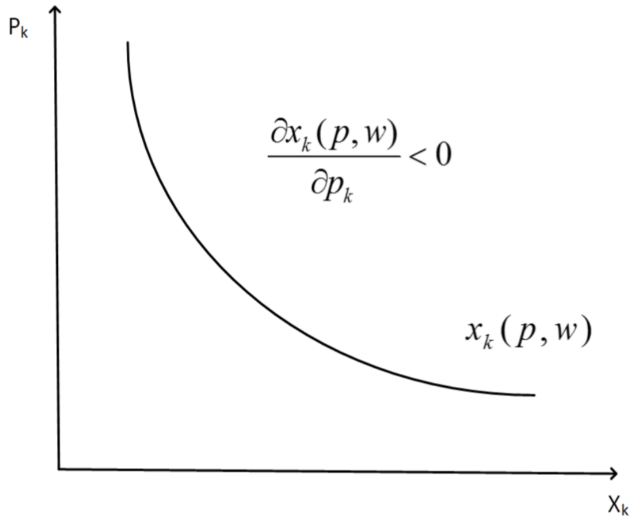
- Cross-price effect:

$$\frac{\partial x_k(p,w)}{\partial p_l} \begin{Bmatrix} > \\ < \end{Bmatrix} 0 \quad \begin{Bmatrix} \text{Substitutes} \\ \text{Complements} \end{Bmatrix}$$

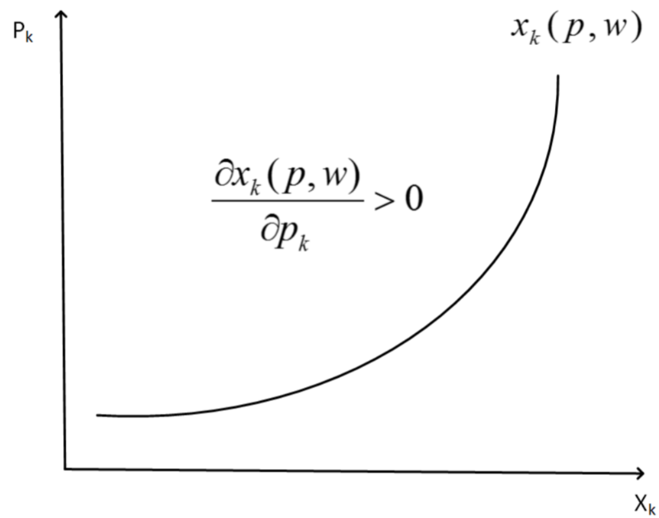
  – *Examples of Substitutes*: two brands of mineral water, such as Sant'Anna vs. Acqua Panna (Disclaimer: I did not receive money from any of the two....)
  – *Examples of Complements*: coffee and sugar.

Walrasian Demand: Price Effects

- Own price effect (inverse demand is graphed, i.e. P in vertical axis and the good in horizontal axis)



Usual good
(law of price holds, if P increases quantity decreases)



Giffen good
(law of price does not hold)

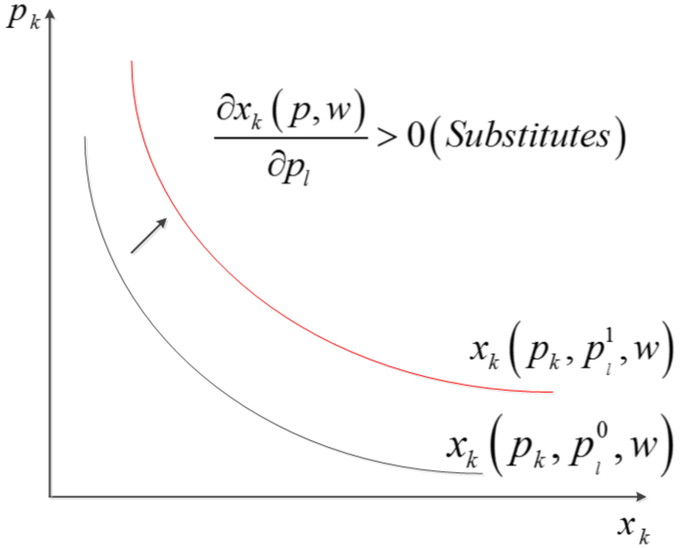
Vertical axis is the demand. We have said that if price increases the quantity increase and the walras' demand is positively sloped.

What if we want to see graphically if demand for one good is the same as the second good?
We want to see how demand depends. On another wealth. We can't use this curve because represent the realtion between quiantity of k and price of k.

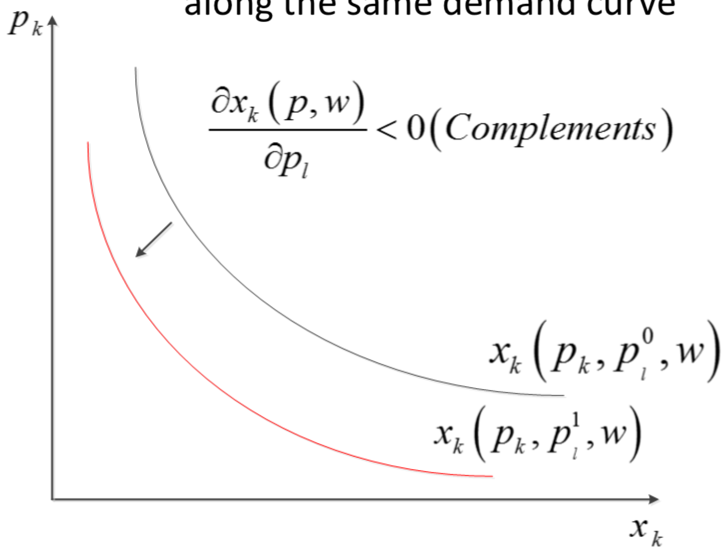
Walrasian Demand: Price Effects

- Cross-price effect

- Two-dimensions graph: change in p_1 means moving to another demand curve, while changes in p_k means moving along the same demand curve



Substitutes



Complements

Warla’s demand. Level 0 of price pk. Walras demand. If other two variable, like price in the other good change, the curves could change up or down. If good increase and the goods are substitute the curve moves up.
For a given pk do you demand more or less pk. So curve goes up right.
Complements good is the opposite. If the price of the other good increase the second one will decrease.

Different goods can be classified using walras’ demand.

Indirect Utility Function

- The Walrasian demand function, $x(p, w)$, is the solution to the UMP (i.e., argmax, i.e. value of the argument that maximizes utility).

- What would be the utility function evaluated at the solution of the UMP, i.e., $x(p, w)$?
  - This is the ***indirect utility function*** (i.e., the highest utility level), $v(p, w) \in \mathbb{R}$, associated with the UMP.
  - It is the "value function" of this optimization problem.

(I.e the function evaluated at the maximum)

If good normal or inferior we expect demand of the good will increase or decreases.
After solving the UMP getting the argmax yesterday, the solution of this problem is called walras demand. We have found this solution called x(p,w). Now we can compute the utilty function of this argument. If we compute utility function at the optimal level.

$u(x(p,w))$ = IND UTILITY FUNCTION

$$\max_{x \geq 0} u(x) \quad \text{such that} \quad x \cdot p \leq w$$

Degree of homogeneity of the indirect utilty function?
What happened to the value function if the prices and the wealth increase by the same proportion? [value alpha]. And we want to see what happen to the maximum likehood. What we have found? Walra’s demeaned is homogeneous of degree 0 since the budget constraint the solution will be the same.
What happen to the utility function if p and w change for the small propotion of alpha. The value of utility doesn’t change so I directed utility function is homogeneous of degree 0.

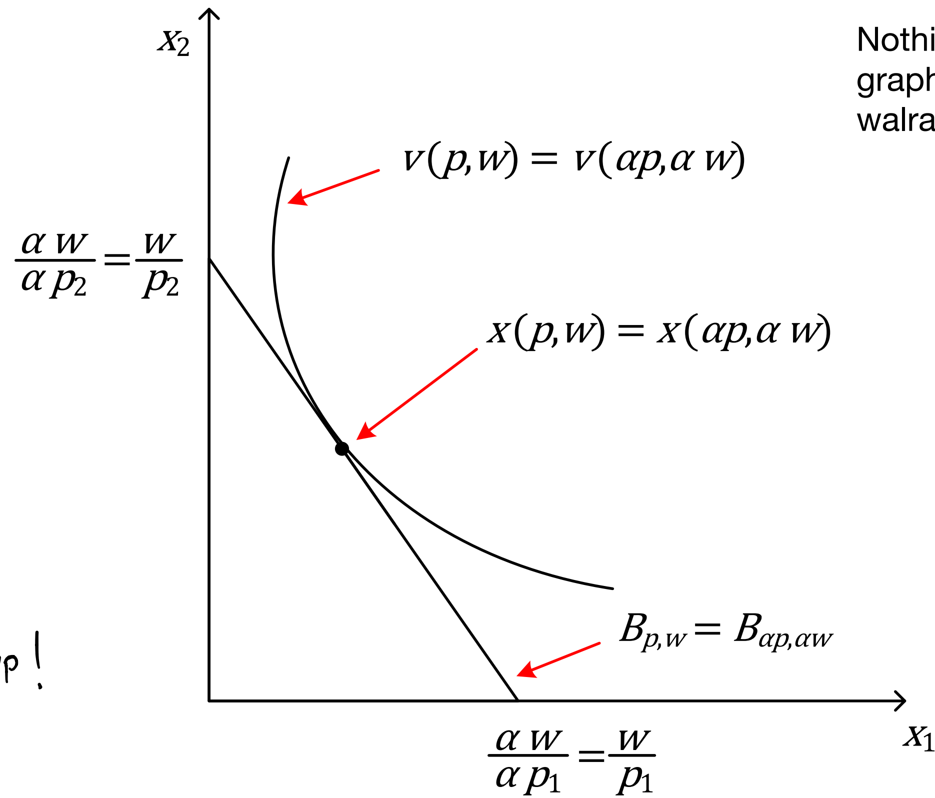
The indirect utilty function is homogeneous of degree 0.

Properties of Indirect Utility Function

- If the utility function is continuous and preferences satisfy LNS over the consumption set $X = \mathbb{R}_+^L$, then the indirect utility function $v(p, w)$ satisfies:

    1) ***Homogenous of degree zero:*** Increasing p and w by a common factor $\alpha > 0$ does not modify the consumer's optimal consumption bundle, $x(p, w)$, nor his maximal utility level, measured by $v(p, w)$.

Properties of Indirect Utility Function



Nothing change graphically: same IC, walras demand ecc..

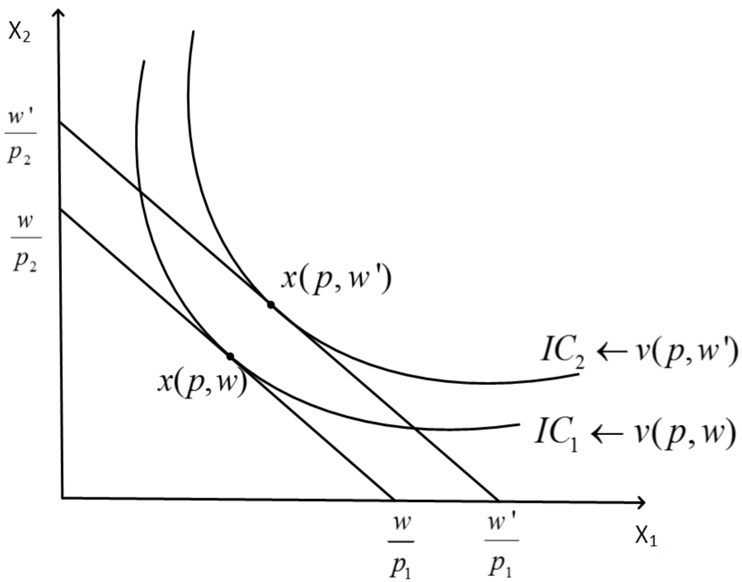
If $w \uparrow$ /
IUF ? shift up!

Properties of Indirect Utility Function

If wealth increase, i will get more

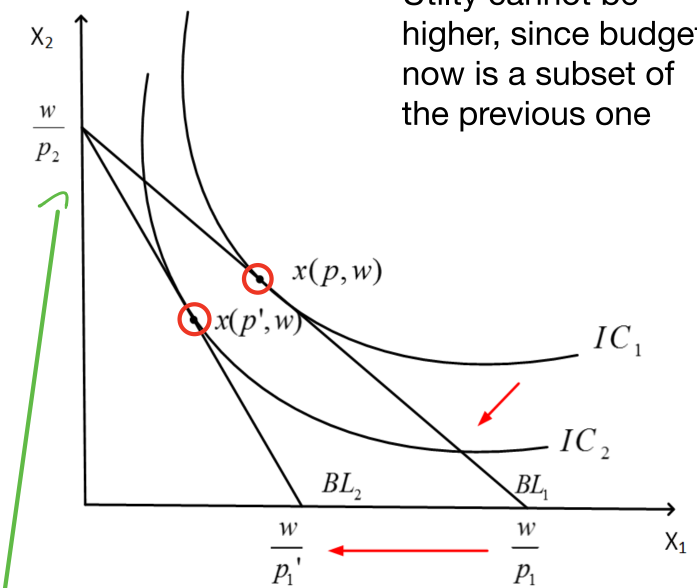
2) Strictly increasing in w:

$v(p, w') > v(p, w)$ for $w' > w$.



3) non-increasing (i.e., weakly decreasing) in p_k

Utilty cannot be higher, since budget now is a subset of the previous one



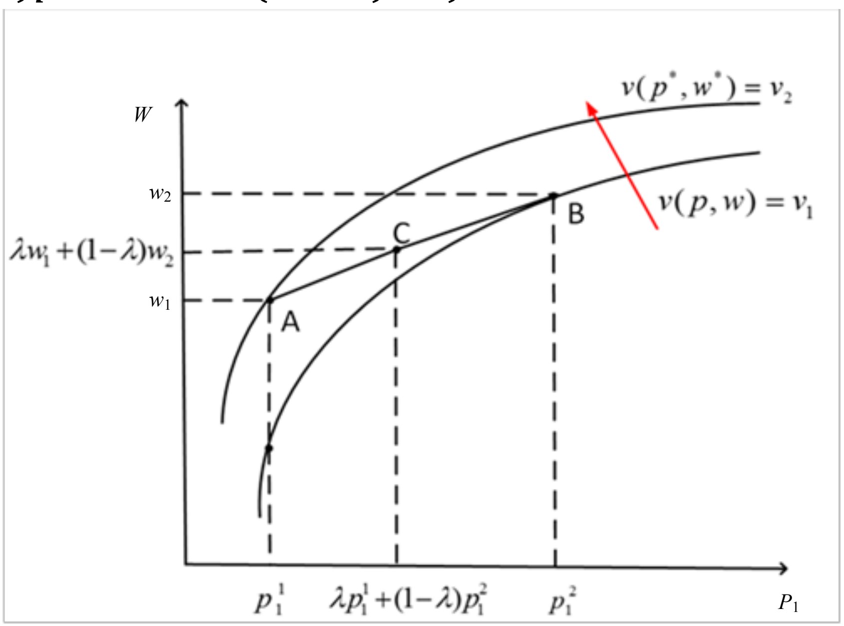
Imagine corner solution, the demand remain the same (x2), the supply will decrease. So is not increasing in pk\

NOT SO IMPORTANT FOR EX!!

Properties of Indirect Utility Function

4) Quasiconvex: The set $\{(p,w): v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

- ***Interpretation I***: If $(p^1, w^1) \gtrsim^* (p^2, w^2)$, then $(p^1, w^1) \gtrsim^* (\lambda p^1 + (1-\lambda)p^2,\ \lambda w^1 + (1-\lambda)w^2)$; i.e., if $A \gtrsim^* B$, then $A \gtrsim^* C$.

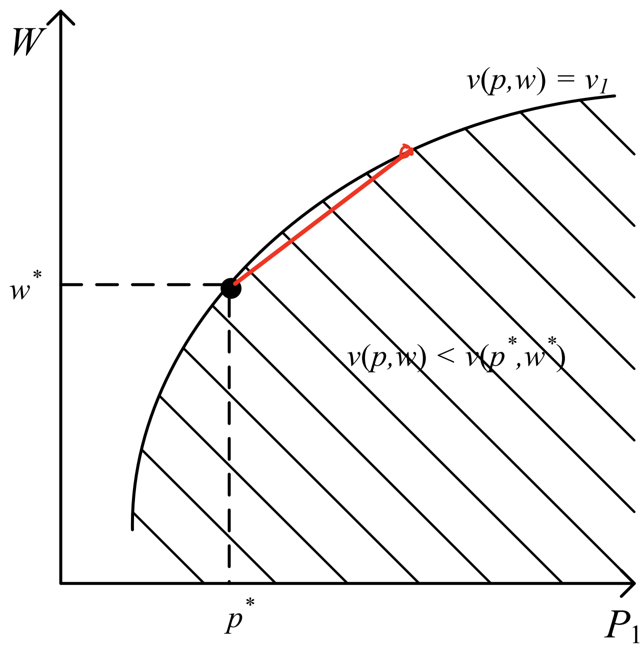


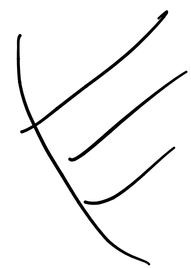
Properties of Indirect Utility Function

- ***Interpretation II***: $v(p, w)$ is quasiconvex if the set of (p, w) pairs for which $v(p, w) < v(p^*, w^*)$ is convex.

UCS -> CONVEX
LWS -> CONCAVE

IUF IS CONCAVE

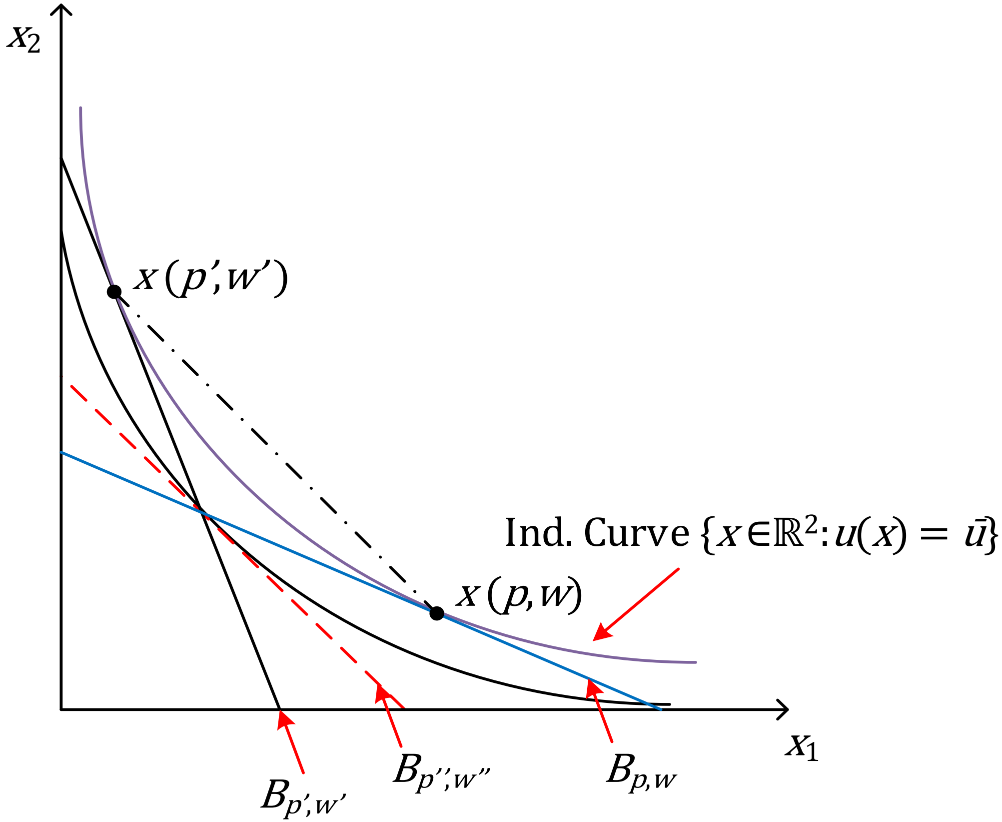




Properties of Indirect Utility Function

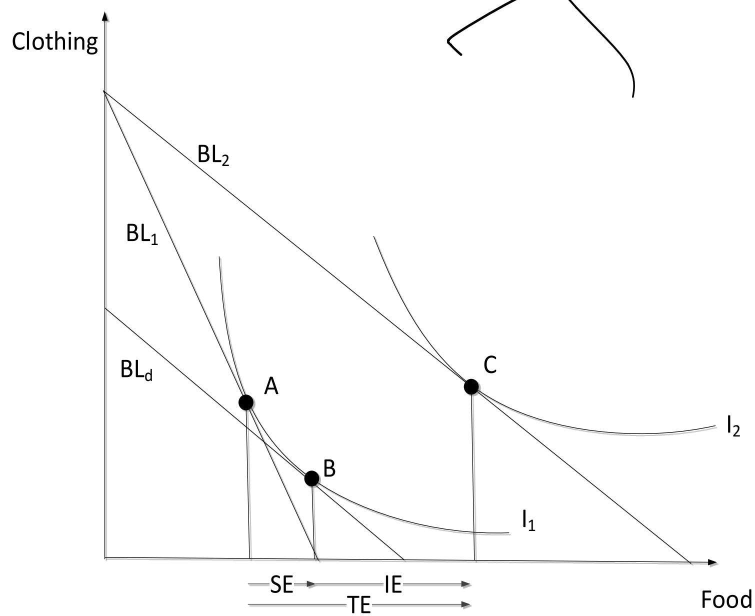
- ***Interpretation III***: Using x_1 and x_2 in the axis, perform following steps:
    1) When $B_{p,w}$, then $x(p,w)$
    2) When $B_{p',w'}$, then $x(p',w')$
    3) Both $x(p,w)$ and $x(p',w')$ induce an indirect utility of $v(p,w) = v(p',w') = \bar{u}$
    4) Construct a linear combination of prices and wealth:
$$\left.\begin{aligned} p'' &= \alpha p + (1-\alpha)p' \\ w'' &= \alpha w + (1-\alpha)w' \end{aligned}\right\} \quad B_{p'',w''}$$
    5) Any solution to the UMP given $B_{p'',w''}$ must lie on a lower indifference curve (i.e., lower utility)
$$v(p'',w'') \leq \bar{u}$$

Properties of Indirect Utility Function



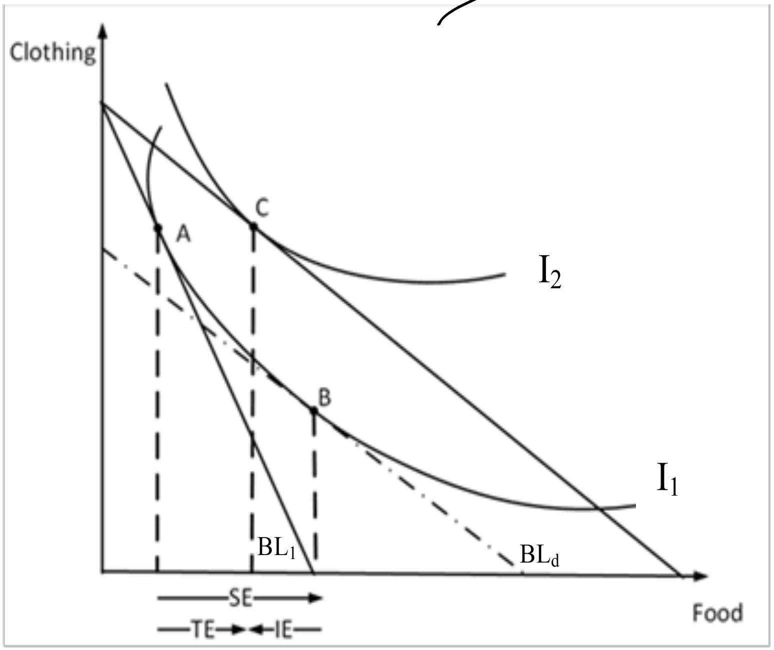
Substitution and Income Effects: Normal Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The substitution effect (SE) moves in the opposite direction as the price change.
  - A reduction in the price of food implies a positive substitution effect.
- The income effect (IE) is positive (thus it reinforces the SE).
  - The good is normal.



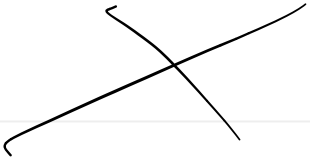
Substitution and Income Effects: Inferior Goods

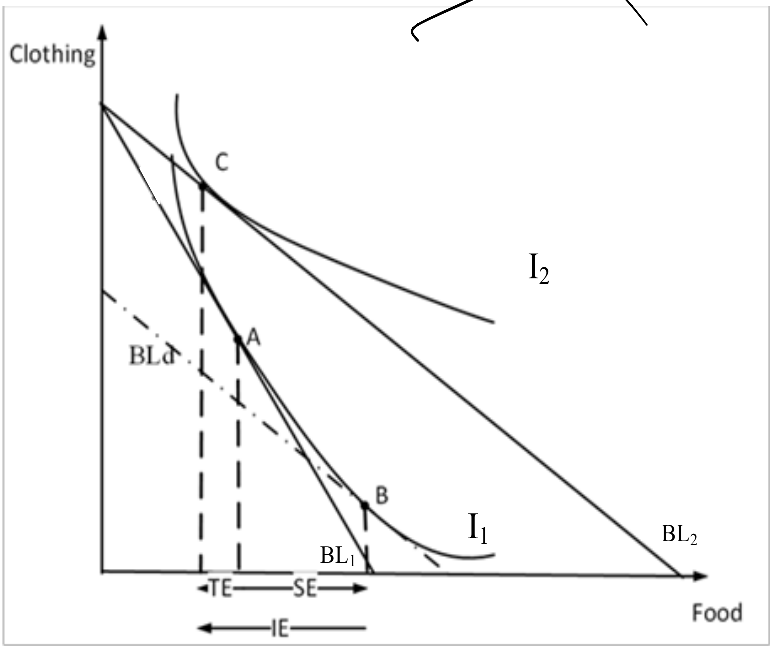
- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is now negative (which partially offsets the increase in the quantity demanded associated with the SE).
  - The good is inferior.
- *Note*: the SE is larger than the IE.



Substitution and Income Effects: Giffen Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is still negative but now completely offsets the increase in the quantity demanded associated with the SE.
  - The good is Giffen good.
- *Note*: the SE is less than the IE.





Substitution and Income Effects

	SE	IE	TE
Normal Good	+	+	+
Inferior Good	+	-	+
Giffen Good	+	-	-

- Not Giffen: Demand curve is negatively sloped (as usual)
- Giffen: Demand curve is positively sloped

ESERCIZI

EX 2 CH.2

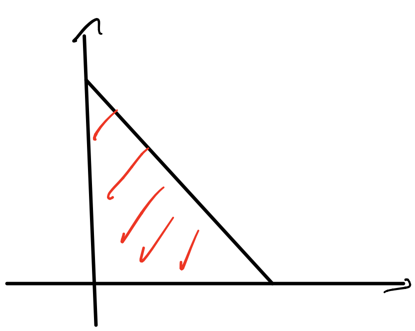
$$u(x_1, x_2) = x_1^{a} x_2^{\frac{1}{2}-a}$$

1. WALRAS DEMAND x_1, x_2
2. RESTRICTION ON a SUCH THAT $x_1, x_2 > 0$

$p_1x_1 + p_2x_2 \leq w$ CONSTRAINT

1) $$L = x_1^{a} x_2^{\frac{1}{2}-a} + \lambda(w - p_1x_1 - p_2x_2)$$

$$\text{FOCS} = \begin{cases} \dfrac{\delta L}{\delta x_1} = a x_1^{a-1} x_2^{\frac{1}{2}-a} - \lambda p_1 = 0 \\ \dfrac{\delta L}{\delta x_2} = \left(\dfrac{1}{2} - a\right) x_2^{-\frac{1}{2}-a} x_1^{a} - \lambda p_2 = 0 \\ \dfrac{\delta L}{\delta \lambda} = w - p_1x_1 - p_2x_2 = 0 \leftarrow \text{WALRAS LAW} \end{cases}$$



CORNER SOLUTION CAN'T BE THE SOLUTION BECAUSE THE $u(\cdot)$ IS A COBB-DOUGLAS BECAUSE IN CORNER UTILITY CAN BE $\emptyset$

NOW WE HAVE 3 EQUATION AND 3 VARIABLES SO:

$$\frac{a x_1^{a-1} x_2^{\frac{1}{2}-a}}{\left(\frac{1}{2}-a\right) x_2^{-\frac{1}{2}-a} x_1^{a}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2} \Rightarrow \frac{a x_1^{a-1} x_2^{\frac{1}{2}-a}}{\left(\frac{1}{2}-a\right) x_2^{-\frac{1}{2}-a} x_1^{a}} = \frac{p_1}{p_2}$$

$$\frac{\alpha}{\frac{1}{2}-\alpha}\cdot\frac{x_2}{x_1}=\frac{p_1}{p_2}$$

$$\frac{\cancel{x_1}^{\alpha-1}}{x_1^{\alpha}} \qquad \frac{x_2^{\frac{1}{2}-\alpha}}{\cancel{x_2}^{-\frac{1}{2}-\alpha}}$$

$$x_2=\frac{p_1}{p_2}\cdot\frac{\frac{1}{2}-\alpha}{\alpha}\,x_1 \qquad \boxed{x_2=\frac{1-2\alpha}{2\alpha}\cdot\frac{p_1}{p_2}\,x_1}$$

THE SUBSTITUTE x_2 IN THE CONSTRAINT

$$w-p_1x_1-p_2x_2=0 \to w-p_1x_1-\cancel{p_2}\left(\frac{1-2\alpha}{2\alpha}\,\frac{p_1}{\cancel{p_2}}\,x_1\right)=0$$

$$w=p_1x_1+\frac{1-2\alpha}{2\alpha}\,p_1\,x_1 \qquad w=\left(1+\left(\frac{1-2\alpha}{2\alpha}\right)\right)x_1p_1$$

$$w=x_1\left(\frac{2\alpha+1-2\alpha}{2\alpha}\right) \to x_1=\frac{2\alpha\,w}{p_1}$$

now find x_2:

$$x_2=\frac{1-2\alpha}{2\alpha}\cdot\frac{p_1}{p_2}\,x_1=\frac{1-2\alpha}{p_2}\cdot w$$

$$\Rightarrow x^*=\left(\underbrace{\frac{2\alpha\,w}{p_1}}_{x_1}\quad\underbrace{\left(\frac{1-2\alpha}{p_2}\right)w}_{x_2}\right)$$

LOOKING AT THIS, CONSIDERING x_1 WHAT ABOUT THE DERIVATIVE? IS $\emptyset$ SO THE TWO GOODS ARE INDEPENDENT

THE SHARE OF INCOME SPENT IN x_1 $=\dfrac{\cancel{p_1}\cdot\dfrac{2\alpha\cdot\cancel{w}}{\cancel{p_1}}}{\cancel{w}}=2\alpha$

THE SHARE OF INCOME SPENT IN x_2 $=\dfrac{\cancel{p_2}\;\dfrac{1-2\alpha}{\cancel{p_2}}\cdot\cancel{w}}{\cancel{w}}=1-2\alpha$

2) CONDITION ON α? α SHOULD BE POSITIVE

$$x^{\alpha} = \frac{2\alpha w}{p_1} \; ; \; \frac{1-2\alpha}{p_2} w$$

$\alpha > 0$

$\rightarrow 1 - 2\alpha > 0 \rightarrow -2\alpha > -1$

$\rightarrow 2\alpha < 1 \qquad \alpha < \frac{1}{2}$

$\alpha \in \left(0, \frac{1}{2}\right)$ FOR INTERIOR CONDITION, BOTH x_1 AND x_2 ARE POSITIVE

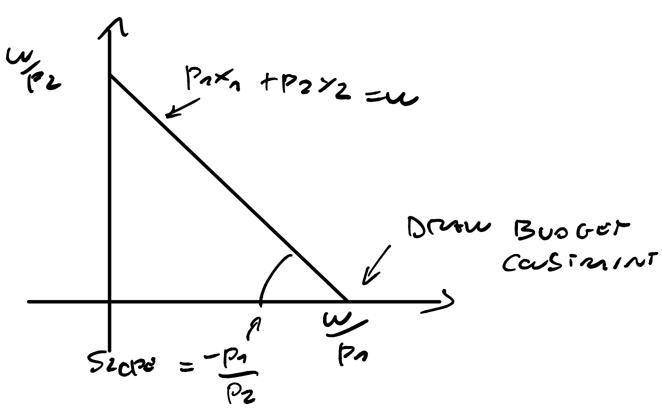
SO ALSO $\alpha \in \left[0, \frac{1}{2}\right]$

$\alpha x_1 + \beta x_2$ IF x_1 OR x_2 IS $\emptyset$ THE VALUE OF UTILITY CAN STILL NOT BE $\emptyset$

$$\boxed{EX. 1 \quad CH. 2}$$

LINEAR FUNCTION

$u(x) = 3x_1 + 4x_2 \quad$ S.T. $\quad p_1 x_1 + p_2 x_2 \leq w$



$$L = 3x_1 + 4x_2 - \lambda(w - p_1 x_1 - p_2 x_2)$$

$$\begin{cases} \frac{\delta L}{x_1} = 3 - \lambda p_1 = 0 \\ \frac{\delta L}{x_2} = 4 - \lambda p_2 = 0 \\ \frac{\delta L}{\lambda} = w + p x_1 + p_2 x_2 \end{cases}$$

|MRS| SLOPE OF B.C.

$$\rightarrow \boxed{\frac{3}{4} = \frac{p_1}{p_2}}$$

THIS CAN BE TRUE ONLY IF $\frac{p_1}{p_2}$ IS $\frac{3}{4}$

(INTERNAL?)

THERE IS NO INTERIOR SOLUTION

$\Rightarrow$ WALRAS DEMAND IS NOT A FUNCTION BUT CORRESPONDENCE THE OPT SOLUTIONS ARE ALL POSITIVE ON THE BUDGET CONSTRAINT

NO TANGENCY POINT BUT THEY OVERLAP

OTHER WAY

UTILITY LEVEL K

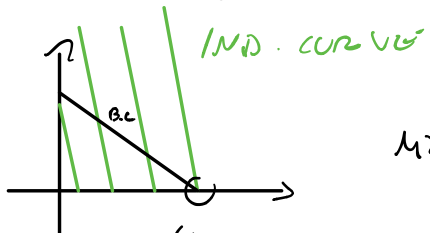
$$3x_1 + 4x_2 = K$$

$$x_2 = -\frac{3x_1}{4} + \frac{K}{4}$$ THEN WHAT IS THE SLOPE? IN ABSOLUTE VALUE

IS THE COEFFICIENTS OF $x_1 = -\frac{3}{4}$ SO $\left|-\frac{3}{4}\right| = \frac{3}{4}$

NOW IMAGINE MRS $> \frac{p_1}{p_2}$ SO $\frac{3}{4} > \frac{p_1}{p_2}$

IND. CURVES ARE ... THAN THE B.C

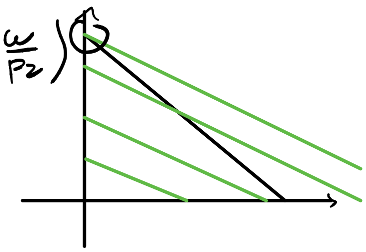


MRS $> \frac{p_1}{p_2}$

1 $\left(\frac{w}{p_1}, 0\right)$

Corner solution with $x_2 = 0$ $\quad x_1 = \frac{w}{p_1}$

$\left(0, \frac{w}{p_2}\right)$



MRS $< \frac{p_1}{p_2}$ $\rightarrow$ corner solution with $x_1 = 0$

$\left(\frac{3}{4}\right)$

$x_2 = \frac{w}{p_2}$

$x^* = \left(0, \frac{w}{p_2}\right)$

Expenditure Minimization Problem

and connection between functions

Expenditure Minimization Problem

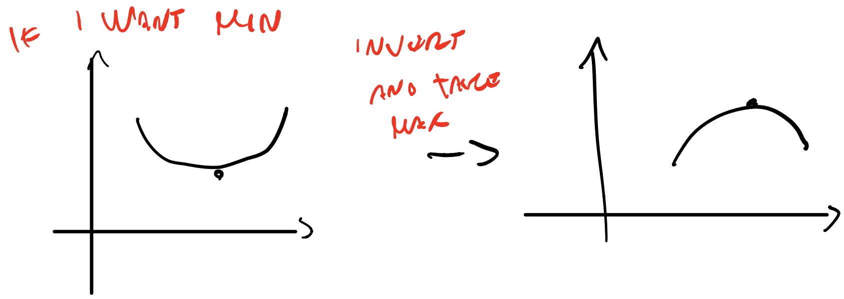
- Expenditure minimization problem (EMP):

$$\min_{x \geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$
$$(\text{i.e. } u(x) - u \geq 0)$$

- Alternative to utility maximization problem
- NB. $\min_{x \geq 0} p \cdot x = \max_{x \geq 0} -(p \cdot x)$ I can set up this as a maximization problem, and use what we already know.

In the previous problem we have the budget constraint and we have ...

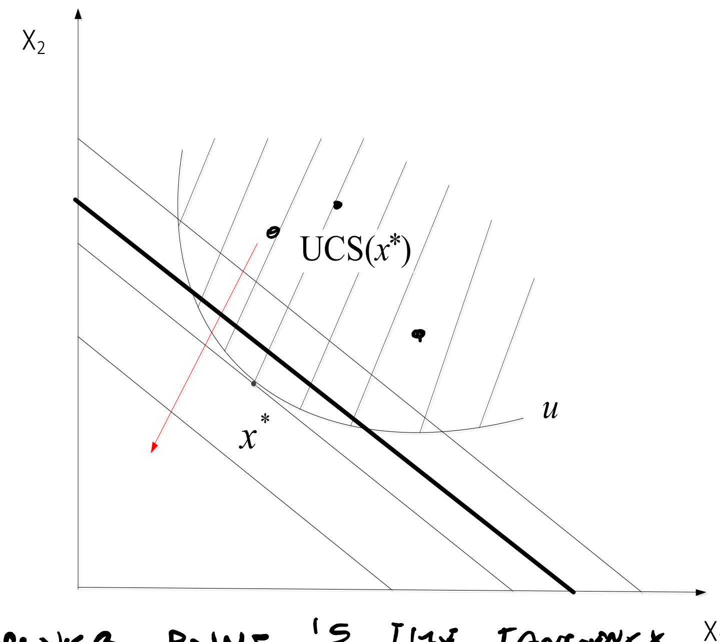
If you want to translate this problem in a optimization problem we can maximise the opposite of the max.



Expenditure Minimization Problem

WE WANT TO FIND BUDGET CONSTRAINT

- Consumer seeks a utility level associated with a particular indifference curve, while spending as little as possible.
- Bundles strictly above x^* cannot be a solution to the EMP:
  - They reach the utility level u
  - But, they do not minimize total expenditure
- Bundles on the budget line strictly below x^* cannot be the solution to the EMP problem:
  - They are cheaper than x^*
  - But, they do not reach the utility level u



LOWER POINT IS THE TANGENCY POINT BETWEEN CONSTRAINT AND LOWEST POSSIBLE BUDGET CONSTRAINT

Expenditure Minimization Problem

VECTORS AND PRODUCT $p_1 x_1 + p_2 x_2$

TAKES OPPOSITE

- Lagrangian

$$L = -p \cdot x + \mu \underbrace{[u(x) - u]}_{\geq 0}$$

$-p_1 x_1 - p_2 x_2 + \mu [u(x_1, x_2)]$

DERIVATIVE

$-p_1 + \mu \dfrac{\partial u}{\partial x_1} \leq 0$

ALSO HERE COMPLEMENTARY SLACKNESS COND

- FOCs (necessary conditions)

$$\frac{\partial L}{\partial x_k} = -p_k + \mu \frac{\partial u(x^*)}{\partial x_k} \leq 0$$

[= 0 for interior solutions]

$=0 \; ; \; x_k > 0$

$<0 \; ; \; x_k = 0$

$k = 1, 2$

$$\frac{\partial L}{\partial \mu} = u(x^*) - u \geq 0$$

Take the opposite of the maximal function.

The second is the constraint and i add the la grangian multiplier (mu) which multiply the budget constrain.

INTERIOR SOLUTION

$x_k > 0$

BY GRAPHICAL SOLUTION

POINT IN UCS ARE NOT OPTIMAL

BUT MUST BE EXACTLY IN THE IC SO = !!

AND WE CAN FOCUS ON

= CASE

$\Downarrow$

$u(x^*) - \bar{u} = 0$

$\downarrow$

THEN MOVE

ON INTERIOR

AND (NEXT SLIDE)

Expenditure Minimization Problem

K = 1,2

SINCE BOTH EQUAL TO $\frac{1}{\mu}$

- For interior solutions,

$$\boxed{p_k = \mu \frac{\partial u(x^*)}{\partial x_k}} \quad \text{or} \quad \frac{1}{\mu} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k}$$

for any good k. This implies,

SLOPE OF BUDGET CON = MRS

SAME COND. OF 1€ IN ONE GOOD THEN I HAVE TO SPEND 1€ IN OTHER GOOD

$$\frac{\frac{\partial u(x^*)}{\partial x_k}}{p_k} = \frac{\frac{\partial u(x^*)}{\partial x_l}}{p_l} \quad \text{or} \quad \frac{p_k}{p_l} = \frac{\frac{\partial u(x^*)}{\partial x_k}}{\frac{\partial u(x^*)}{\partial x_l}}$$

$$\frac{1}{\mu} = \frac{\frac{\partial u}{\partial x_1}}{p_1}$$

$$\frac{1}{\mu} = \frac{\frac{\partial u}{\partial x_2}}{p_2}$$

SO THEY ARE THE SAME

- The consumer allocates his consumption across goods until the point in which the marginal utility per dollar spent on each good is equal across all goods (i.e., same “bang for the buck”).

- That is, the slope of indifference curve is equal to the slope of the budget line. **(i.e. the “usual tangency condition”)**

EMP: Hicksian Demand

- The bundle $x^* \in \operatorname{argmin} p \cdot x$ (the argument that solves the EMP) is the ***Hicksian demand***, which depends on p and u (while **Walrasian demand depends on *p* and *w***),

$$x^* \in h(p, u)$$

- Recall that if such bundle x^* is unique, we denote it as $x^* = h(p, u)$ (i.e. it is a function not a correspondance).

Walras demand is the solution of maximisation problem. Similar we get the same with minimum problem and is called the Hicksian demand.

Walras demand depends on the price and the wealth that are the parameter in the budget constraint. While x is the choice variable.

$x(p, w)$ $\quad x \to$ choice

price $\quad$ wealth

Parameters appearing? Price parameter, is u parameter? Yes.
Hicksian depend on price and utility! So it's different.

Solution is unique.... set of bundle???
[24]

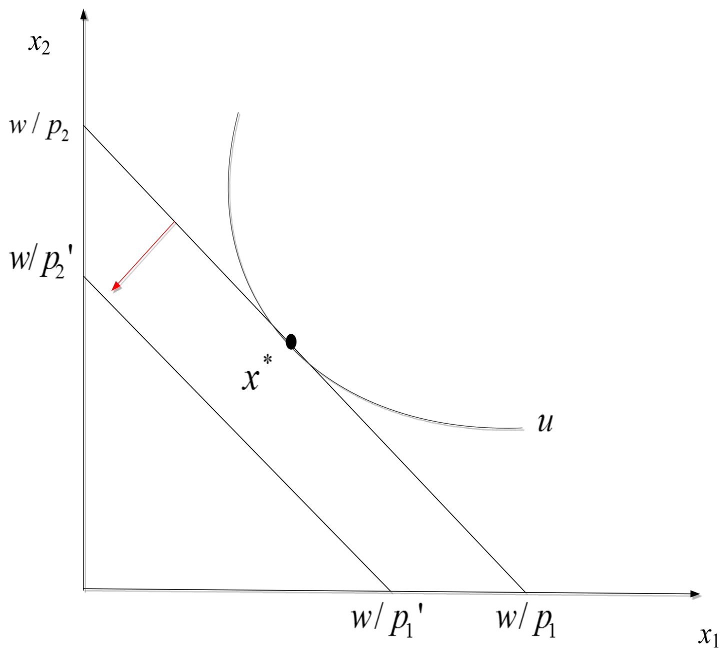
If both price and u increase by alpha then ratio between price doesn't change. Bundle does't change but expenditure does change! P X* —> alpha P X*.
To reach that utilty level you spend more!

Properties of Hicksian Demand

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}^L_+$. Then for $p \gg 0$, $h(p,u)$ satisfies: is just increasing p not u
  1) ***Homog(0)*** in p, i.e., $h(p,u) = h(\alpha p, u)$ for any p, u, and $\alpha > 0$
     - If $x^* \in h(p,u)$ is a solution to the problem
       $$\min_{x \geq 0} p \cdot x$$
       then it is also a solution to the problem
       $$\min_{x \geq 0} \alpha p \cdot x$$
     - *Intuition*: a common change in all prices does not alter the slope of the consumer's budget line.

Properties of Hicksian Demand

- x^* is a solution to the EMP when the price vector is $p = (p_1, p_2)$.
- Increase all prices by factor α
  $$p' = (p_1', p_2') = (\alpha p_1, \alpha p_2)$$
- Downward (parallel) shift in the budget line, i.e., the slope of the budget line is unchanged.
- But I have to reach utility level u to satisfy the constraint of the EMP!
- Spend more to buy bundle $x^*(x_1^*, x_2^*)$, i.e.,
  $$p_1'x_1^* + p_2'x_2^* > p_1x_1^* + p_2x_2^*$$
- Hence, $h(p,u) = h(\alpha p, u)$

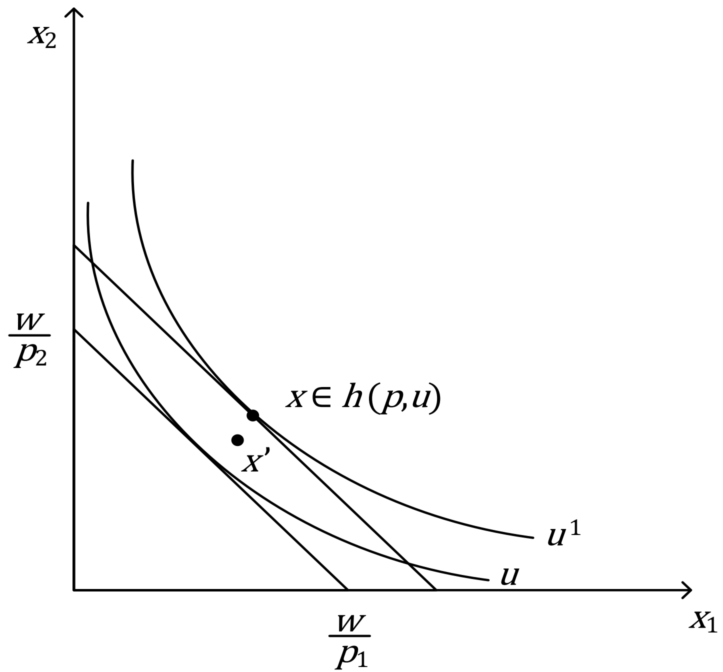


Properties of Hicksian Demand

2) No excess utility:

for any optimal consumption bundle $x \in h(p, u)$, utility level satisfies $u(x) = \bar{u}$.

(That is the level of utility fixed in the constraint)



NB. Equivalent of Walras' Law in UMP (constraint holds with equality)

Properties of Hicksian Demand

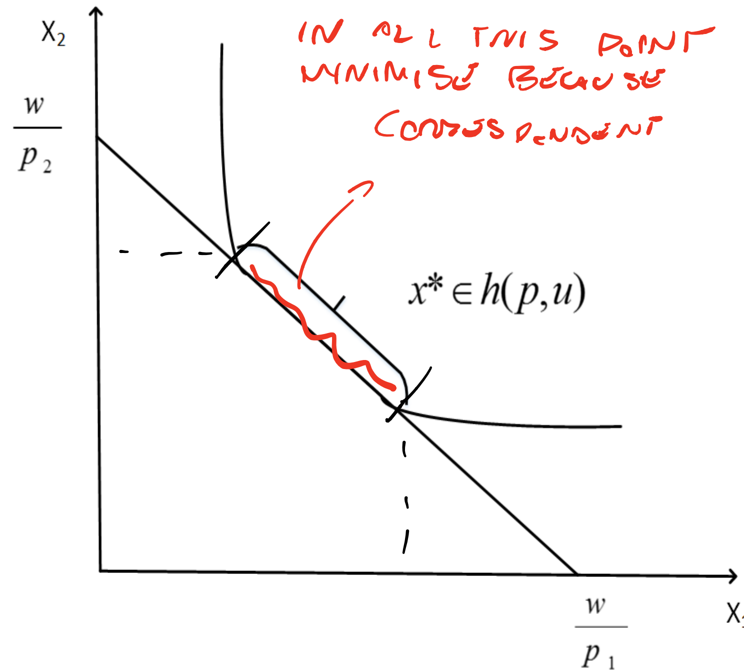
- *Intuition*: Suppose there exists a bundle $x \in h(p,u)$ for which the consumer obtains a utility level $u(x) = u^1 > u$, which is higher than the utility level u he must reach when solving EMP.
- But we can then find another bundle $x' = x\alpha$, where $\alpha \in (0,1)$, very close to x ($\alpha \to 1$), for which $u(x') > u$.
- Bundle x':
  - is cheaper than x since it contains fewer units of all goods; and
  - exceeds the minimal utility level u that the consumer must reach in his EMP.
- We can repeat that argument until reaching bundle x.
- In summary, for a given utility level u that you seek to reach in the EMP, bundle $h(p,u)$ does not exceed u. Otherwise you can find a cheaper bundle that exactly reaches u.

Properties of Hicksian Demand

CORRESPONDENTS

3) Convexity:

If the preference relation is convex, then $h(p,u)$ is a convex set.

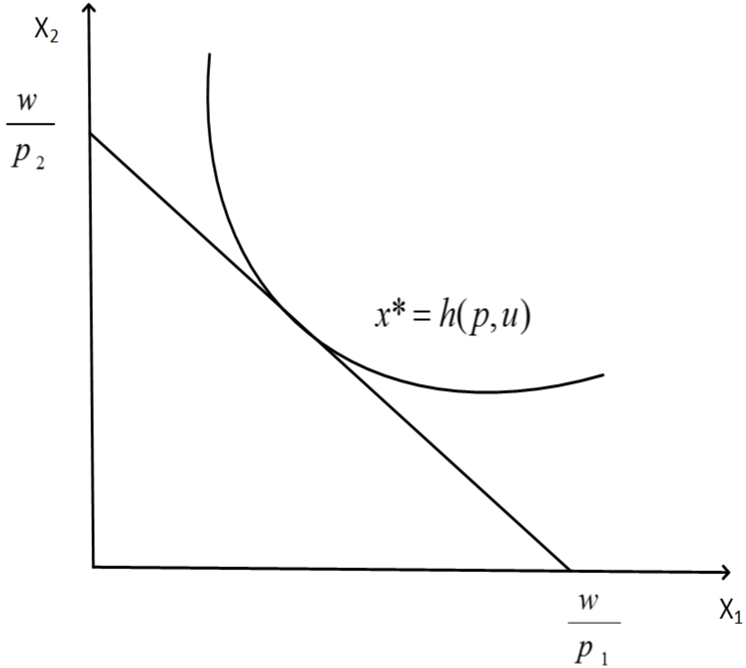


Properties of Hicksian Demand

IF CONVEX, SOLUTION IS UNIQUE

4) ***Uniqueness:***

If the preference relation is strictly convex, then $h(p,u)$ contains a single element.



COMPENSATED DEMAND

PRODUCT MUST BE SO THIS HOLDS FOR EVERY GOOD IN BUNDLE

$p' > p$ OR $p' < p$

$>$ $\quad\quad\quad\quad\quad\quad\quad\quad$ <0

$$(p'-p) \cdot [h(p',w) - h(p,w)] \leq 0$$

<0 $\quad\quad\quad\quad\quad\quad\quad\quad$ >0

IF P INCR HICK DEMAND INCR.

SAMEONE IN OPPOSITE DIRECTION

Properties of Hicksian Demand

- ***Compensated Law of Demand:*** for any change in prices p and p',

$$(p' - p) \cdot [h(p', u) - h(p, u)] \leq 0$$

  - *Implication*: for every good k,

$$(p'_k - p_k) \cdot [h_k(p', u) - h_k(p, u)] \leq 0$$

  - This is true for Hicksian (also named "compensated") demand, but not necessarily true for Walrasian demand (which is uncompensated). This means that movements in prices and movements in quantities must go in **opposite direction**.
    - The following will be clear later, when we introduce income and substitution effects:
    - Recall the figures on Giffen goods, where a decrease in p_k in fact decreases $x_k(p, u)$ when wealth was left uncompensated.
    - Meaning: changes in prices and changes in compensate demand always go in opposite directions (if price increases demand falls, if price falls demand increases)

$p \cdot x \to$ EXPENDITURE

SUBSTITUTE

$x(p, w)$

The Expenditure Function

- Plugging the result from the EMP, $h(p, u)$, into the objective function, $p \cdot x$, we obtain the value function of this optimization problem,

$$p \cdot h(p, u) = e(\underline{p, u})$$

THIS IS CALLED EXPENDITURE FUNCTION

where $e(p, u)$ represents the ***minimal expenditure*** that the consumer needs to incur in order to reach utility level u when prices are p.

This is called expenditure function.

Properties of Expenditure Function

- Suppose that $u(\cdot)$ is a continuous function, satisfying LNS defined on $X = \mathbb{R}_+^L$. Then for $p \gg 0$, $e(p, u)$ satisfies:

$e(\alpha p, u) = (\alpha p) h(\alpha p, u) = \alpha\, p \cdot h(p, w) =$

$= \alpha \cdot e(p, w)$ -> This is homog of deg 1.

1) Homog(1) in p,

$\alpha > 0$

$$e(\alpha p, u) = (\alpha p) h(\alpha p, u) = \alpha[p \cdot h(p, u)] = \alpha \cdot e(p, u)$$

for any p, u, and $\alpha > 0$.

- We know that the optimal bundle is not changed when all prices change, since the optimal consumption bundle in $h(p, u)$ satisfies homogeneity of degree zero.
- Such a price change just makes it more or less expensive to buy the same bundle.

Properties of Expenditure Function

→ INCR UTILITY CAN MOVE TO HIGH I.C. SO BUDGET WILL BE HIGHER

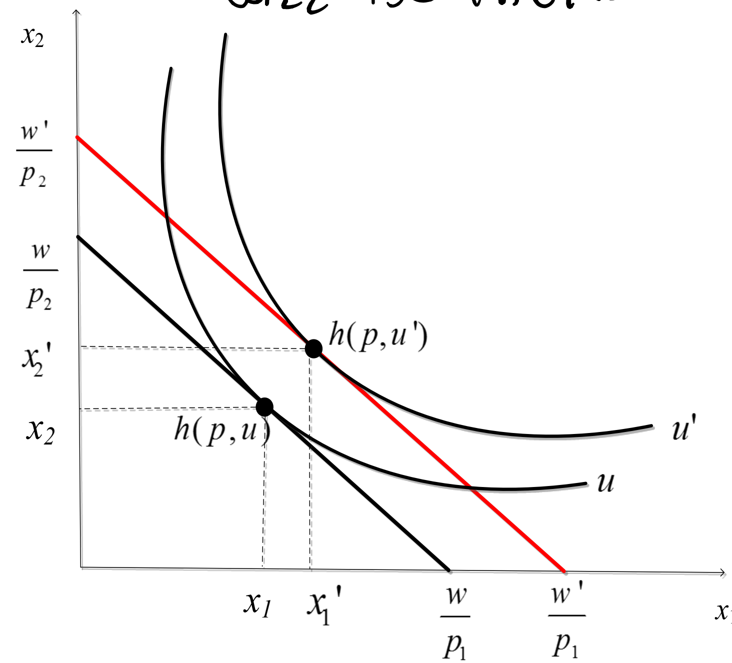
2) *Strictly increasing in u:* For a given price vector, reaching a higher utility requires higher expenditure:

$$p_1x_1' + p_2x_2' > p_1x_1 + p_2x_2$$

where $(x_1, x_2) = h(p,u)$ and $(x_1', x_2') = h(p,u')$.

Then,

$$e(p,u') > e(p,u)$$



IF P↑, MINIMAL EXP ↑

Properties of Expenditure Function

3) *Non-decreasing in p_k for every good k:*

Higher prices mean higher expenditure to reach a given utility level.

- Let $p' = (p_1, p_2, \dots, p'_k, \dots, p_L)$ and $p = (p_1, p_2, \dots, p_k, \dots, p_L)$, where $p'_k > p_k$.
- Let $x' = h(p', u)$ and $x = h(p, u)$ from EMP under prices p' and p, respectively.
- Then, $p' \cdot x' = e(p', u)$ and $p \cdot x = e(p, u)$. No

SI MC

$$e(p', u) = p' \cdot x' \;\geq\; p \cdot x' \;\geq\; p \cdot x = e(p, u)$$

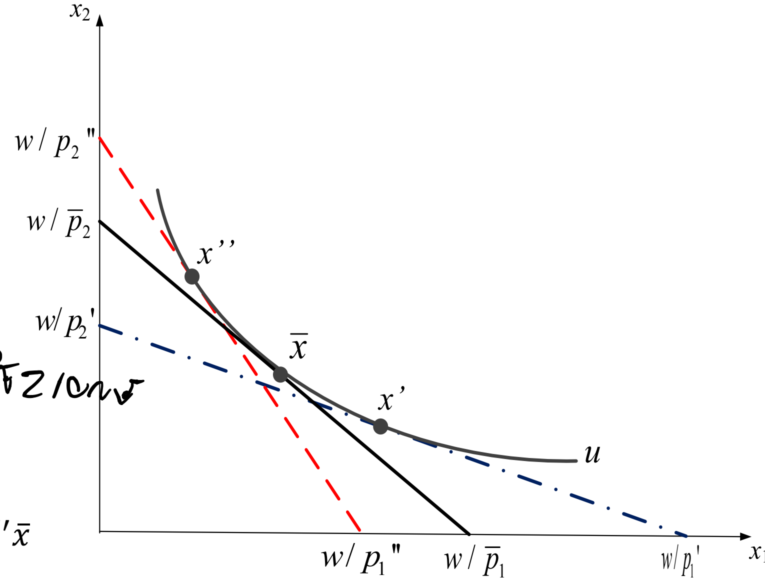
- 1st inequality due to $p' \geq p$
- 2nd inequality: at prices p, bundle x minimizes EMP.

Properties of Expenditure Function

4) Concave in p:

Let $x' \in h(p',u) \Rightarrow p'x' \leq p'x$
$\forall x \neq x'$, e.g., $p'x' \leq p'\bar{x}$
and
$x'' \in h(p'',u) \Rightarrow p''x'' \leq p''x$
$\forall x \neq x''$, e.g., $p''x'' \leq p''\bar{x}$
where $\bar{x} = \alpha x' + (1-\alpha)x''$

NO DIMOSTRAZIONE



This implies

$$\alpha p'x' + (1-\alpha)p''x'' \leq \alpha p'\bar{x} + (1-\alpha)p''\bar{x}$$

$$\alpha \overbrace{p'x'}^{e(p',u)} + (1-\alpha) \overbrace{p''x''}^{e(p'',u)} \leq [\underbrace{\alpha p' + (1-\alpha)p''}_{\bar{p}}]\bar{x}$$

$$\alpha e(p',u) + (1-\alpha)e(p'',u) \leq e(\bar{p},u)$$

as required by concavity

Connections

Relationship between the Expenditure and Hicksian Demand

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_+^L$. For all p and u,

WE CAN DERIVE HICKSIAN FROM EXPENDITURE

$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) \text{ for every good } k$$

This identity is "***Shepard's lemma***": if we want to find $h_k(p,u)$ and we know $e(p,u)$, we just have to differentiate $e(p,u)$ with respect to prices.

IN EXERCISES HE COULD GIVE US EXPENDITURE IN MINIMUM → WE CAN COMPUTE HICKSIAN USING DER IN RESPECT TO p_k

- *Proof*: three different approaches
    1) the support function
    2) first-order conditions
    3) the envelope theorem (See Appendix 2.2)

Proof of Shephard's lemma (using "Envelope theorem")

$$e(p,u) = \min_{x \geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

To see how e(.) changes when a parameter p_k changes we can use the Langrangian

$L = -(p \cdot x) + \mu(u(x) - u)$ (remember we set it as a max problem)

In particular

$$\frac{\partial e(p,u)}{\partial p_k} = -\left[\frac{\partial L}{\partial p_k}|_{x=x^*(p)}\right] = -\frac{\partial[-p \cdot x(p)) + \mu(u\big(x(p)\big) - u]}{\partial p_k}|_{x=x^*(p)}$$

$$= -[-x_k(p) - p\frac{\partial x}{\partial p_k} + \mu\frac{\partial u}{\partial x}\frac{\partial x}{\partial p_k}]|_{x=x^*(p)}$$

But $-p + \mu\frac{\partial u}{\partial x} = 0$ from FOCs then $\frac{\partial e(p,u)}{\partial p_k} = x_k(p)|_{x=x^*(p)} = h_k(p,u)$

(NB. p, $x(p)$, $\frac{\partial u}{\partial x} = \nabla u\big(x(p)\big)$, $\frac{\partial x}{\partial p_k} = D_{p_k}x(p)$ are vectors, while μ a scalar)

- Take opposite
- Write lagrangian

The opt will be x* so computing minimum deriving la grandina in respect of pk. The values of the problem computed in the opt should be the same. I take der of Exp in respect to pk that will be der of L with respect to pk.

Next i take a minus since I translated the min problem in the max problem.
X is a function of p since if we change p then will change the opt solution that is x*. We write x as a function of p. Also x is a function in p computing the derivative.

With a composite function we have first to derive in respect to the second function moltiply by the derive the second function in respect with the parameter par.

$$\frac{\partial u}{\partial x(p)}(x\ p) \cdot \frac{\partial x(p)}{\partial p}$$

$$\frac{\partial x}{\partial p_k}\left(-p + \mu \frac{\partial u}{\partial x}\right)$$ LOOK AT EMP IN WHICH $p = \mu \frac{\partial u}{\partial x}$

$$\frac{\partial x}{\partial p_k}(\overset{0}{-p + p})$$ SO $h_k(p, u)$

THIS IS ENV THEOREM

If we have opt problem you can forget all the der involving the constraint, you can just derive in the Expedncicture function the part that is related to the objective function.

Relationship between Hicksian and Walrasian Demand

- We can formally relate the Hicksian and Walrasian demand as follows:
  - Consider $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_+^L$.
  - Consider a consumer facing $(\bar{p}, \bar{w})$ and attaining utility level $\bar{u}$ (i.e. solution of UMP) [SOLUTION OF UMP]
  - Note that $\bar{w} = e(\bar{p}, \bar{u})$, i.e. the min expenditure that the consumer bear to reach utility $\bar{u}$ is $\bar{w}$. In addition, we know that for any (p, u), $h_l(p, u) = x_l(p, \underbrace{e(p, u)}_{w})$. Differentiating this expression with respect to p_k, and evaluating it at $(\bar{p}, \bar{u})$, we get:

$\bar{u}$ max UMP $x(\bar{p}, \bar{w})$ $\bar{w}$

COMPOSITE FUNCTION

$$\frac{\partial h_l(\bar{p}, \bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial p_k} + \boxed{\frac{\partial x_l(\bar{p}, e(\bar{p}, \bar{u}))}{\partial e(\bar{p}, \bar{u}))} \boxed{\frac{\partial e(\bar{p}, \bar{u})}{\partial p_k}}}$$

Utilty maximisation problem: How much consumer spend in opt solution in this UMP?
W (barrato). So u(bar) is the max utility in UMP.
To reach u maximising u and the level of wealth then it must be the case is the w(bar)

[illegible] EQUAL THEN HICKSIAN DEMAND

I have p bar and w bar. Reach level of utilty b bar and w bar. What is the expenditure of this walras demand? Is the w bar. Now I'm saying, what is the min exp to reach

$$h_e(\bar{p}, \bar{w}) = x_e(\bar{p}, \bar{e}(\bar{p}, \bar{w}))$$

WALRAS DEMAND ON P AND W. REPLACE WITH IDENTITY [illegible]
AND THIS IS $\bar{w}$

WE HAVE EXP OPT PROBLEM. IMAGINE TO SOLVE [illegible]
GIVING MAXIMUM, SO THIS IS UTILITY

$\ell > k$

$\ell \neq k$

Relationship between Hicksian and Walrasian Demand

- Using the fact that $\frac{\partial e(\bar{p},\bar{u})}{\partial p_k} = h_k(\bar{p},\bar{u})$ (Shepard's lemma),

(SHEPPARD LEMMA)

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial p_k} + \frac{\partial x_l(\bar{p},e(\bar{p},\bar{u}))}{\partial e(\bar{p},\bar{u}))} h_k(\bar{p},\bar{u})$$

I CAN REPLACE $\bar{w}$ WITH EXPENDITURE

- Finally, since $\bar{w} = e(\bar{p},\bar{u})$ and $h_k(\bar{p},\bar{u}) = x_k\big(\bar{p},e(\bar{p},\bar{u})\big) = x_k(\bar{p},\bar{w})$, then

$$\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k} = \frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k} + \frac{\partial x_l(\bar{p},\bar{w})}{\partial \bar{w}} x_k(\bar{p},\bar{w})$$

HOW I CAN LINK HICKSIAN DEM W/ IN WALRAS DEMAND

I CAN REPLACE e WITH w

Slutsky equation correspond to total effect and income effect. So

$$\underbrace{\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k}}_{SE} = \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k}}_{TE} + \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial w} x_k(\bar{p},\bar{w})}_{IE}$$

$$\underbrace{\frac{\partial x_l}{\partial p_k}}_{TE} = \frac{\partial h_l}{\partial p_k} - \underbrace{\frac{\partial x_l}{\partial w} \cdot x_k}_{IE}$$

Tot diff of the two

Relationship between Hicksian and Walrasian Demand

- This is the so called **Slutsky equation:** The total effect of a price change on Walrasian demand can be decomposed into a substitution effect and an income effect:

$$\underbrace{\frac{\partial h_l(\bar{p},\bar{u})}{\partial p_k}}_{SE} = \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial p_k}}_{TE} + \underbrace{\frac{\partial x_l(\bar{p},\bar{w})}{\partial \bar{w}} x_k(\bar{p},\bar{w})}_{IE}$$

Or, more compactly, $SE = TE + IE$ or TE = SE - IE

Where SE = **substitution effect**

TE = **total effect**

IE = **income effect**

TE, IE, SE

- ***Total Effect:*** measures how the quantity demanded is affected by a change in the price of good l, when we leave the wealth uncompensated (Walras demand is also called **uncompensated demand**).
- ***Substitution Effect***: measures how the quantity demanded is affected by a change in the price of good l, after the wealth adjustment which allows the consumer to reach the same utility as before the price change. Is given by Hicksian demand that is also called **compensated demand**.
  - That is, the substitution effect only captures the change in demand due to variation in the price ratio, but abstracts from the larger (smaller) purchasing power that the consumer experiences after a decrease (increase, respectively) in prices.
- ***Income Effect:*** measures the change in the quantity demanded as a result of the wealth adjustment.

IF P ONE GOOD MORE EXPENS

THEN I WILL BUY LESS SECOND GOOD

Rent increase I'll go to the second house, i consume a little bit houses. This means that we are left with less income to buy less good. So increasing price of one good will reduce the consumption of others good even if you don't change the consumption of one good.

Inflation is an exemple. If i consume the same bundle ...[1.31]
So this is the income effect.

Substitution effect relate to the fact of compensate the Hicksian demand. When computing Hicksian demand we gave a utilty level .. to the price before. How the bundle changes when we keep the consumer in the same IC as the prices changes. So neutralising the effect on well. Slutsky ...

This is the slutsky equation. In the left wee have SE that is the change in Hicksian demand. The change in the Hicksian demand depend in the price change = TE + IE.

We can compute this effect for each good: der first good with respect to price of first good or der in the quantity and changing price. With 2 good we have 4 derivatives. This 4 can be put in a matrix called Slutsky matrix.

A generic term slk(p,w)

Slutsky matrix

- All these derivatives can be collected into a matrix, the so called ***Slutsky (or substitution) matrix***

$$S(p,w) = \begin{bmatrix} s_{11}(p,w) & \cdots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \cdots & s_{LL}(p,w) \end{bmatrix}$$

Cross price effect out of the main diagonal

where each element in the matrix is

new w effect

$$s_{lk}(p,w) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

$$\frac{\partial h_l}{\partial p_k}$$

Implications of WARP: Slutsky Matrix

Just know this
about WARP

- ***Proposition:*** If preferences satisfy LNS and strict convexity, and they are represented with a continuous utility function, then the Walrasian demand $x(p, w)$ generates a Slutsky matrix, $S(p, w)$, which is symmetric.

- The above assumptions are really common.
  - Hence, the Slutsky matrix will then be symmetric.

- However, the above assumptions are not satisfied in the case of preferences over perfect substitutes (i.e., preferences are convex, but not strictly convex).

Implications of WARP: Slutsky Equation

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect}} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\text{Total effect}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\text{Income effect}}$$

- ***Total Effect:*** measures how the quantity demanded is affected by a change in the price of good l, when we leave the wealth uncompensated.
- ***Income Effect:*** measures the change in the quantity demanded as a result of the wealth adjustment.
- ***Substitution Effect***: measures how the quantity demanded is affected by a change in the price of good l, after the wealth adjustment.
  - That is, the substitution effect only captures the change in demand due to variation in the price ratio, but abstracts from the larger (smaller) purchasing power that the consumer experiences after a decrease (increase, respectively) in prices.

Why is useful to decompose total effect changing? We see how quantity changes depending on the characteristics of the goods.

Implications of WARP: Slutsky Matrix

If weak (WARP) .. holds then substitution effect is negative —> Hicksian demand decrease

- Let us focus now on the signs of the IE and SE (implied by WARP, i.e. of the utilities that we will use) in case of $P_l \uparrow$
- Non-positive substitution effect, $s_{ll} \leq 0$:

WALRASIAN ↑ ⇒ NEGATIVE

SE always non positive

↓

$$\underbrace{s_{ll}(p,w)}_{\text{substitution effect }(-)} = \underbrace{\frac{\partial x_l(p,w)}{\partial p_l}}_{\substack{\text{Total effect:}\\ (-)\text{ usual good}\\ (+)\text{ Giffen good}}} + \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_l(p,w)}_{\substack{\text{Income effect:}\\ (+)\text{ normal good}\\ (-)\text{ inferior good}}}$$

Giffen: if price increase, demand increases so this derivative increase

- Substitution Effect = Total Effect + Income Effect

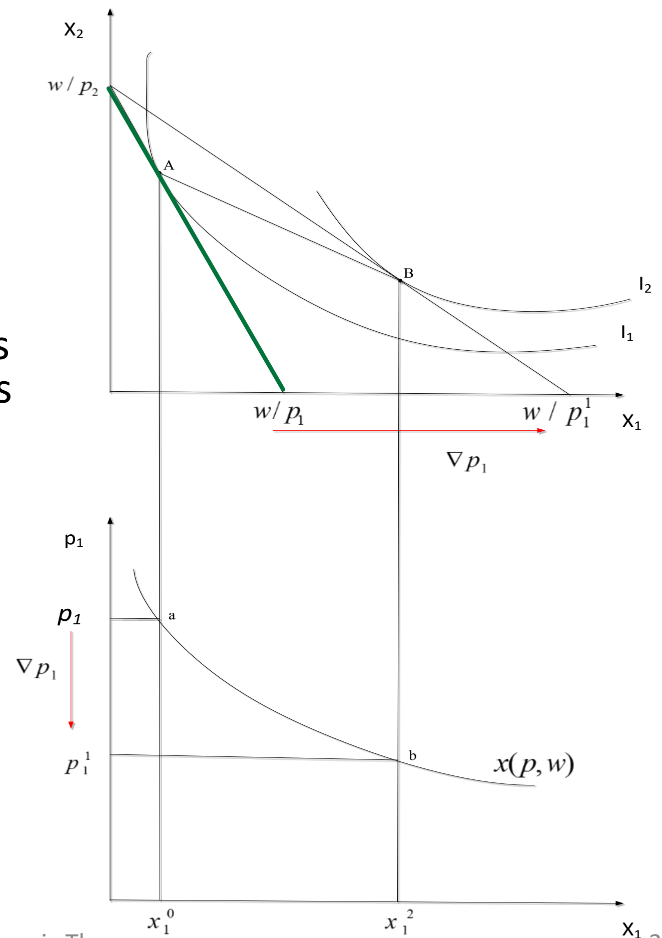
$\Rightarrow$ Total Effect = Substitution Effect - Income Effect

If SE decrease and TE positive mean that IE should be negative and greater than TE. So x(p,w) should be > 0 so derivative is negative. Giffen good can only be inferior good by definition. But not only inferior good are Giffen. If income increase i call it normal.

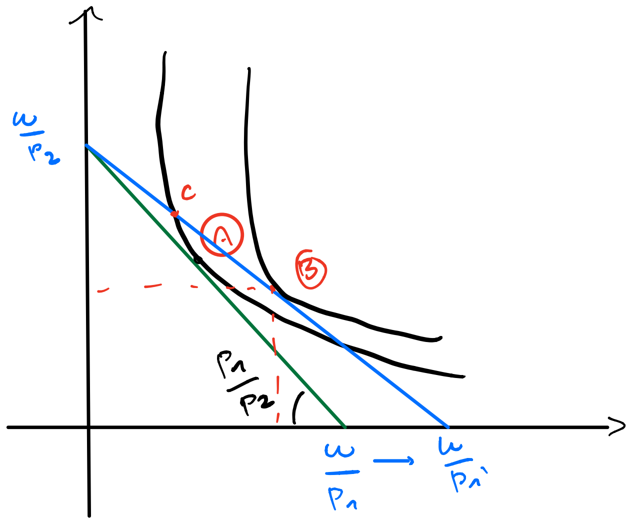
Decompose the two effect graphically.

Graphical representation: Slutsky Equation

- Reduction in the price of x_1.
  - It enlarges consumer's set of feasible bundles.
  - He can reach an indifference curve further away from the origin.
- The Walrasian demand curve indicates that a decrease in the price of x_1 leads to an increase in the quantity demanded.
  - This induces a negatively sloped Walrasian demand curve (so the good is "normal").
- The increase in the quantity demanded of x_1 as a result of a decrease in its price represents the ***total effect (TE)***.



We start from a given budget constraint with price p1. The solution of consumer problem is the tangency point between the IC and the budget constraint. We call this point A.



• IF $p_1 \downarrow$: p_1'

→ x_1 USUAL GOOD SINCE QUANTITY INCR.

IF C NEW TANG POINT, THEN WE HAVE GIFFEN GOOD ($p \uparrow$, DEMAND $\uparrow$)

GIFFEN GOOD

$p \uparrow$ $q \uparrow$

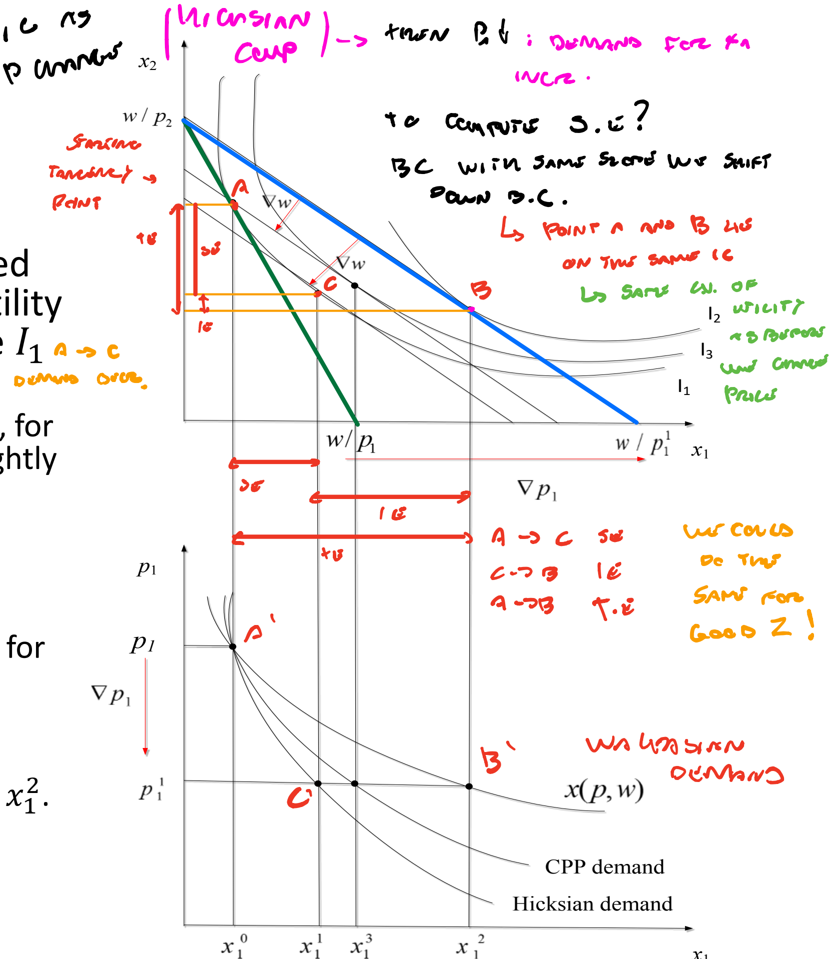
→ P AND Q GO IN THE SAME DIRECTION

$p \downarrow$ $q \downarrow$

Graphical representation: Slutsky Equation

same IC as before p changes

- Reduction in the price of x_1.
  - ***Hicksian wealth compensation*** (i.e., "constant utility" demand curve)?
- The consumer's wealth level is adjusted so that she can still reach her initial utility level (i.e., the same indifference curve I_1 as before the price change).
  - The Hicksian demand curve reflects that, for a given decrease in p_1, the consumer slightly increases her consumption of good one.
- In summary, a given decrease in p_1 produces:
  - A small increase in the Hicksian demand for the good, i.e., from x_1^0 to x_1^1.
  - (neglect CCP)
  - A substantial increase in the Walrasian demand for the product, i.e., from x_1^0 to x_1^2.



FOCUS ON GOOD x_1 AND COMPARE A AND C.

A TO B FOR WALRASIAN DEMAND

A TO C IS HICKSIAN DEMAND

FOR WALRASIAN DEMAND THE CHANGING DEMAND IS LARGER

⇓ WHAT DOES IT IMPLY?

POINT C' IS LOWER THAN A' AND B'

SO WALRASIAN DEMAND IS HIGHER

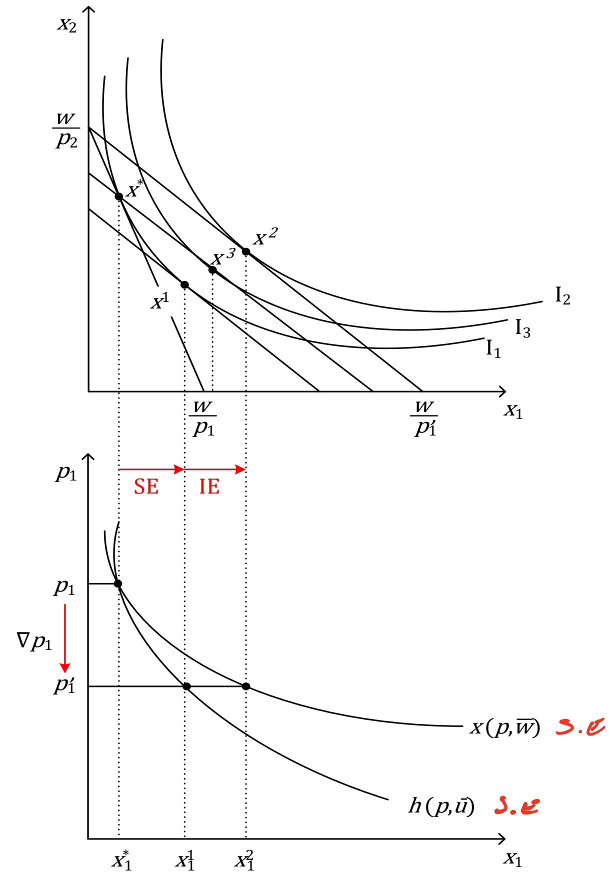
Implications of WARP: Slutsky Equation

- A decrease in price of x_1 leads the consumer to increase his consumption of this good, Δx_1, but:
  - The Δx_1 which is solely due to the price effect (measured by the Hicksian demand curve) is smaller than the Δx_1 measured by the Walrasian demand, $x(p, w)$, which also captures wealth effects.

Relationship between Hicksian and Walrasian Demand

$P_1 \downarrow$: $h \uparrow$ $x \uparrow$

- When income effects are positive (*normal goods*), then the Walrasian demand $x(p, w)$ is **above** the Hicksian demand $h(p, u)$.
  - The Hicksian demand is *steeper* than the Walrasian demand.



$P_1 \downarrow$: IE SE

YOU ARE RICHER
SINCE PURCHASING POWER
INCREASES (INFLATION)

S.E ALWAYS DRIVE CONSUMPTION UP
IF GOOD NORMAL

IF GOOD INFERIOR P↓ D↓
I.E DRIVE CONSUMPTION DOWN

IF GOOD IS INCREASING DIR OF SE
WILL BE HICKSIAN DEMAND ABOVE
OR BELOW WALRASIAN?
ABOVE

Relationship between Hicksian and Walrasian Demand

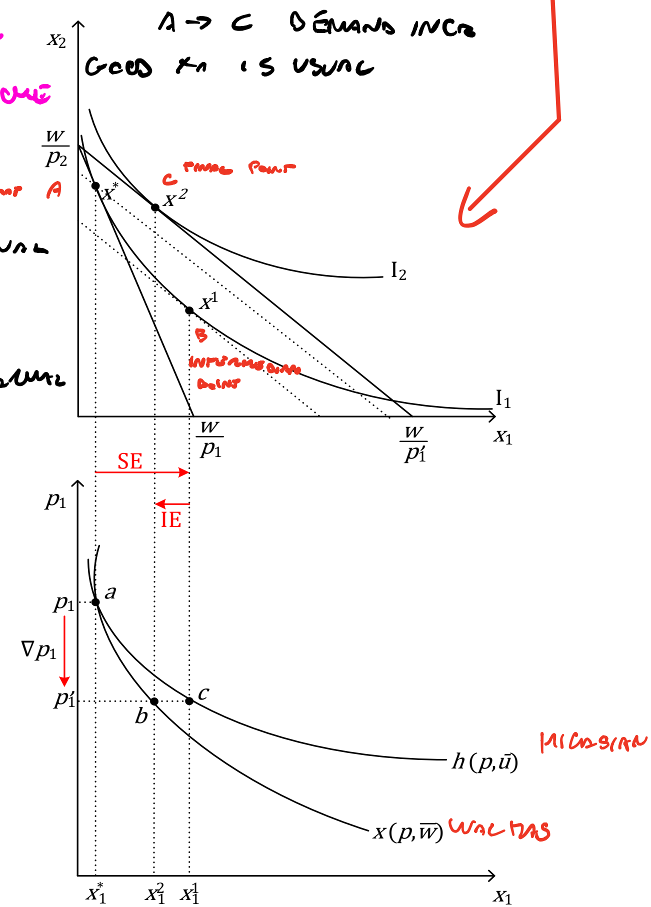
- When income effects are negative (*inferior goods*), then the Walrasian demand $x(p, w)$ is **below** the Hicksian demand $h(p, u)$.
  - The Hicksian demand is *flatter* than the Walrasian demand.

USUAL PRICE
NORMAL INCOME

$\frac{\delta x_1}{\delta p_1} < 0$ USUAL

$\frac{\delta x_1}{\partial w} > 0$ NORMAL

HENCE $p \downarrow$ D $\uparrow$ SO



A → B SUBSTITUTION EFFECT (CONSUMPT INC)

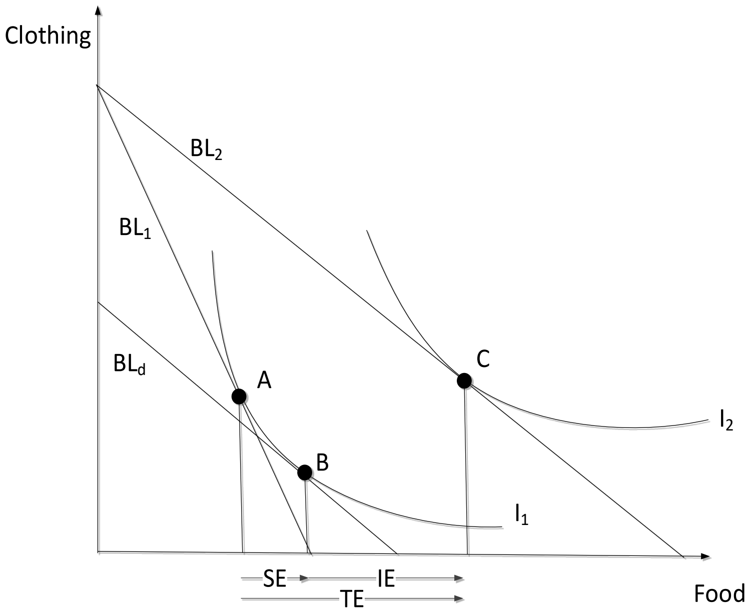
B → C INCOME EFFECT (CONSUMP DECRE)

SE IE ↓ GOOD IS INFERIOR → SE GO CONTRARY TO IE

HICKSIAN > WALRAS

Substitution and Income Effects: Normal Goods

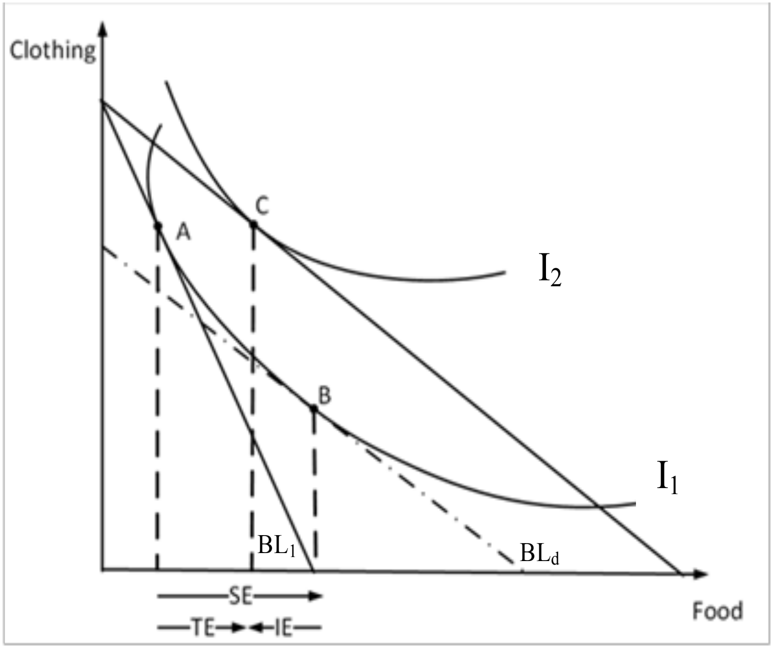
- Decrease in the price of the good in the horizontal axis (i.e., food).
- The substitution effect (SE) moves in the opposite direction as the price change.
  - A reduction in the price of food implies a positive substitution effect.
- The income effect (IE) is positive (thus it reinforces the SE).
  - The good is normal.



SE, IE SAME DIRECTION

Substitution and Income Effects: Inferior Goods

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is now negative (which partially offsets the increase in the quantity demanded associated with the SE).
  - The good is inferior.
- *Note*: the SE is larger than the IE (Law of price still holds)

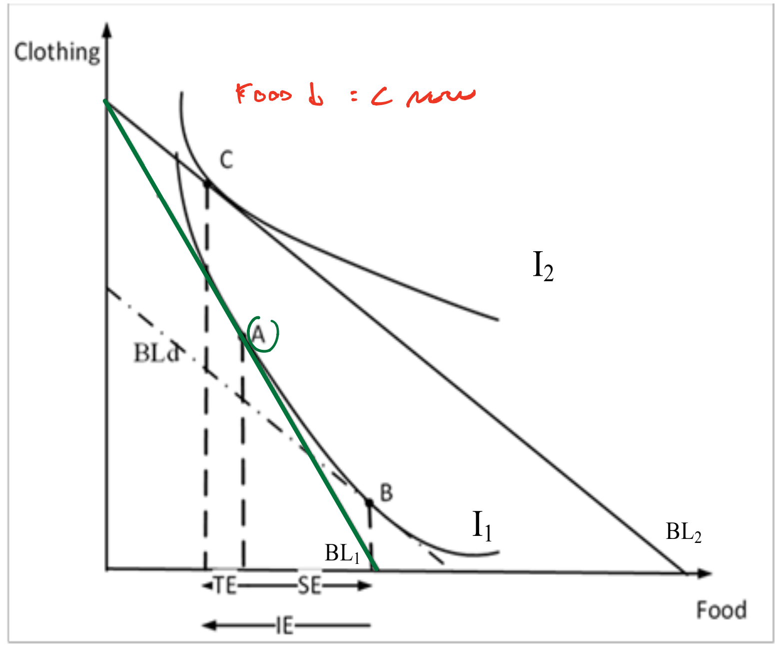


TE < SE

Substitution and Income Effects: Giffen Goods

→ WHEN YOU ARE POOR YOU CONSUME IT A LOT!

- Decrease in the price of the good in the horizontal axis (i.e., food).
- The SE still moves in the opposite direction as the price change.
- The income effect (IE) is still negative but now completely offsets the increase in the quantity demanded associated with the SE.
  - The good is Giffen good.
- *Note*: the SE is less in absolute value than the IE (Law of price does not hold)



FOOD ↓ : C NOW

FOOD IS GIFFEN GOOD.

IN POINT B COMPENSATION DEMAND

TE NEGATIVE.

WHEN YOU GET RICHER YOU CONSUME IS VERY LOW

Substitution and Income Effects (e.g. effects P_l on x_l)

IE P↓

	SE	IE	TE
Normal Good	↑	↑	↑
Inferior Good	↑	↓	↑
Giffen Good	↑	↓	↓

- Not Giffen: Demand curve is negatively sloped (as usual)
- Giffen: Demand curve is positively sloped

Substitution and Income Effects

- ***Summary:***
  1) SE is negative (since $\downarrow p_1 \Rightarrow \uparrow x_1$, they move in opposite directions)
     - SE < 0 does not imply $\downarrow x_1$ just implies that **the two move in opposite directions**
  2) If good is inferior, IE < 0. Then,

$$\mathrm{TE} = \underbrace{\mathrm{SE}}_{-} \underbrace{- \underbrace{\mathrm{IE}}_{-}}_{+} \quad \Rightarrow \quad \text{if } \ |\mathrm{IE}| \begin{Bmatrix} > \\ < \end{Bmatrix} |\mathrm{SE}|\text{, then } \begin{Bmatrix} \mathrm{TE}(+) \\ \mathrm{TE}(-) \end{Bmatrix}$$

For a price decrease $\downarrow p_1$, this implies

$$\begin{Bmatrix} \mathrm{TE}(+) \\ \mathrm{TE}(-) \end{Bmatrix} \Rightarrow \begin{Bmatrix} \downarrow x_1 \\ \uparrow x_1 \end{Bmatrix} \qquad \begin{matrix} \text{Giffen good} \\ \text{Non-Giffen good} \end{matrix}$$

  3) Hence,
     a) A good can be inferior, but not necessarily be Giffen
     b) But all Giffen goods must be inferior.

NB. The signs of SE and IE are opposite if we consider $\downarrow p_1$ or $\uparrow p_1$

Relationship between the Expenditure and Hicksian Demand

- The relationship between the Hicksian demand and the expenditure function $\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$ can be further developed by computing the second derivative. That is,

$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_k} = \frac{\partial h_k(p,u)}{\partial p_k}$$

or

$$D_p^2 e(p,u) = D_p h(p,u)$$

- Since $D_p h(p,u)$ provides the Slutsky matrix, $S(p,w)$, then

$$S(p,w) = D_p^2 e(p,u)$$

Thus the **Slutsky matrix can be obtained from the observable Walrasian demand (rather than from the unobservable Hicksian or compensated demand)**.

Relationship between Walrasian Demand and Indirect Utility Function

- Let's assume that $u(\cdot)$ is a continuous function, representing preferences that satisfy LNS and are strictly convex and defined on $X = \mathbb{R}_+^L$. Suppose also that $v(p, w)$ is differentiable at any $(p, w) \gg 0$. Then,

$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p, w) \text{ for every good } k$$

RESULT IS WALRASIAN DEMAND

- This is ***Roy's identity*** (I don't do this proof, is ex. 28 Ch. 2)
- Powerful result, since in many cases it is easier to compute the derivatives of $v(p, w)$ than solving the UMP with the system of FOCs. Hint. Having the indirect utility function allows you to derive the Walrasian demand functions.

IUtilty is walrasian demand on maximum.
Roy identity to derive walrasian demand just computation ratio of the two derivative.
[1.02

Taking stock: Summary of Relationships

- The Walrasian demand, $x(p, w)$, is the solution of the UMP.
  - Its value function is the indirect utility function, $v(p, w)$.
- The Hicksian demand, $h(p, u)$, is the solution of the EMP.
  - Its value function is the expenditure function, $e(p, u)$.

Summary of Relationships

The UMP | **The EMP**

$$
\begin{tikzcd}
x(p,w) \arrow[d, "(1)"'] & & h(p,u) \arrow[d, "(1)"] \\
v(p,w) & & e(p,u)
\end{tikzcd}
$$

Summary of Relationships

- Relationship between the value functions of the UMP and the EMP (lower part of figure):

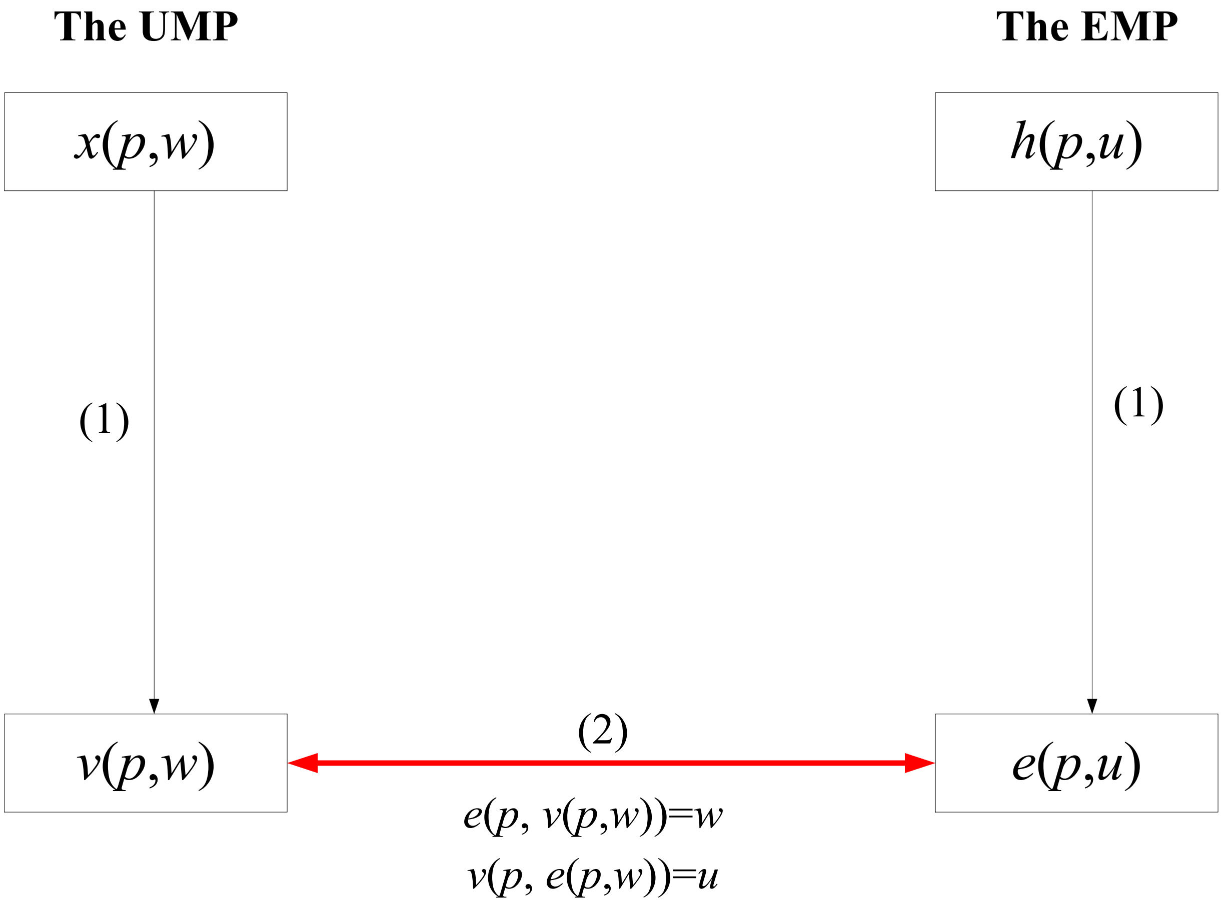
  EMP
  - $e(p, v(p, w)) = w$, i.e., the minimal expenditure needed in order to reach a utility level equal to the maximal utility that the individual reaches at her UMP, $u = v(p, w)$, must be w.

  UMP
  - $v(p, e(p, u)) = u$, i.e., the indirect utility that can be reached when the consumer is endowed with a wealth level w equal to the minimal expenditure she optimally bear in the EMP, i.e., $w = e(p, u)$, is exactly u.

In the expenditure prices and utilty in constraint. Since EMP the expenditure function will be function of price and utilty.

IUF depends on wealth and price.
What maximise price p and wealth w. When we give max utilty level in price p and wealth w and by definition is w.
We can do the same with Indirect utilty function.

Summary of Relationships



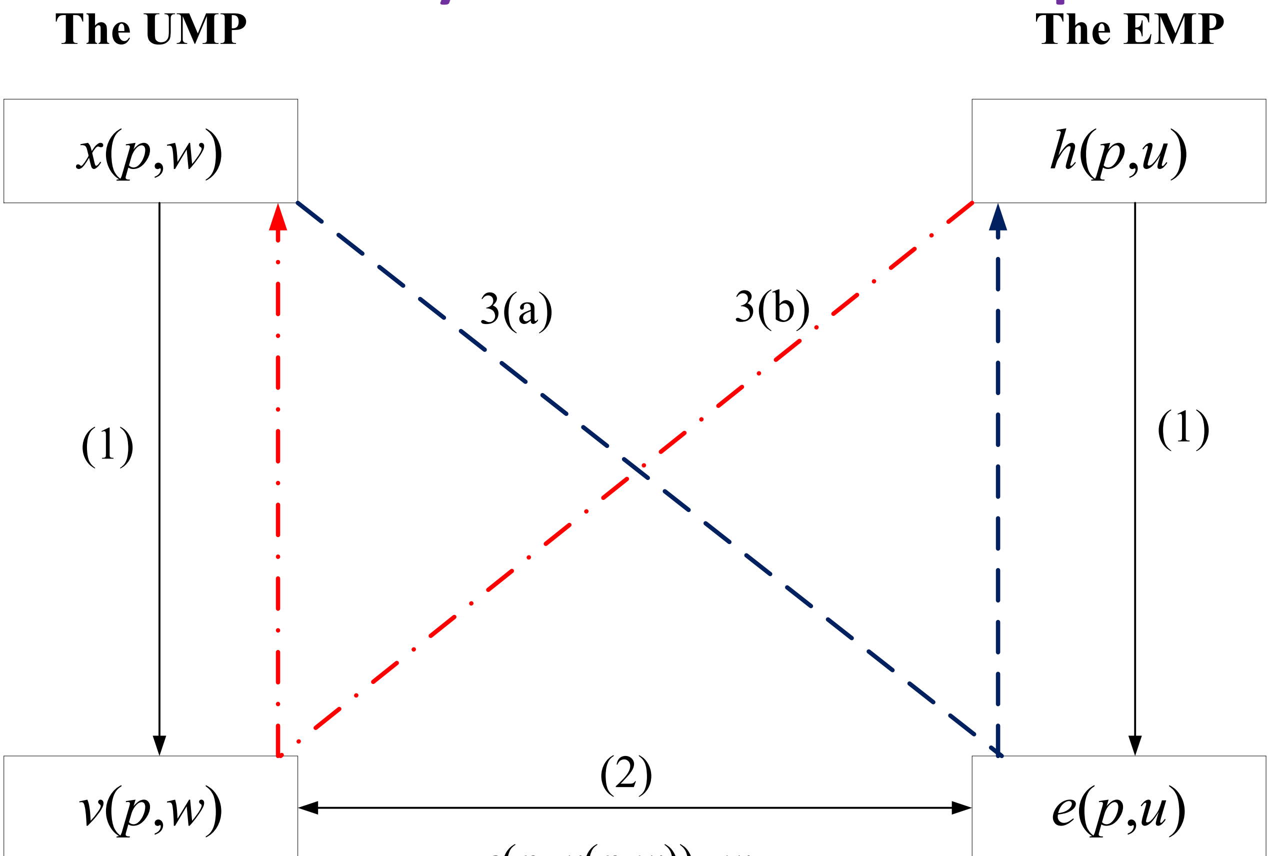
Summary of Relationships

- Relationship between the argmax of the UMP (the Walrasian demand) and the argmin of the EMP (the Hicksian demand):
  - $x(p, e(p, u)) = h(p, u)$, i.e., the (uncompensated) Walrasian demand of a consumer endowed with an adjusted wealth level w (equal to the expenditure she optimally bear in the EMP), $w = e(p, u)$, coincides with his Hicksian demand, $h(p, u)$.
  - $h(p, v(p, w)) = x(p, w)$, i.e., the (compensated) Hicksian demand of a consumer reaching the maximum utility of the UMP, $u = v\ (p, w)$, coincides with his Walrasian demand, $x(p, w)$.

H = W if UMP

W = H if EMP

Summary of Relationships



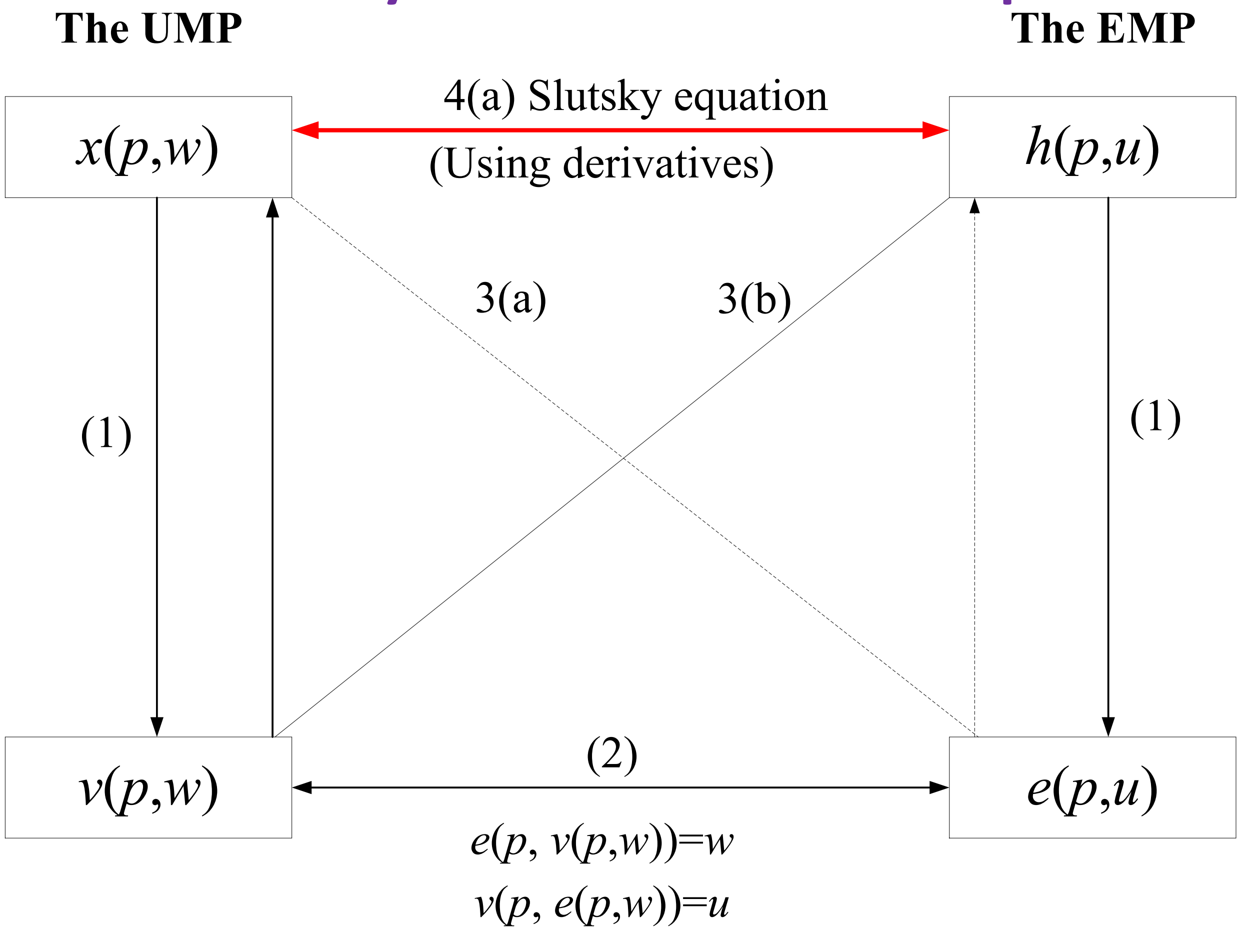
$e(p, v(p,w)) = w$

$v(p, e(p,w)) = u$

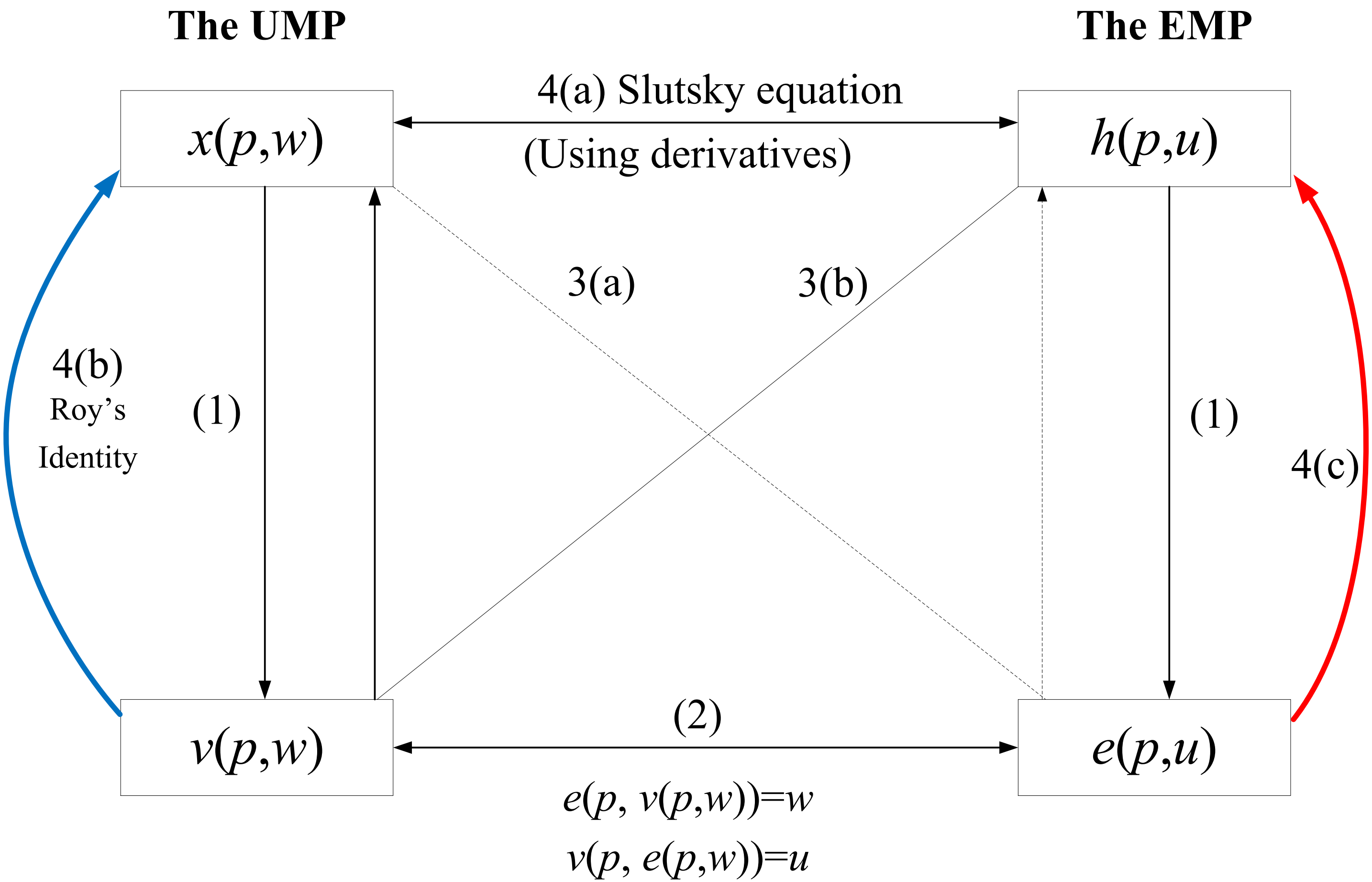
Summary of Relationships

- Finally, we can also use:
  - The ***Slutsky equation***:
$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$
to relate the derivatives of the Hicksian and the Walrasian demand.
  - ***Shepard's lemma***:
$$\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u)$$
to obtain the Hicksian demand from the expenditure function.
  - ***Roy's identity***:
$$-\frac{\frac{\partial v(p,w)}{\partial p_k}}{\frac{\partial v(p,w)}{\partial w}} = x_k(p,w)$$
to obtain the Walrasian demand from the indirect utility function.

Summary of Relationships



Summary of Relationships



Take away

- It is time to study hard guys!!!

- To defuse Micro:

A physicist, a chemist and an economist are stranded on an island, with nothing to eat. A can of soup washes ashore.
The physicist says, "Lets smash the can open with a rock."
The chemist says, "Lets build a fire and heat the can first."
The economist says, "Lets assume that we have a can-opener..."

ESERCIZI

EX. 1) FINDING WALRASIAN DEMAND

UTILITY = $\ln x_1 + x_2$ $\qquad p_1 x_1 + p_2 x_2 \leq w$ BUDGET CONSTRAINT

WALRASIAN → MAX $\ln x_1 + x_2$

SUCH THAT $p_1 x_1 + p_2 x_2 \leq w$
AS BUDGET CONSTRAINT

$$\mathcal{L} = \ln x_1 + x_2 + \lambda (w - p_1 x_1 - p_2 x_2)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} + \lambda p_1 \leq 0 \qquad x_1 \geq 0 \text{ WITH CS}$$

$$\frac{\partial L}{\partial x_2} = 1 - p_2 \lambda \leq 0 \qquad x_2 \geq 0 \text{ WITH CS}$$

$$\frac{\partial L}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0$$

INTERIOR SOLUTION → FOCUS ON 1° DERIVATIVE x_1, x_2 NEEDS

$$\begin{cases} \frac{1}{x_1} = \lambda p_1 \\ 1 = p_2 \lambda \end{cases} \qquad x_1^* = \frac{p_2}{p_1}$$

OPTIMAL SINCE ONLY x_1 IS IN THIS EQUATION

WALRASIAN DEMAND FOR x_2 → REPLACE x_1 IN BUDGET CONSTRAINT

$$w - p_1 \frac{p_2}{p_1} - p_2 x_2 = 0 \qquad w - p_2 = p_2 x_2$$

$$x_2^* = \frac{w}{p_2} - 1$$

WE FIND WALRASIAN DEMAND → WE SHOULD CHECK FOR CORNER SOLUTION

(IF $x_1 \cdot x_2$ → NO CORNER SINCE $x_1 = 0$ THEN $x_2 = 0$ → NO CORNER)

(IF $x_1 = 0$ FEASIBLE CORNER SOLUTION)

$x_1 = \frac{w}{p_2}$ AND $x_2 = 0$ BUT THIS CANNOT BE OPT SOLUTION.

IN GENERAL IT'S OK TO FOCUS ON INTERIOR SOLUTION AT THE EXAM

2.) NOW FIND IND. UTILITY FUNCTION → UTILITY ON WALRASIAN DEMAND

SAME PARAMETER OF UMP

$$V(p_1, p_2, w) = \ln x_1^* + x_2^* = \ln \frac{p_2}{p_1} + \frac{w}{p_2} - 1 =$$

$$= \ln p_2 - \ln p_1 + \frac{w}{p_2} - 1$$

3.) EXP. FUNCTION → MIN EXPENDITURE TO REACH A GIVEN UTILITY LEVEL

⇓

SOLUTION OF EMP

SO WE MINIMISE

ALSO

EXP FROM IUF

MIN EXPENDITURE TO REACH u (UTILITY LV.) IS THE WEALTH

$$u = \ln p_2 - \ln p_1 + \frac{w}{p_2} - 1 \qquad e(p_1, p_2, u)$$

$$\left[u - \ln p_2 + \ln p_1 + 1\right] p_2 = e(p_1, p_2, u)$$

IN CASE EASY FROM IUF TO ISOLATE W SO WE CAN GET EXPENDITURE FUNCTION

IN CASE $(w^{1/2} + w)$ NOT EASY

4) FIND HICKSIAN DEMAND OF BOTH GOODS

SOLUTION OF EMP BUT IN THIS CASE WE CAN USE

SHEPPARD $\rightarrow \dfrac{\delta \,..}{\delta p}$

CONTINUE OF 1

EXPENDITURE FUNCTION

$$e = p_2 \cdot u - p_2 \ln\left(\frac{p_2}{p_1}\right) + p_1 =$$

$$p_2 \cdot u - p_2 \underbrace{\left[\ln p_2 - \ln p_1\right]}_{\parallel} + p_2$$

$$p_2 \ln p_2 - p_2 \ln p_1$$

$$h_1(p_1, p_2, u),\ h_2(p_1, p_2, u)$$

$$\frac{\partial e}{p_1} = h_1 \qquad \frac{\partial e}{p_2} = h_2$$

- $$\frac{\partial e}{p_1} = \frac{p_2}{p_1} \quad \rightsquigarrow \quad -p_2 \cdot \left(-\frac{1}{p_1}\right) = \frac{p_2}{p_1}$$

- $$\frac{\partial e}{p_2} = u + 1 - \ln p_2 - \cancel{p_2} \cdot \frac{1}{\cancel{p_2}} + \ln p_1 =$$

$$= u + 1 + \ln p_1 - \ln p_2 - 1 = u + \ln \frac{p_1}{p_2}$$

- $$\frac{\partial h_2}{\partial p_2} = -\frac{p_1}{p_2^2} < 0$$

- $$\frac{\partial h_1}{\partial p_1} = \frac{-p_2}{p_1^2}$$

CROSS PRICE EFFECT	CROSS (WALRASIAN DEMAND)	NET (HICKSIAN DEMAND)
COMPLEMENTS	$\frac{\partial x_1}{\partial p_2} < 0$ $p_2 \uparrow$ $x_1 \downarrow$ (OPPOSITE DIRECTION)	$\frac{\partial h_1}{\partial p_2} < 0$
SUBSTITUTES	$\frac{\partial x_1}{\partial p_2} > 0$ $p_2 \uparrow$ $x_1 \uparrow$ (SAME DIRECTION)	$\frac{\partial h_1}{\partial p_2} > 0$

→ THIS DEFINITION CAN BE ASYMMETRIC

$$\frac{\partial h_1}{\partial p_2} = \frac{1}{p_1} > 0$$ (PRICE CAN'T BE NEGATIVE) => NET SUBSTITUTES

SO THEY ARE

$$h_1 = \frac{p_2}{p_1}$$

$$\frac{\partial h_2}{\partial p_1} = \frac{1}{p_1} > 0$$ ⇝ NET SUBSTITUTES

$$h_2 = u + \ln\left(\frac{p_1}{p_2}\right) = u + \ln p_1 - \ln p_2$$

SYMMETRIC

LOOK AT THE WALRASIAN DEMAND

$\boxed{Ex}$ 1. b

$$u = \sqrt{x_1} + \sqrt{x_2} = x_1^{1/2} + x_2^{1/2}$$

$$\max_{x_1, x_2 > 0} \quad u = x_1^{1/2} + x_2^{1/2}$$

$$\text{s.t} \quad x_1 p_1 + x_2 p_2 \leq w$$

$$L \cdot x_1^{1/2} + x_2^{1/2} + \lambda (w - p_1 x_1 + p_2 x_2)$$

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = \dfrac{1}{2} x_1^{-1/2} - \lambda p_1 \leq 0 & x_1 \geq 0 \\ \dfrac{\partial L}{\partial x_2} = \dfrac{1}{2} x_2^{-1/2} - \lambda p_2 \leq 0 & x_2 \geq 0 \\ \dfrac{\partial L}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \end{cases}$$

$$\boxed{\frac{\frac{1}{2} x_1^{-1/2}}{\frac{1}{2} x_2^{-1/2}}} = \frac{\lambda p_1}{\lambda p_2} \quad \text{slope of I.C.}$$

MRS

$$\Downarrow$$

$$\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{p_1}{p_2} \qquad x_2 = \left(\frac{p_1}{p_2}\right)^2 \cdot x_1$$

$$w - p_1 x_1 - p_2 \left(\frac{p_1}{p_2} \right)^2 x_1 = 0$$

$$w - p_1 x_1 - \frac{p_1^2}{p_2} x_1 = 0$$

$$x_1 \left(p_2 + \frac{p_1^2}{p_2} \right) = w$$

$$x_1^* = \frac{w}{p_1 + \frac{p_1^2}{p_2}} = \frac{w}{p_1 \left(1 + \frac{p_1}{p_2}\right)}$$

$$x_2^* = \left(\frac{p_1^2}{p_2} \right) x_1 = \frac{p_1^2}{p_2} \cdot \frac{w}{p_1 \left(1 + \frac{p_1}{p_2}\right)} = \frac{p_1}{p_2^2} \cdot \frac{w}{\left(1 + \frac{p_1}{p_2}\right)}$$

INDIRECT UTILITY FUNCTION

$$u(x_1^*, x_2^*) = V(p_1, p_2, w) =$$

$$\left(\frac{w}{p_1 \left(1 + \frac{p_1}{p_2}\right)} \right)^{\frac{1}{2}} + \left(\frac{w}{\frac{p_1}{p_2} \left(1 + \frac{p_1}{p_2}\right)} \right)^{\frac{1}{2}}$$

$h_1, h_2 \longrightarrow$ SOLUTION OF EMP

$$\min_{x_1, x_2 \geq 0} \quad p_1 x_1 + p_2 x_2 \quad \text{S.T} \quad x_1^{1/2} + x_2^{1/2} \geq u$$

$$\downarrow$$

$$\max_{x_1, x_2 \geq 0} \quad -(p_1 x_1 + p_2 x_2) \quad \text{S.T.} \quad \ldots \quad \geq 0$$

$$L = -p_1 x_1 - p_2 x_2 + \lambda \left(x_1^{1/2} + x_2^{1/2} - u \right)$$

INTERIOR SOLUTION

FOC

$$\frac{\partial L}{\partial x_1} = -p_1 + \lambda \frac{1}{2} x_1^{-1/2} = 0$$

$$\frac{\partial L}{\partial x_2} = -p_2 + \lambda \frac{1}{2} x_2^{-1/2} = 0$$

$$\longrightarrow$$

$$\frac{\partial L}{\partial \lambda} = x_1^{1/2} + x_2^{1/2} = u$$

$$\longrightarrow \frac{\lambda \left(\frac{1}{2} x_1^{-1/2} \right)}{\lambda \left(\frac{1}{2} x_2^{-1/2} \right)} = \frac{p_1}{p_2}$$

$$\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{p_1}{p_2} \qquad x_2 = \left(\frac{p_1}{p_2}\right)^2 \cdot x_1$$

$$x_2^{1/2} = \frac{p_1}{p_2} \cdot x_1^{1/2}$$

$$x_1^{1/2} + \left(\frac{p_1}{p_2}\right) x_1^{1/2} = u$$

$$\left(1 + \frac{p_1}{p_2}\right) x_1^{1/2} = u$$

$$\Rightarrow x_1 = \left(\frac{u}{1+\frac{p_1}{p_2}}\right)^2 = h_1$$

$$x_2 = \left(\frac{p_1}{p_2}\right)^2 \cdot \left(\frac{u}{1+p_1/p_2}\right)^2$$

EXPENDITURE FUNCTION

$$e(p_1, p_2, u) = p_1 h_1 + p_2 h_2 =$$

$$= p_1 \cdot \left(\frac{u}{1 \cdot \frac{p_1}{p_2}}\right)^2 + p_2 \left(\frac{p_1}{p_2}\right)^2 \left(\frac{u}{1+\frac{p_1}{p_2}}\right)^2$$

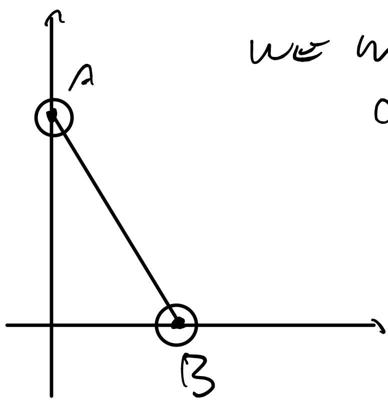
EX #

I.C. of S.C WITH PERFECT SUBSTITUTES

WALRAS DEMAND $u(x_1, x_2) = \underline{x_1 + 2x_2}$

S.T $p_1 x_1 + p_2 x_2 \leq w$

LINEAR
↓
CORNER SOLUTION



WE HAVE TO COMPARE THE SLOPE OF B.C AND I.C.

$$|MRS| > \frac{p_1}{p_2} \rightarrow \text{B} \quad x_2 = 0, \quad x_1 = \frac{w}{p_1}$$

$$|MRS| < \frac{p_1}{p_2} \rightarrow \text{A} \quad x_1 = 0, \quad x_2 = \frac{w}{p_2}$$

EXAMPLE $p_1 = 1, p_2 = 3$ → WE ARE IN CASE B

$$\frac{1}{2} > \frac{1}{3} \Rightarrow x_1^* = w, \quad x_2^* = 0$$

WALRASIAN DEMAND OF GOOD 2

WHAT HAPPEN IF $P_2 = 4$?

$\frac{1}{2} > \frac{1}{4} \rightsquigarrow |MRS| > \frac{P_1}{P_2} \Rightarrow$ CORNER SOLUTION $x_1^* = w \quad x_2^* = 0$

$\boxed{\text{TOTAL EFFECT OF PRICE CHANGE}}$ ON GOOD 1 AND GOOD 2 ?

$TE\ x_1 = w - w = 0$

END POINT (AFTER THE PRICE CHANGE)

$TE\ x_2 = 0 - 0 = 0$

$\boxed{\text{SUBSTITUTION EFFECT}}$

CASE 2 $\quad P_2 \downarrow \quad P_2 = 3 \rightsquigarrow P_2\ \frac{1}{2}$

OPTIMAL SOLUTION WITH $P_1 = 1 \quad P_2 = \frac{1}{2}$

$$|MRS| < \frac{P_1}{P_2}$$

$\frac{1}{2} < 2 \rightarrow$ OPTIMAL SOLUTION IS $x_1^* = 0$

$x_2^* = \frac{w}{1/2} = 2w$

NEW SOLUTION $(0, 2w)$

$\boxed{\text{TOTAL EFFECT}}$

$TE: x_1 = 0 - w = -w$

$TE: x_2 = 2w - 0 = 2w$

(END POINT - START POINT)

? I prove this

$(x_1, x_2) \succsim (y_1, y_2) \longrightarrow x_1 \geq y_1 - 1$

compare 1° components of bundles

MONOTHONICITY

- (x_1, x_2) vs $(a x_1, x_2)$
- (x_1, x_2) vs $(x_1, a x_2)$
- (x_1, x_2) vs $(a x_1, a x_2)$

- $a x_1$ $\quad x_1 - 1$ $\quad$ $\underline{a > 1}$ $\quad$ then $\quad a x_1 > x_1 - 1$

  so $(a x_1, x_2) \succsim (x_1, x_2)$

- $x_1 \quad x_1 - 1 \rightarrow x_1 > x_1 - 1$ so $(x_1, a x_2) \succsim (x_1, x_2)$

- is like case 1!

STRONG MONOTONE MAINLY BECAUSE $a x_1 > x_1 - 1$

YESTERDAY

T.E $x_1 = -w$ $\qquad$ $A(w, 0)$

T.E $x_2 = 2w$ $\qquad$ $C(0, 2w)$

WE COMPUTE T.E. BY TAKING DIFF. BETWEEN A AND C

NOW COMPUTE SUBSTITUTION EFFECT (HICKSIAN COMPENSATION)

WE HAVE TO GIVE SO MANY AS BEFORE UTILITY LEVEL BEFORE PRICE CHANGE

$u(x_1, x_2) = x_1 + 2x_2$

UTILITY LEVEL? ON A $\quad U(x_1, x_2) = w$

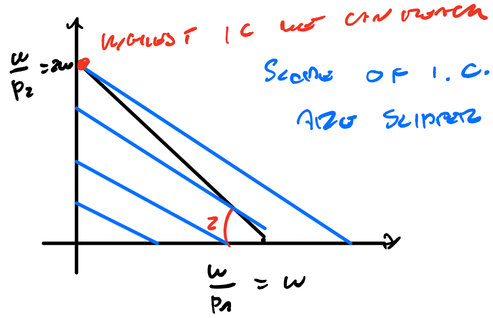
$p_1 = 1 \quad p_2 = \frac{1}{2}$ AFTER PRICE CHANGE

HICKSIAN COMPENSATION

$$\begin{cases} \underbrace{x_1 + 2x_2}_{\text{UTILITY LEVEL}} = w \\ \frac{1}{2} < 2 \end{cases}$$

NO TANGENCY CONDITION

DER x_1 / DER x_2 $\quad$ RATIO PRICE



SO CORNER SOLUTION

$x_2 = 2w = h_1$

$x_1 = 0 = h_2$

$B(0, 2w)$ INTERMEDIATE POINT.

TO SUMO FROM A TO B $(B - A)$

SE $x_1 = 0 - w = -w$

SE $x_2 = 2w - 0 = 2w$

S.E. AND TE ARE EQUAL

INCOME EFFECT B TO C $(C - B)$

IE $x_1 = 0 - 0 = 0$

IE $x_2 = 2w - 2w = 0$

IN PERFECT SUBSTITUTES I.E. IS $\emptyset$

$TE = SE + IE$ BY SUMMING UP THIS WE OBTAIN TOTAL EFFECT

$TE_{x_1} = -w + 0 = -w$

$TE_{x_2} = 0 + 2w = 2w$

WE GET STARTING T.E. SO IS THE CASE

EXAMPLE

INCOME & SUBSTITUTION EFFECT OF FUNCTION WELL BEHAVE (COBB-DOUGLAS)

$$\max U(x_1, x_2) = x_1^{1/2} x_2^{1/3} \qquad P_1 = 3 \quad P_2 = 2$$

$x_1, x_2 \geq 0$ s.t. $3x_1 + 2x_2 = 50$ $\qquad w = 50$

$$L = x_1^{\frac{1}{2}} x_2^{1/3} + \lambda(50 - 3x_1 - 2x_2)$$

NOT CORNER SINCE WITH U & WE GET $v \to [40, \infty]$ $\Downarrow$ WE EXCLUDE CORNER SOLUTION

$$\text{FOC} = \frac{\partial L}{\partial x_1} = \frac{1}{2} x_1^{-1/2} x_2^{1/3} - 3\lambda = 0 \qquad x_1 > 0$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{3} x_2^{-1/3} x_1^{1/2} - 2\lambda = 0 \qquad x_2 > 0$$

$$\frac{\partial L}{\partial \lambda} = 50 - 3x_1 - 2x_2$$

$$\frac{\frac{1}{2}}{\frac{1}{3}}\left(\frac{1}{2}\; x_1^{-1/2} \cdot x_1^{-1/2}\; x_2^{1/3}\; x_2^{2/3}\right) = \frac{3}{2}$$

$$\boxed{\cancel{\frac{3}{2}} \frac{x_2}{x_1} = \cancel{\frac{3}{2}}} \rightarrow x_2 = x_1$$

NOW WE REPLACE THIS IN BUDGET CONSTRAINT

$$50 - 3x_1 - 2x_1 = 0 \qquad x_1^* = \frac{50}{5} = 10 \qquad x_2^* = 10$$

$$A(10, 10)$$

WHAT MEANS IF $\boxed{p_2 = 4}$

$$\downarrow$$

$$MRS = \frac{p_1}{p_2}$$

SO $\begin{cases} |MRS| = \frac{p_1}{p_2} \\ \text{BUDGET CONSTRAINT} \end{cases}$ $\begin{cases} \cancel{\frac{3}{2}} \frac{x_2}{x_1} = \frac{\cancel{3}}{4} \\ 50 - 3x_1 - 4x_2 = 0 \end{cases}$

UTILITY REMAIN
THE SAME

$$\downarrow$$

PRICE APPEAR ONLY ON
BUDGET CONSTRAINT

$$\begin{cases} x_2 = \frac{1}{2} x_1 \\ 50 - 3x_1 - 2x_1 = 0 \end{cases} \qquad \begin{cases} x_1 = 10 \\ x_2 = 5 \end{cases}$$

NEW POINT AFTER
PRICE CHANGE

NOW COMPUTE T.E x_1 AND x_2

$$C(10, 5)$$

$$(C-A) \left| \begin{array}{ll} \text{TE} & x_1 = 10 - 10 = 0 \\ \text{TE} & x_2 = 5 - 10 = -5 \end{array} \right.$$

FOR x_1 T.E. NOT CHANGE.

FOR x_2 CHANGE OF -5

$x_1^* = 10 \quad x_2^* = 10 \quad u(10, 10) = 10^{\frac{1}{2}+\frac{1}{3}} = 10^{\frac{5}{6}}$

$A(10, 10)$

to COMPUTE S.E. → HICKSIAN DEMAND

$x_1^{\frac{1}{2}} x_2^{\frac{1}{3}} = 10^{\frac{5}{6}}$

$|MRS| = \frac{p_1}{p_2}$ → AFTER PRICE CHANGE

$\frac{3}{2}\frac{x_2}{x_1} = \frac{3}{4} \to x_2 = \frac{1}{2}x_1$

SAME TANGENCY POSITION

$x_1^{\frac{1}{2}} \cdot \left(\frac{1}{2}x_1\right)^{\frac{1}{3}} = 10^{\frac{5}{6}}$

$x_2 = \frac{1}{2}x_1$

$x_1^{\frac{1}{2}+\frac{1}{3}} = 10^{\frac{5}{6}} \left(\frac{1}{2}\right)^{-\frac{1}{3}}$

$x_2 = \frac{1}{2}x_1$

SQUARE ROOT! $\quad x^2 = 25 \quad x^{2\cdot\frac{1}{2}} = 25^{\frac{1}{2}}$

$x = 5$

$x_1^{\frac{5}{6}\cdot\frac{6}{5}} = 10^{\frac{5}{6}\cdot\frac{6}{5}} \left(\frac{1}{2}\right)^{-\frac{1}{3}\cdot\frac{6}{5}}$

SO $\quad x_1^* = 10\left(\frac{1}{2}\right)^{-\frac{2}{5}} \qquad x_2^* = 5\left(\frac{1}{2}\right)^{-\frac{3}{5}}$

$B\left(10\left(\frac{1}{2}\right)^{-\frac{2}{5}}, 5\left(\frac{1}{2}\right)^{-\frac{2}{5}}\right)$

(B-A)

S.E $x_1 = \dots$

SE $x_2 = \dots$

(C-B)

IE $x_1 = \dots$

IE $x_2 = \dots$

CHECK THAT SUMMING S.E + IE = T.E. !
SOLUTIONS ARE OK AGREE

ENGLE CURVE → LINK QUANTITY WITH WEALTH (OR INCOME)

$$U(x_1, x_2) = x_1^{1/2} x_2^{2/3} \qquad p_1 = 3 \quad p_2 = 2 \quad w = 50$$

$$\text{s.t} \quad 3x_1 + 2x_2 = w$$ VARIABLE

RELATION BETWEEN x AND w

$x_1 = f(w)$
$x_2 = f(w)$

SO SHOULD BE LINES

w IS CONSTANT → TAKE w AS VARIABLE IN B.C.

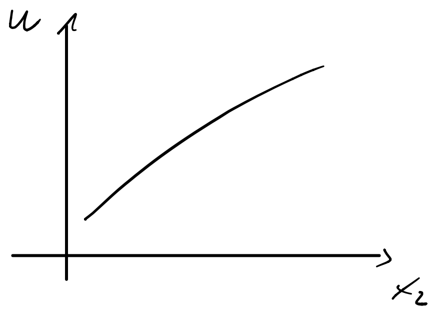
$$\begin{cases} |MRS| = \dfrac{p_1}{p_2} \\ \text{B.C} \end{cases} \qquad \begin{cases} \dfrac{\cancel{3/2}\, x_2}{x_1} = \cancel{3/2} \rightarrow x_2 = x_1 \\ \end{cases}$$

ENGLE C. FOR x_2

$$\begin{cases} x_2 = x_1 \\ 3x_1 + 2x_1 = w \end{cases} \qquad \begin{cases} x_2 = x_1 \\ 5x_1 = w \end{cases} \qquad \begin{matrix} \rightarrow x_2 = \dfrac{w}{5} \\ \rightarrow x_1 = \dfrac{w}{5} \end{matrix}$$

ALSO ISOLATING w

$w = 5x_1 \qquad w = 5x_2$



ENGEL CURVE FOR GOOD 1

INC ↑ D ↑ → NORMAL GOOD

$$\frac{\text{DER DEMAND}}{\text{DER } w} \rightarrow \frac{\partial x_2}{\partial w} = \boxed{\frac{1}{5} > 0}$$

NORMAL

IF DER $\frac{\delta x_k}{\delta w} < 0$

INFERIOR GOOD

EX SET II

d) $X(p_x, p_y, w) = 200 - 4p_x - 1.5\, p_y + 0.008 w$

↓ DEMAND FOR X

GOODS ARE GROSS SUBSTITUTE OR GROSS COMPLEMENTS WITH RESPECT TO y?

- IF I GET X GROSS SUB THEN I GET GROSS FOR Y
- IF NOT WELL BEHAVE Y IS IND. FROM X!

EX

$Y = 6p_y \longrightarrow$ Y IND FROM X SINCE DOES NOT APPEAR p_x

CROSS PRICE DERIVATIVE

$$\frac{\partial x}{\partial p_y} = -1.5 < 0$$ → SO 2 GOODS ARE GROSS COMPLEMENTS

ARE THE NET COMPLEMENTS OR NET SUBSTITUTES?

→ HICKSIAN DEMAND

IS x NORMAL OR INFERIOR?

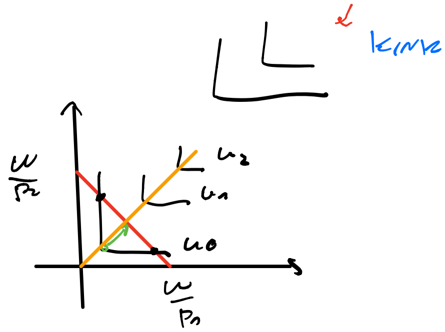
$$\frac{\partial x}{\partial w} = 0.008 > 0$$ SO GOOD IS NORMAL

INCOME & SUBSTITUTION EFFECT WITH PERFECT COMPLEMENTS

KINK

$$u(x_1, x_2) = \min\{x_1, 2x_2\}$$

$x_1, x_2 \geq 0$ s.t. $p_1x_1 + p_2x_2 \leq w$

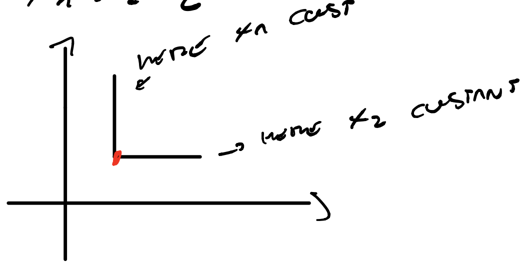


I PUT FIRST COMPONENT EQUALS TO THE SECOND TO FIND KINK

I CAN MOVE IN THIS DIR INWHICH IC TOUCH B.C WITH ONLY ONE POINTS

(KINK SOLUTION)

$x_1 = 2x_2$



$x_1 = 2x_2 \rightarrow x_2 = \frac{1}{2} x_1$ SLOPE $= \frac{\delta x_2}{\delta x_1} = \frac{1}{2}$

FIND WALRASIAN DEMAND SINCE SOLUTION IS IN THE KINK

$$\begin{cases} \text{KINK} \rightarrow x_2 = \frac{1}{2} x_1 \\ \text{B.C.} \rightarrow p_1 x_1 + p_2 x_2 = w \end{cases} \quad \Big\} \; p_1 x_1 + \frac{p_2}{2} x_1 = w$$

WE SOLVE EVEN IF WE DON'T HAVE PRICES

$$\Big\{ \left(p_1 + \frac{p_2}{2}\right) x_1 = w \qquad \begin{cases} x_2^* = \dfrac{w/2}{p_1 + \frac{p_2}{2}} \\ x_1^* = \dfrac{w}{p_1 + \frac{p_2}{2}} \end{cases}$$

$$A \left(\frac{w}{p_1 + \frac{p_2}{2}} , \frac{w/2}{p_1 + \frac{p_2}{2}} \right)$$

WHAT TO DO TO FIND UTILITY IN WALRASIAN DEMAND?

CHECK LAST LECTURE

u(WALRASIAN) $\Rightarrow$ WE GET INDIRECT UTILITY.

$$v\left(\frac{w}{2} \dots\right)$$

WHAT HAPPEN IF PRICE DECREASES?

$p_2 \uparrow$ $\quad \overset{\text{NEW}}{p'_2 = 4} \quad \overset{\text{OLD}}{p_2 = 2} \quad p_1 = 1$

$$\begin{cases} x_1 = \frac{1}{2} x_2 \quad (\text{KINK}) \longrightarrow x_1 + 2x_1 = w \\ 1 \cdot x_1 + 4x_2 = w \end{cases}$$

$x_1^* = \frac{w}{3}$

$x_2^* = \frac{w}{6}$

OLD

$A\left(\frac{w}{2}, \frac{w}{4}\right)$

$C\left(\frac{w}{3}, \frac{w}{6}\right)$

T.E. FROM A TO C

T.E $x_1 = \frac{w}{3} - \frac{w}{2} = -\frac{w}{6}$

T.E $x_2 = \frac{w}{6} - \frac{w}{4} = -\frac{w}{12}$

S.E.?

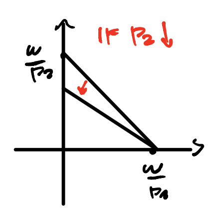
$\downarrow$

HICKSIAN COMPENSATION

1. SAME UTILITY
2. B.L. MUST BE TANGENT TO OLD UTILITY LV OF I.C.

NEW B. LINE WILL BE TANGENT TO OLD UTILITY LEVEL

B.C $= x_1 + 4x_2 = w$



$$\begin{cases} x_2 = \frac{1}{2} x_1 \\ \min\{x_1, 2x_2\} = \frac{w}{2} \end{cases}$$

$x_1 = 2x_2 = \frac{w}{2} \qquad x_1^* = \frac{w}{2} \qquad x_2^* = \frac{w}{4}$

$B\left(\frac{w}{2}, \frac{w}{4}\right)$

A to B

S.E $x_1 = 0$

S.E. $x_2 = 0$

NO S.E SINCE THE GOODS ARE CONSUMED IN THE SAME PROPORTION

B to C

$B\left(\frac{w}{2}, \frac{w}{4}\right)$ $C\left(\frac{w}{3}, \frac{w}{6}\right)$

I. E.

$IE\ x_1 = \frac{w}{3} - \frac{w}{2} = -\frac{w}{6}$

$IE\ x_2 = \frac{w}{6} - \frac{w}{4} = -\frac{w}{12}$

$T.E = IE$ BECAUSE $SE = 0$ FOR BOTH GOODS

WE CANNOT SUBSTITUTE ONE WITH THE EACH OTHER

Advanced Microeconomic Theory

Chapter 3: Welfare evaluation

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

Measuring the Welfare Effects of a Price Change

Measuring the Welfare Effects of a Price Change

- How can we measure the welfare effects of:
  - a price decrease/increase
  - the introduction of a tax/subsidy

    $P\uparrow$ $P\downarrow$

- Why not use the difference in the individual's utility level, i.e., from u^0 to u^1?
  - Two problems:
    1) *Within a subject criticism*: Only ranking matters (ordinality), not the difference;
    2) *Between a subject criticism*: Utility measures would not be comparable among different individuals.

- Instead, we will pursue **monetary evaluations** of such price/tax changes.

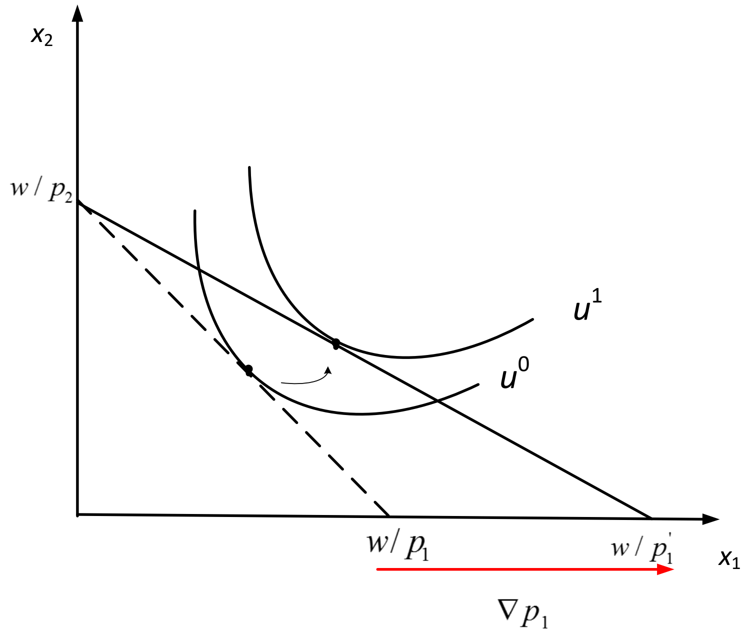
How to evaluate the welfare with different level of utilty? In reality different guys have different utilty function.

2) utility may be different between individuals.

We use money to evaluate welfare

Measuring the Welfare Effects of a Price Change

- Consider a price decrease from p_1^0 to p_1^1.
- We cannot compare u^0 to u^1.
- Instead, we will find a **money-metric** measure of the consumer's welfare change due to the price change.



Measuring the Welfare Effects of a Price Change

How much money i will have to transfer to the consumer after a price change(decrease or increase) to be as well off as before the price change

- ***Compensating Variation (CV)***:
  - How much money a consumer would **be willing to give up** *after* a reduction in prices to be just as well off as *before* the price decrease (After-Before, AB)

same utility level

- ***Equivalent Variation (EV)***:
  - How much money a consumer **would need** *before* a reduction in prices to be just as well off as *after* the price decrease (Before-After, BA)

We could use Hicksian demand or expenditure function

Hoping with Lower price is better than with higher prices. This means that after price decrease we have higher utility level. After price change utility level was lower.
To let the guy reach the same utility level before the price decrease the guy should have more or less income? We have to reduce the income.

If we consider a increase in price is the opposite. Willing to give up is only fro reduction in price.
Transfer can be positive or negative. Positive mean increasing income, negative decreasing income.

Measuring the Welfare Effects of a Price Change

- Two approaches:
  1) Using expenditure function
  2) Using the Hicksian demand

CV using Expenditure Function

$p\hat{x} = w$ $\qquad e(p^0, u^0) = w$

- $CV(p^0, p^1, w)$ using $e(p, u)$: Utilty level solving the UMP

$$CV(p^0, p^1, w) = \boxed{\overset{=w}{e(p^1, u^1)}} - e(p^1, u^0)$$

- The amount of money the consumer is willing to give up *after* the price decrease (after price level is p^1 and her utility level has improved to u^1) to be just as well off as *before* the price decrease (reaching utility level u^0).

CV is new price and new unity level - new price and old utilty level.
The vector of prices goes from p0 to p1, so wealth remain the same. So BC remain the same.

→ HICKSIAN COMPENSATION

CV using Expenditure Function

HOW MUCH MONEY YOU WILL TRANSFER TO CONSUMER AFTER PRICE CHANGE

$P_1 \Downarrow$ $P_2 \Leftrightarrow$

NEW PRICE

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust final wealth (*after* the price change) to make the consumer as well off as *before* the price change
4) Difference in expenditure:

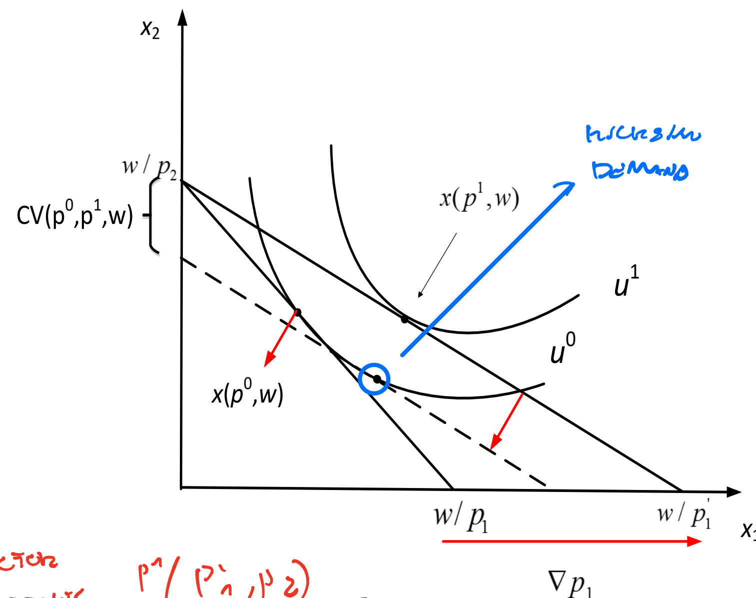
IN $x(p^1,w)$ $w =$

$$CV(p^0, p^1, w) = \underbrace{e(p^1, u^1)}_{\text{at } B_{p^1,w}} - \underbrace{e(p^1, u^0)}_{\text{dashed line}}$$

→ VECTOR OF PRICES

$p^1(p'_1, p_2)$ → $p_1 h_1 + p_2 h_2$

$p^0(p_1, p_2)$



This is **Hicksian wealth compensation**!

HOW MUCH MONEY TRANSFER BEFORE PRICE CHANGE TO BE WELL OFF AFTER PRICE CHANGE

EV using Expenditure Function

- $EV(p^0, p^1, w)$ using $e(p, u)$:

MINIMUM EXPENDITURE WITH OLD PRICES. ALSO UMP BEFORE PRICE CHANGES

$$EV(p^0, p^1, w) = \boxed{e(p^0, u^1)} - e(p^0, u^0)$$

SO WE WANT TO FIND THIS

$= w$

- The amount of money the consumer needs to receive *before* the price decrease (at the initial price level p^0 when her utility level is still u^0) to be just as well off as *after* the price decrease (reaching utility level u^1).

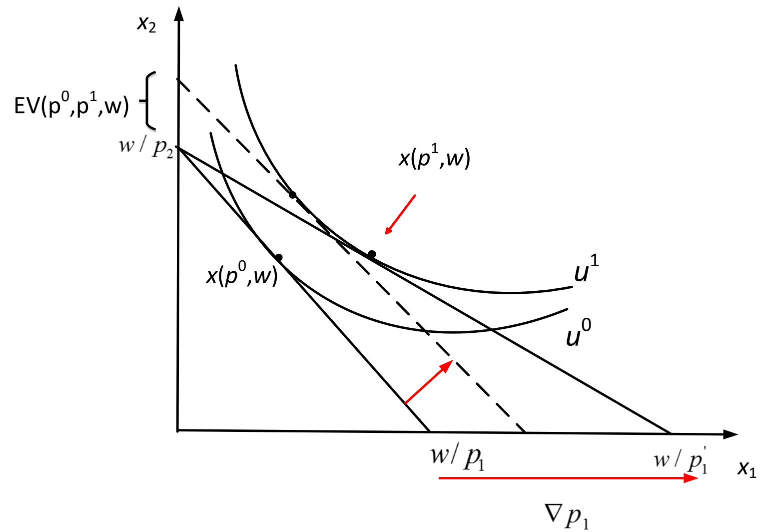
HOW WE FIND $e(p^0, u^1)$? SEE IT GRAPHICALLY

EV using Expenditure Function

$p_1 \downarrow$

1) When $B_{p^0,w}$, $x(p^0, w)$
2) ∇p_1 and $x(p^1, w)$ under $B_{p^1,w}$
3) Adjust initial wealth (*before* the price change) to make the consumer as well off as *after* the price change
4) Difference in expenditure:

$$EV(p^0, p^1, w) = \underbrace{e(p^0, u^1)}_{\text{dashed line}} - \underbrace{e(p^0, u^0)}_{\text{at } B_{p^0,w}} = w$$



NOW WE ARE ABLE TO DO ALL

EXAM 1 EXERCISES

CV using Hicksian Demand

new — old

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

GOOD 1!

FOR OTHER GOOD DEMAN THE SAME

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0)$$

$$= w - e(p^1, u^0)$$

$e(p^0, u^0) =$

UTILITY FOUND SOLVING UMP WITH P0

(since $e(p^1, u^1)$= $e(p^0, u^0)$=w)

$$= e(p^0, u^0) - e(p^1, u^0) \ (*)$$

PRIMITIVE OF DERIVATIVE OF EXP. FUNCTION

HICKSIAN

$$= \int_{p_1^1}^{p_1^0} \boxed{\frac{\partial e(p_1, \bar{p}_{-1}, u^0)}{\partial p_1}} dp_1 \ (**)$$

(since (**) is the solution of (*))

$$= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^0) \, dp_1$$

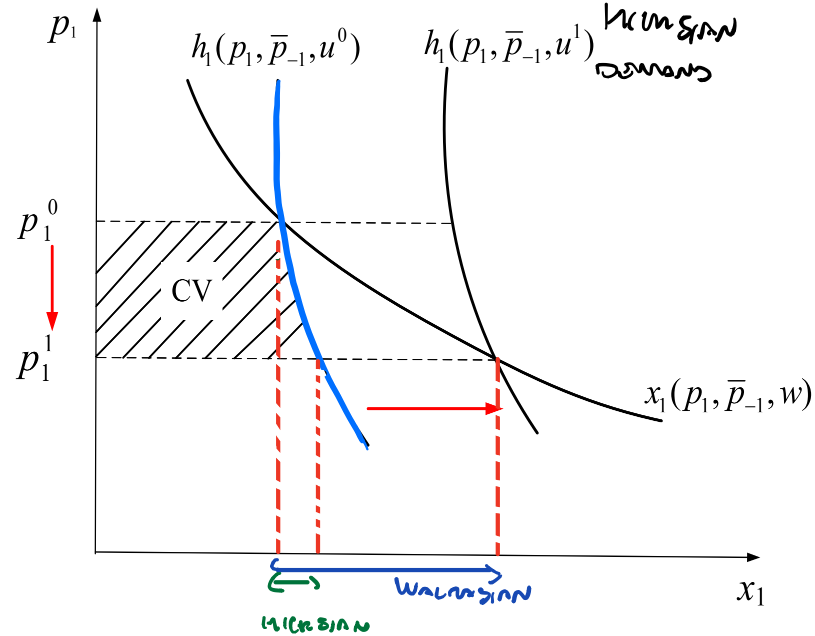
DERIVATIVE THEN IS HICKSIAN COMPENSATED DEMAND

GRAPHICALLY IS AREA UNDER HICKSIAN COMP DEMAND

CV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, CV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^0, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.

NORMAL GOOD! WALRASIAN DEM IS FLATTER THAN HICKSIAN DEMAND



SO CHANGE IN WALRASIAN IS LARGER THAN HICKSIAN

EV using Hicksian Demand

- From the previous definitions we know that, if $p_1^1 < p_1^0$ and $p_k^1 = p_k^0$ for all $k \neq 1$, then

OLD NEW UTI OLD PRICE

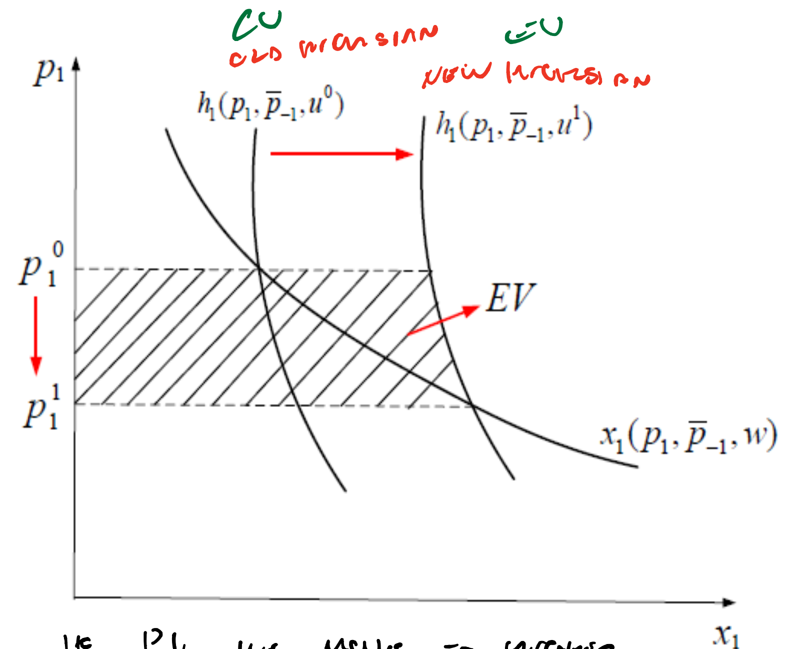
$$
\begin{aligned}
EV(p^0, p^1, w) &= e(p^0, u^1) - \boxed{e(p^0, u^0)} \\
&= e(p^0, u^1) - w \\
&= e(p^0, u^1) - e(p^1, u^1) \\
&= \int_{p_1^1}^{p_1^0} \frac{\partial e(p_1, \bar{p}_{-1}, u^1)}{\partial p_1} \, dp_1 \\
&= \int_{p_1^1}^{p_1^0} h_1(p_1, \bar{p}_{-1}, u^1) \, dp_1
\end{aligned}
$$

w

$w = e(p^1, u^1)$

EV using Hicksian Demand

- The case is:
  - Normal good
  - Price decrease
- Graphically, EV is represented by the area to the left of the Hicksian demand curve for good 1 associated with utility level u^1, and lying between prices p_1^1 and p_1^0.
- The welfare gain is represented by the shaded region.



IF P↓ WE MOVE TO HIGHER UTILITY LEVEL $U_1 > U_0$ SO WE CONSIDERING HICKSIAN DEMAND SHIFTED TO THE RIGHT

What about a price increase?

- The Hicksian demand associated with initial utility level u^0 (before the price increase, or before the introduction of a tax) experiences an inward shift when the price increases, or when the tax is introduced, since the consumer's utility level is now u^1, where $u^0 > u^1$. Hence,

$$h_1(p_1, \bar{p}_{-1}, u^0) > h_1(p_1, \bar{p}_{-1}, u^1)$$

What about a price increase?

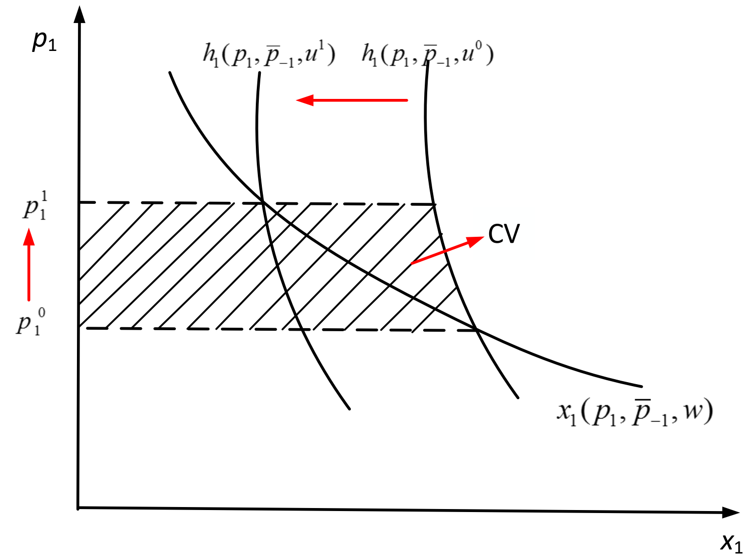
- The definitions of CV and EV would now be:
  - CV: the amount of money that a consumer would need *after* a price increase to be as well off as *before* the price increase.
  - EV: the amount of money that a consumer would be willing to give up *before* a price increase to be as well off as *after* the price increase.
- Graphically, it looks like the CV and EV areas have been reversed:
  - CV is associated to the area below $h_1(p_1, \bar{p}_{-1}, u^0)$ as usual
  - EV is associated with the area below $h_1(p_1, \bar{p}_{-1}, u^1)$.

What about a price increase?

welfare
after price
increase

- CV is always associated with $h_1(p_1, \bar{p}_{-1}, u^0)$

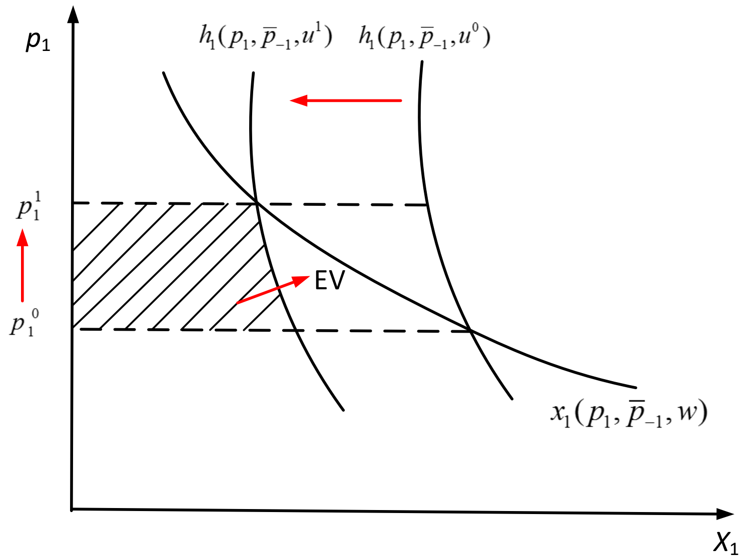
- $CV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$



What about a price increase?

- EV is always associated with $h_1(p_1, \bar{p}_{-1}, u^1)$

- $EV(p^0, p^1, w) = \int_{p_1^0}^{p_1^1} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$



this are usefull for welfare analysis introducing tax

Introduction of a Tax

- The introduction of a **tax** can be analyzed as a **price increase**.
- *The main difference*: we are interested in the area of CV and EV that is *not* related to tax revenue.

  how much state receive with taxes
- Tax revenue is:

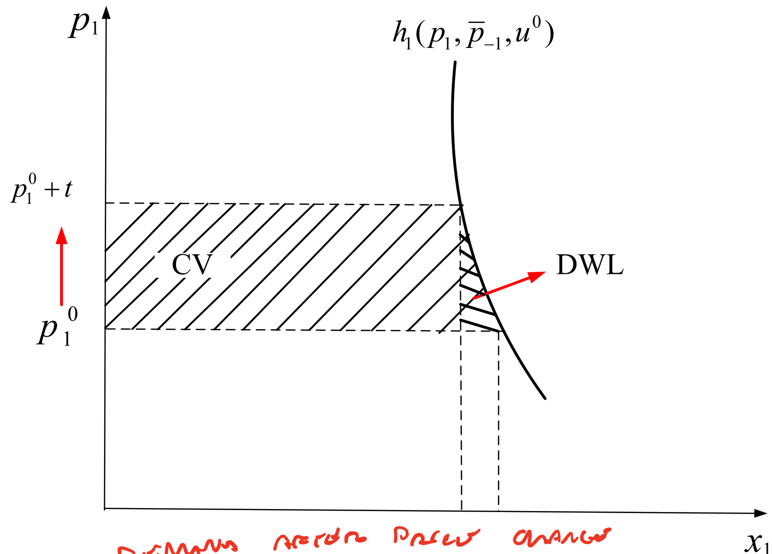
  per unit tax

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^0) \text{ (using CV)}$$

$$T = \underbrace{[(p_1^0 + t) - p_1^0]}_{t} \cdot h(p_1, \bar{p}_{-1}, u^1) \text{ (using EV)}$$

Introduction of a Tax

- CV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^0)$:
$CV(p^0, p^1, w)$
$$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^0)\, dp_1$$



DEMAND AFTER PRICE CHANGE

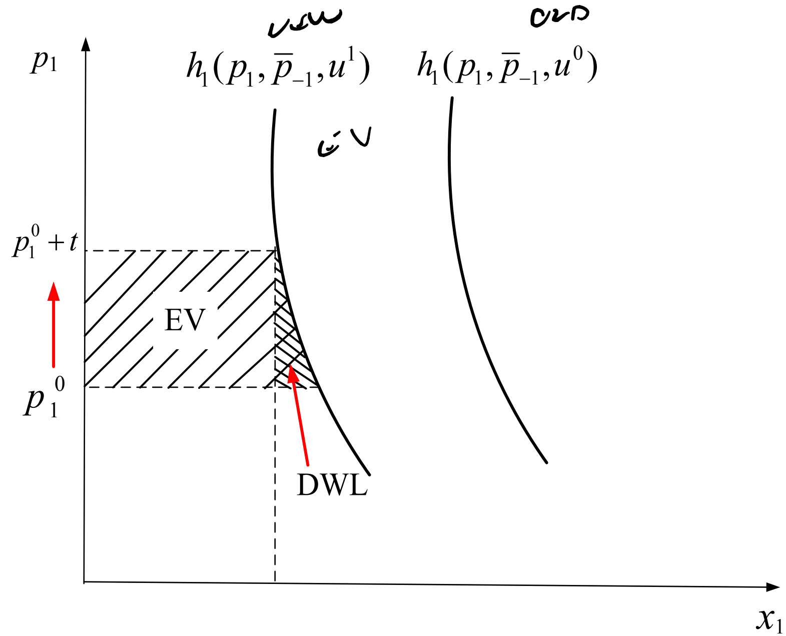
- Welfare loss (DWL) is the area of the CV not transferred to the government via tax revenue:

$DWL = CV - T$

TAX REVENUE = TAX · HICKSIAN DEMAND

Introduction of a Tax

- EV is measured by the large shaded area to the left of $h(p_1, \bar{p}_{-1}, u^1)$:
  $EV(p^0, p^1, w)$
  $$= \int_{p_1^0}^{p_1^0+t} h_1(p_1, \bar{p}_{-1}, u^1)\, dp_1$$
- Welfare loss (DWL) is the area of the EV not transferred to the government via tax revenue:
  $$DWL = EV - T$$

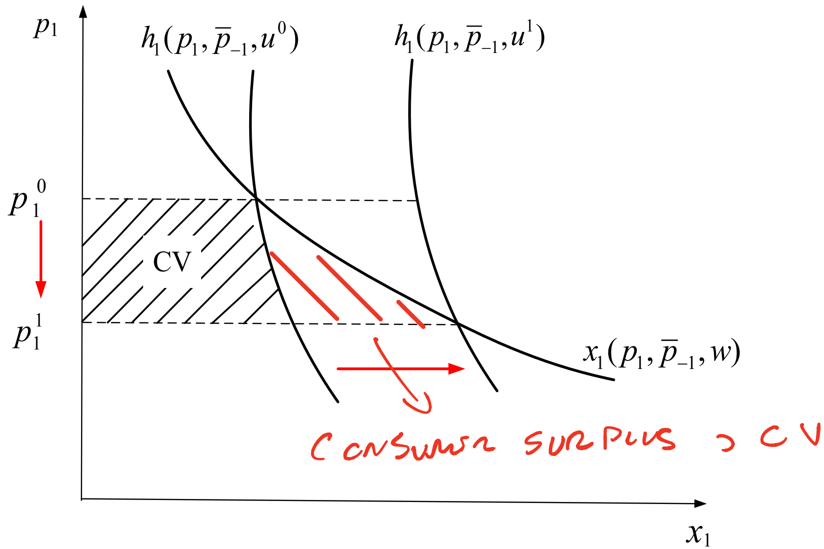


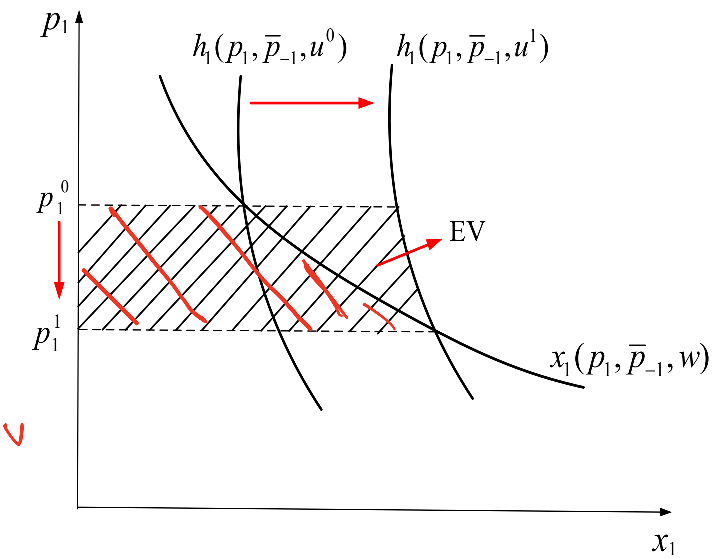
Why not use the Walrasian demand?

- Walrasian demand is easier to observe, so we could use the variation in consumer's surplus as an approximation of welfare changes.
- This is only valid when income effects are *zero*:
  - Recall that the Walrasian demand measures both income and substitution effects resulting from a price change, while
  - **The Hicksian demand measures only substitution effects from such a price change**.
- Hence, there will be a difference between CV and Consumer Surplus (CS), and between EV and CS (area under the Walrasian demand, between prices).

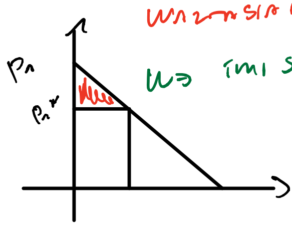
Why not use the Walrasian demand?

- Normal goods (i.e. W-demand flatter than H-demand)





$$CV < CS \qquad\qquad CS < EV$$



WALRASIAN

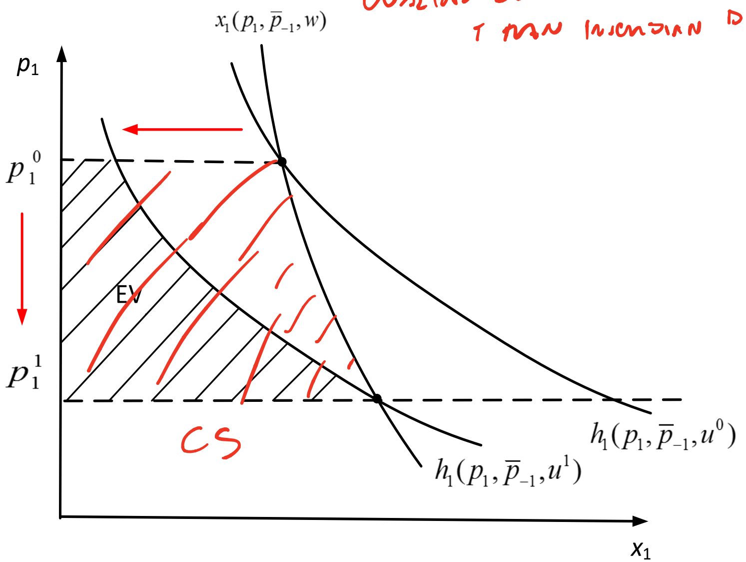
WE THIS AREA IS THE CONSUMER SURPLUS

REDUCTION IN PRICE

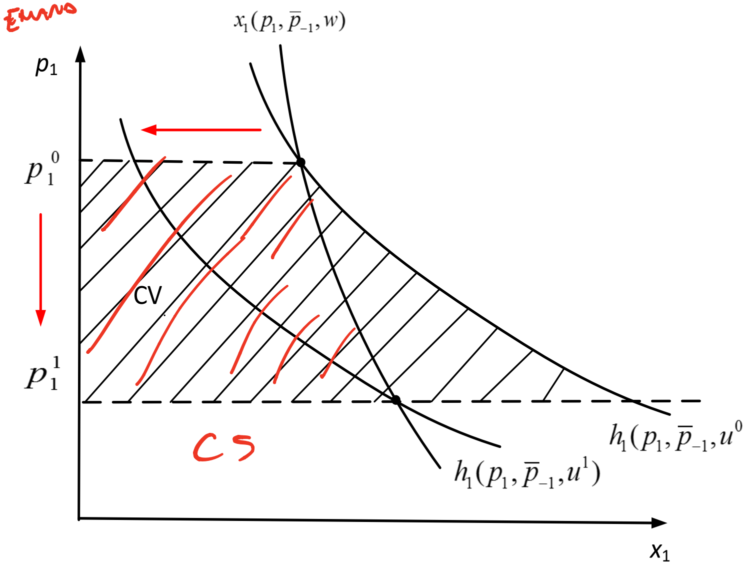
Why not use the Walrasian demand?

- Inferior goods: (i.e. H-demand flatter than W-demand)

WALRASIAN DEMAND IS STEEPER THAN HICKSIAN DEMAND



$$EV < CS$$



$$CS < CV$$

SUMMARISE

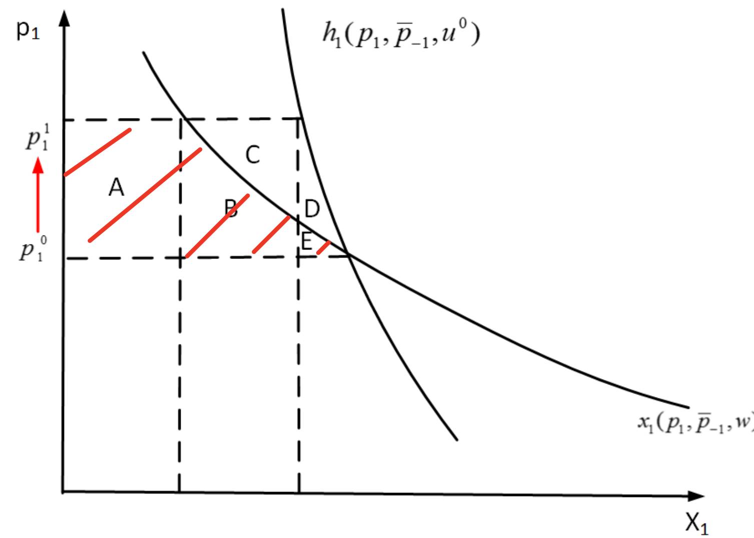
Why not use the Walrasian demand?

- For normal goods:
  - Price decrease: $CV < CS < EV$
  - Price increase: $CV > CS > EV$
- For inferior goods we find the opposite ranking:
  - Price decrease: $CV > CS > EV$
  - Price increase: $CV < CS < EV$
- NOTE: consumer surplus is also referred to as the ***area variation*** (AV).

When can we use the Walrasian demand?

- When the price change is small (using AV):
  - $CV = A + B + C + D + E$
  - $CS = A + B + E$
  - Measurement error from using CS (or AV) is $C + D$

Area under demand



When can we use the Walrasian demand?

- The measurement difference between CV (and EV) and CS, $C + D$, is relatively small:
  1) When **income effects are small**:
     - Graphically, $x(p, w)$ and $h(p, u)$ almost coincide.
     - The welfare change using the CV and EV coincide too.
  2) When the **price change is very small**:
     - The error involved in using *AV*, i.e., areas $C + D$, as a fraction of the true welfare change, becomes small. That is,

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{CV} = 0$$

When can we use the Walrasian demand?

- However, if we measure the approximation error by $\frac{C+D}{DW}$, where $DW = D + E$, then

$$\lim_{(p_1^1 - p_1^0) \to 0} \frac{C + D}{DW}$$

does not necessarily converge to zero.

Application of IE and SE

TE = SE + IE → SE = TE − IE

- From the Slutsky equation, we know

$$\frac{\partial h_1(p,u)}{\partial p_1} = \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)$$

BUDGET ↑ PRICE CHANGE

- Multiplying both terms by $\frac{p_1}{x_1}$,

$$\frac{\partial h_1(p,u)}{\partial p_1}\boxed{\frac{p_1}{x_1}} = \frac{\partial x_1(p,w)}{\partial p_1}\boxed{\frac{p_1}{x_1}} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}$$

And multiplying all terms by $\frac{w}{w} = 1$,

$$\underbrace{\frac{\partial h_1(p,u)}{\partial p_1}\frac{p_1}{x_1}}_{\substack{\text{Substitution Price} \\ \text{elasticity of demand} \\ \tilde{\varepsilon}_{p,Q}}} = \underbrace{\frac{\partial x_1(p,w)}{\partial p_1}\frac{p_1}{x_1}}_{\substack{\text{Price elasticity} \\ \text{of demand} \\ \varepsilon_{p,Q}}} + \boxed{\underbrace{\frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}\frac{w}{w}}_{?}}$$

Elasticity of walrasian demand with respect to price

$$\left[\frac{\partial x_1}{\partial w}\frac{w}{\partial x_1}\right]\frac{x_1 p_1}{w}$$

SHARE ON THE BUDGET
HOW T ON GOES ↑

Elasticity is the percentage change of a variable divided by the percentage in a second variable.

$$\varepsilon_x = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} \longrightarrow \frac{\Delta x}{\Delta p} \cdot \frac{p}{x} \rightarrow \frac{\partial x}{\partial p} \cdot \frac{p}{x}$$

To get elasticity we moltiply both side by the same ratio (p1/x1)
Also then multiply by w/w for the last term (w/w which is 1) but convenient.

For elasticity then, we can write this in this way

Application of IE and SE

- Rearranging the last term, we have

$$\frac{\partial x_1(p,w)}{\partial w} x_1(p,w)\frac{p_1}{x_1}\frac{w}{w}$$

$$= \underbrace{\frac{\partial x_1(p,w)}{\partial w}\frac{w}{x_1}}_{\substack{\text{Income elasticity}\\ \text{of demand}\\ \varepsilon_{w,Q}}} \cdot \underbrace{\frac{p_1 x_1(p,w)}{w}}_{\substack{\text{Share of budget}\\ \text{spent on good 1, } \theta}}$$

- We can then rewrite the Slutsky equation in terms of elasticities as follows

$$\underbrace{\tilde{\varepsilon}_{p,Q}}_{SE} = \underbrace{\varepsilon_{p,Q}}_{TE} + \underbrace{\varepsilon_{w,Q} \cdot \theta}_{IE}$$

If income very close to 0 then SE = TE. So if eps is 0 no income effect or if income effect is very small. So this one case we can use walrasian demand instead of Hicksian demand to do welfare analysis. Also if share of budget spent on good 1 is closer to 0

Application of IE and SE

- ***Example***: consider a good like housing, with $\theta = 0.4$, $\varepsilon_{w,Q} = 1.38$, and $\varepsilon_{p,Q} = -0.6$.
- Therefore,

Much smaller than walrasian demand! (1.38)

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta = -0.6 + 1.38 \cdot 0.4 = \boxed{-0.05}$$

- If price of housing rises by **10%**, and consumers do not receive a wealth compensation to maintain their welfare unchanged, consumers reduce their consumption of housing by 6%.
- However, if consumers receive a wealth compensation, the housing consumption will only fall by 0.5%.
  - Intuition: Housing is such an important share of my monthly expenses, that higher prices lead me to significantly reduce my consumption (if not compensated), but to just slightly do so (if compensated).

Share on the budget is not small in housing. In this example testa is 0.4 so 40% of IE. So this term is not close to 0. We can use walrasian demand instead of Hicksian demand to have some infos about elasticity of housing with respect to income.

What does elasticity of 1.38 means?
You cannot by a piece of house so we can measure it with square feet. So 1.38 if your income increase by 1% the demand for housing increase 1.38% so demand increase more than demand in proportion.

This means that elasticity is not small at all. So we can predict and we expect and increase of 10% in prices. So when we only consider substitution effect and walrasian demand (uncompensated demand). In this case we already have the estimati which is -0.6. So if price increase 1% the demand for housing decrease for 0.6%.

We can compute compensated demand in price change.
First of all we get the substitution elasticity that we can get from the parameter.

Application of IE and SE

- Other useful lessons from the previous expression

$$\tilde{\varepsilon}_{p,Q} = \varepsilon_{p,Q} + \varepsilon_{w,Q} \cdot \theta$$

- Price-elasticities very close $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$ if
  – Share of budget spent on this particular good, θ, is very small (Example: garlic).
  – The income-elasticity is really small (Example: pizza).

- Advantages if $\tilde{\varepsilon}_{p,Q} \simeq \varepsilon_{p,Q}$:
  – The Walrasian and Hicksian demand are very close to each other. Hence, $CV \simeq EV \simeq CS$.

Application of IE and SE

- You can read sometimes "in this study we use the change in CS to measure welfare changes due to a price increase given that income effects are negligible"
  - What the authors are referring to is:
    - Share of budget spent on the good is relatively small and/or
    - The income-elasticity of the good is small
- Remember that our results are not only applicable to price changes, but also to changes in the sales taxes.
- For which preference relations a price change induces no income effect? Quasilinear.

Application of IE and SE

- In 1981 the US negotiated voluntary automobile export restrictions with the Japanese government.
- Clifford Winston (1987) studied the effects of these export restrictions:
  - Car prices: p_{Jap} was 20% higher with restrictions that without. p_{US} was 8% higher with restrictions than without.
  - What is the effect of these higher prices on consumer’s welfare?
  - Would you use CS? Probably not, since both θ and $\varepsilon_{w,Q}$ are relatively high.

Imaging import tax. So what happen to the consumer? The demand decreases since the prices increases and we are going to replace with internally goods.
On average we tend to replace internal good instead of abroad good but prices will increase.

We can evaluate in advance to evaluate the introduction of import tax.

Application of IE and SE

- Winston did not use CS. Instead, he focused on the CV. He found that CV = -$14 billion.
  - *Intuition*: The wealth compensation that domestic car owners would need after the price change (after setting the export restrictions) in order to be as well off as they were before the price change is $14 billion.
- This implies that, considering that in 1987 there were 179 million car owners in the US, the wealth compensation per car owner should have been $14,000/$179 = $78.
- Of course, this is an underestimation, since we should divide over the new number of car owners (lower) during the period of export restriction was active (not the number of all current car owners).

Application of IE and SE

- Jerry Hausmann (MIT) measures the welfare gain consumers obtain from the price decrease they experience after a Walmart store locates in their locality/country.
- He used CV. Why? Low-income families spend a non-negligible part of their budget in Wal-Mart.
- Result: welfare improvement of 3.75%.

Advanced Microeconomic Theory

Chapter 3: Gross and net complements and substitutes, and substitutability across goods

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

Gross/Net Complements and Gross/Net Substitutes

Perfect substitute we are looking for the crossite

Demand Relationships among Goods

- So far, we were focusing on the SE and IE of varying the price of good k on the demand for good k.

- Now, we analyze the SE and IE of varying the price of good k on the demand for other good j.

Demand Relationships among Goods

- For simplicity, let us start our analysis with the two-good case.
  - This will help us graphically illustrate the main intuitions.

- Later on we generalize our analysis to $N > 2$ goods.

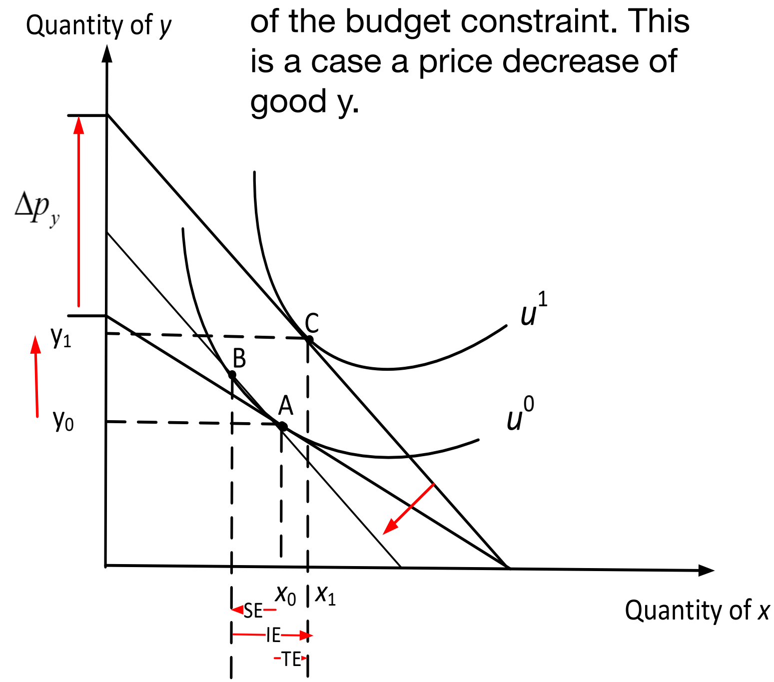
Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *small* that the consumer purchases more x and more y.
  - In this case, we call x and y ***gross complements***.

$$\boxed{\frac{\partial x}{\partial p_y} < 0}$$

NEGATIVE DERIVATIVE

If y increase BC shift up or down. So there is a rotation of the budget constraint. This is a case a price decrease of good y.



C is the new walrasian demand and account for total effect. Moving A to C. The price of Py decrease and quantity of x increase so TE is positive. What about demand for y? Increases. Demand of both increase due to decrease in price of y.

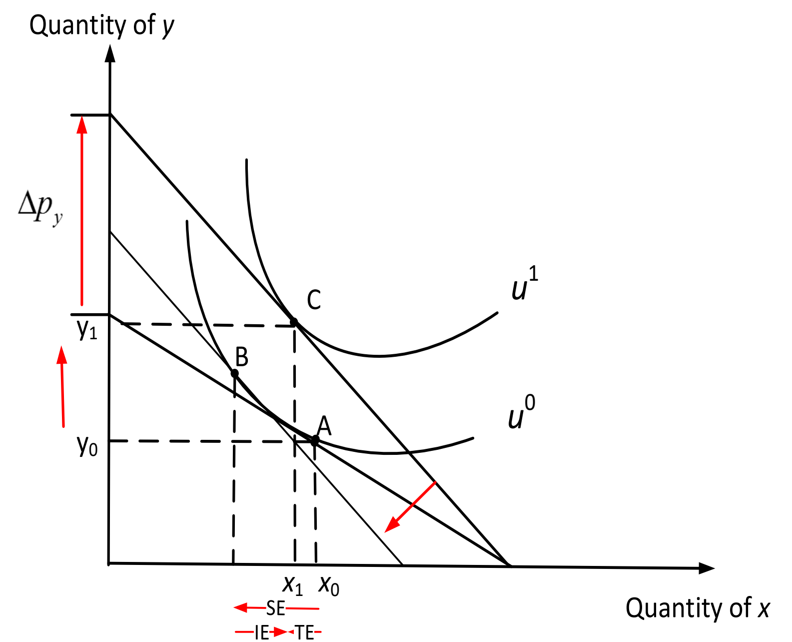
Are the two good complement or substitute?
So see the walrasian or Hicksian? WALRASIAN and they are complements. If der negative they are moving in opposite direction: if py decrease the demand for x increases.

From A to B demand for X decrease, but demand of y increases. So this is the SE for good y.
From B to C we can find income effect: are the two good normal or inferior? Demand for X increase so x is normal. Demand for Y increase so also Y is normal.

Demand Relationships among Goods: The Two-Good Case

- When the price of y falls, the substitution effect may be so *large* that the consumer purchases less x and more y.
  - In this case, we call x and y ***gross substitutes***.

$$\boxed{\frac{\partial x}{\partial p_y} > 0}$$



POSITIVE DERIVATIVE → SAME DIRECTION

Demand Relationships among Goods: The Two-Good Case

- A mathematical treatment
  - The change in x caused by changes in p_y can be shown by a Slutsky-type equation:

$$\underbrace{\frac{\partial x}{\partial p_y} = \underbrace{\frac{\partial h_x}{\partial p_y}}_{SE\ (+)} - \underbrace{y\frac{\partial x}{\partial w}}_{\substack{IE: \\ (-)\text{ if } x \text{ is normal} \\ (+)\text{ if } x \text{ is inferior}}}}_{\text{Combined effect (ambiguous)}}$$

$SE > 0$ is not a typo: Δp_y induces the consumer to buy more of good x, if his utility level is kept constant. Graphically, we are moving along the same indifference curve.

Demand Relationships among Goods: The Two-Good Case

- Or, in elasticity terms

$$\varepsilon_{x,p_y} = \underbrace{\tilde{\varepsilon}_{x,p_y}}_{SE\ (+)} - \underbrace{\theta_y \varepsilon_{x,w}}_{\substack{IE: \\ (-)\text{ if } x \text{ is normal} \\ (+)\text{ if } x \text{ is inferior}}}$$

where θ_y denotes the share of income spent on good y. The combined effect of Δp_y on the observable Walrasian demand, $x(p,w)$, is ambiguous.

Demand Relationships among Goods: The Two-Good Case

- ***Example***: Let's show the SE and IE across different goods for a Cobb-Douglas utility function $\underline{u(x,y) = x^{0.5}y^{0.5}}$.
  - The Walrasian demand for good x is → FOR X

    OPT. DEMAND

$$x(p,w) = \frac{1}{2}\frac{w}{p_x} \quad = \emptyset$$

  - The Hicksian demand for good x is → HICKSIAN DEMAND

$$h_x(p,u) = \frac{\sqrt{p_y}}{\sqrt{p_x}} \cdot u$$

Is X gross complement or substitute with respect to y?
What we have to do?
WE CAN USE DERIVATIVE OF x with respect to Py $\longrightarrow$ $\frac{\partial x}{\partial P_y} = 0$

By looking at the walrasian demand the consumption of x and y is independent.
Let's see the income and the substitution effect for these cobb Douglas.

If we look at the Hicksian demand: What would you conclude between the relationship between X or Y (are they net complement or substitutes?)
The derivative here is

$$\frac{\partial h_x}{\partial P_y} = \frac{1}{2} P_y^{-1/2} \, P_x^{-1/2} \, u \underline{> 0}$$

THEY ARE NET SUBSTITUTES
SINCE DER IS > 0

Effect Walrasian demand is 0
What about income effect? Is the same as the substitution effect since TE = 0 of increasing Py. So IE = SE and opposite.
So the effect on walrasian demand is 0 and we can prove this if we compute the SE of the derivative here (sopra).

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - First, not that differentiating $x(p, w)$ with respect to p_y, we obtain
$$\frac{\partial x(p,w)}{\partial p_y} = 0$$
i.e., variations in the price of good y do not affect consumer's Walrasian demand.
  - But,
$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}} \neq 0$$
  - How can these two (seemingly contradictory) results arise?

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Answer: the SE and IE completely offset each other.
  - **Substitution Effect**: Given

$$\frac{\partial h_x(p,u)}{\partial p_y} = \frac{1}{2}\frac{u}{\sqrt{p_x p_y}},$$

plug Walrasian demands for x and y in u(x,y) to get the indirect utility function $u = \frac{1}{2}\frac{w}{\sqrt{p_x p_y}}$, and replace it in the expression above to obtain a SE of $\frac{1}{4}\frac{w}{p_x p_y}$.

  - **Income Effect**: —> Right side of slushy equation

$$-y\frac{\partial x}{\partial w} = -\overset{\text{at Walrasian}}{\left(\frac{1}{2}\frac{w}{p_y}\right)}\overset{\text{Walrasian}}{\left(\frac{1}{2}\frac{1}{p_x}\right)} = -\frac{1}{4}\frac{w}{p_x p_y}$$

Demand Relationships among Goods: The Two-Good Case

- ***Example*** (continued):
  - Therefore, the total effect is

$$\overbrace{\frac{\partial x(p,w)}{\partial p_y}}^{TE} = \overbrace{\frac{\partial h_x}{\partial p_y}}^{SE} - \overbrace{y\frac{\partial x}{\partial w}}^{IE}$$
$$= \frac{1}{4}\frac{w}{p_x p_y} - \frac{1}{4}\frac{w}{p_x p_y} = 0$$

  - Intuitively, this implies that the substitution and income effect completely offset each other.

Demand Relationships among Goods: The Two-Good Case

- Common mistake:
  - "$\frac{\partial x(p,w)}{\partial p_y} = 0$ means that good x and y cannot be substituted in consumption. That is, they must be consumed in fixed proportions (perfect complents). Hence, this consumer's utility function is a Leontief type."

- No! We just showed that

$$\frac{\partial x(p,w)}{\partial p_y} = 0 \implies \frac{\partial h_x}{\partial p_y} = y\frac{\partial x}{\partial w}$$

i.e., the SE and IE completely offset each other.

Demand Relationships among Goods: The N-Good Case

- We can, hence, generalize the Slutsky equation to the case of $N > 2$ goods as follows:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial w}$$

for any i and j.

- The change in the price of good j induces IE and SE on good i.

Asymmetry of the Gross Substitute and Complement

$\frac{\partial X}{\partial P_Y}$ $\frac{\partial Y}{\partial P_X}$

- Two goods are ***substitutes*** if one good may replace the other in use.
  - Example: tea and coffee, butter and margarine
- Two goods are ***complements*** if they are used together.
  - Example: coffee and cream, fish and chips.
- The concepts of gross substitutes and complements include both SE and IE.
  - Two goods are gross substitutes if $\frac{\partial x_i}{\partial p_j} > 0$.
  - Two goods are gross complements if $\frac{\partial x_i}{\partial p_j} < 0$.

Asymmetry of the Gross Substitute and Complement

- The definitions of gross substitutes and complements are *not* necessarily symmetric.
  - It is possible for x_1 to be a substitute for x_2 and at the same time for x_2 to be a complement of x_1.
- Let us see this potential asymmetry with an example.

We can get that x Perfect Comp to y but not the contrary.

Asymmetry of the Gross Substitute and Complement

EXAMPLE

- Suppose that the utility function for two goods is given by

$$\underline{U(x,y) = \ln x + y}$$

QUASILINEAR UTILITY!

- The Lagrangian of the UMP is

$$\underline{L = \ln x + y + \lambda(w - p_x x - p_y y)}$$

- The first order conditions are

$$\left\{\begin{aligned} \frac{\partial L}{\partial x} &= \frac{1}{x} - \lambda p_x = 0 \\ \frac{\partial L}{\partial y} &= \overset{1}{\cancel{x}} - \lambda p_y = 0 \\ \frac{\partial L}{\partial \lambda} &= w - p_x x - p_y y = 0 \end{aligned}\right.$$

OPT DEMAND OF X

$$x^* = \frac{p_y}{p_x}$$

$$w - \cancel{p_x}\frac{p_y}{\cancel{p_x}} - p_y y = 0$$

OPT DEMAND OF Y

$$y^* = \frac{w - p_x}{p_y}$$

Asymmetry of the Gross Substitute and Complement

- Manipulating the first two equations, we get

$$\frac{1}{p_x x} = \frac{1}{p_y} \implies p_x x = p_y$$

- Inserting this into the budget constraint, we can find the Marshallian demand for y

$$\underbrace{p_x x}_{p_y} + p_y y = w \implies p_y y = w - p_y \implies$$

$$y = \frac{w - p_y}{p_y}$$

THEN CHECK CROSS COMP OR SUBSTITUTE

$$\frac{\partial x}{\partial p_y} = \frac{\lambda}{p_x} > 0 \quad G.S$$

$$\boxed{\text{ASYMMETRIC!!}}$$

$$\frac{\partial y}{\partial p_x} = 0$$ INDEPENDENT!

Asymmetry of the Gross Substitute and Complement

- An increase in p_y causes a decline in spending on y
  - Since p_x and w are unchanged, spending on x must rise $\left(\frac{\partial x}{\partial p_y} > 0\right)$.
    - Hence, x and y are gross substitutes.
  - But spending on y is independent of p_x $\left(\frac{\partial y}{\partial p_x} = 0\right)$.
    - Thus, x and y are neither gross substitutes nor gross complements.
  - This shows the asymmetry of gross substitute and complement definitions.
    - While good y is a gross substitute of x, good x is neither a gross substitute or complement of y.

EX 7. MID TERM

$U(x,y) = xy \qquad P_x = 1 \qquad P_y = 1 \qquad P_x' = 4$

FIND WALRASIAN DEMAND AND INDIRECT UTILITY FUNCTION

$$p_1 x_1 + p_2 x_2 \leq w$$

$$L = xy + \lambda(w - p_x x_1 - p_y x_2)$$

$$\begin{cases} \dfrac{\partial L}{\partial x} = y - \lambda p_x = 0 \\ \dfrac{\partial L}{\partial y} = x - \lambda p_y = 0 \\ \dfrac{\partial L}{\partial \lambda} = w - p_1 x_1 - p_2 x_2 = 0 \end{cases}$$

$$\frac{x}{y} = \frac{p_x}{p_y} \rightarrow y = \frac{p_x}{p_y} x \rightarrow w - p_1 x - p_y \frac{p_x}{p_y} x = 0$$

$$\boxed{x^* = \frac{w}{2p_x} \qquad y^* = \frac{w}{2p_y}}$$

WALRASIAN DEMAND FUNCTION

WHAT HAPPEN IF PRICE INCREASE? I JUST REPLACE IT IN WALRAS DEMAND INSTEAD OF REDOING THE UMP SOLUTION FOR INCREASED PRICE!

$x^* = \frac{48}{2} = 24$ $\quad$ $A(24, 24)$

$y^* = \frac{48}{2} = 24$

VALUE
INDIRECT UTILITY → VALUE OF THE MAXIMUM

SO REPLACE A IN UTILITY

$U(24, 24) = 24^2$

INDIRECT UTILITY FUNCTION? REPLACE WALRAS DEMAND INTO UTILITY FUNCTION!

$$U(x, y) = x \cdot y = \frac{w}{2p_x} \cdot \frac{w}{2p_y} = \frac{w^2}{4 p_x p_y}$$

FIND HICKSIAN DEMAND AND EXPENDITURE FUNCTION

IN GENERAL TO FIND HICKSIAN DEMAND?

MINIMISATION PROBLEM $\rightarrow$ MINIMISE EXPENDITURE

$$\min_{x, y \geq 0} \quad p_x X + p_y Y$$

$$\text{s.t. } x \cdot y = U = 2u^2$$

WE CAN AVOID DOING THIS!

REWRITE IND U. F. AS $v(x,y)$

$$v(x,y) = \frac{w^2}{4p_x p_y}$$

HOW MUCH GUY SPENT?

REPLACE e INTO w

$$v(x,y) = \frac{e^2(p_x, p_y, w)}{4p_x p_y}$$

THIS FUNCTION CAN BE INVERTED

$$e(p_x, p_y, w) = \left(v(x,y) \cdot 4 p_x p_y\right)^{1/2} =$$

$$= u^{1/2} \cdot 2 (p_x p_y)^{1/2}$$

SOLVING EMP WE CAN GET SAME RESULT

INVERTING $v(x,y) = \frac{e^2(p_x, p_y, w)}{4p_xp_y}$

IF WE SOLVE EMP WE GET BEFORE ARGUMENT AND THEN VALUE OF THE FUNCTION HERE WE ARE DOING THE OPPOSITE

$$\frac{\partial e}{\partial p_x} = u^{1/2} \cdot 2\left(\frac{1}{2}\right) p_x^{-1/2} p_y^{1/2} = \underline{u^{1/2}\left(\frac{p_y}{p_x}\right)^{1/2}} = h_x$$

$$\frac{\partial e}{\partial p_y} = u^{1/2}\, 2 \cdot \left(\tfrac{1}{2} p_y^{-1/2} p_x^{1/2}\right) = \underline{u^{1/2}}\left(\frac{p_x}{p_y}\right)^{1/2} = h_y$$

TO GET EXACT VALUE REPLACE VALUES

GET WALRASIAN DEMAND

$p_x = 4 \quad p_y = 1$

UMP? WE CAN REPLACE WITH PREVIOUS

$w = 48$

$x^* = \frac{w}{2p_y} = \frac{48}{8} = 6 \quad \Rightarrow \quad ((6, 24)$

$y^* = \frac{w}{2p_y} = 24$

SHOW WHAT HAPPEN TO BC BECAUSE OF PRICE CHANGE

(USUALLY EXACT SCALE IS NOT IMPORTANT)

TOTAL, SUBSTITUTION AND INCOME EFFECT

TOTAL => A TO C

$$TE\ X = \overset{C_X}{6} - \overset{A_X}{24} = -18$$

$$TE\ Y = \overset{C_Y}{24} - \overset{A_Y}{24} = 0$$

SUBSTITUTION => NEW MAXIMISATION PROBLEM

COMPUTING HICKSIAN FOR COMPENSATED DEMAND

UTILITY IS SAME AFTER CHANGING PRICE

$$\begin{cases} \\ x \cdot y = 24^2 \end{cases}$$ → SAME U BEFORE PRICE CHANGE

SLOPE OF IC TANG SLOPE OF B.C. ??

JUST EXPLOIT TANG CONDITION ON

$$\frac{Y}{X} = \frac{P_X}{P_Y}$$

MRS = SLOPE OF BC

$$\begin{cases} \frac{Y}{X} = \frac{P'_X}{P_Y} \\ X \cdot Y = 24^2 \end{cases} \rightarrow \frac{Y}{X} = \frac{P_X}{P_Y} \rightarrow Y = \frac{P_X X}{Y}$$

$$X \cdot Y = u \rightarrow X \cdot \left(\frac{P_X X}{Y}\right) = u$$

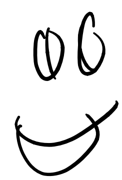
$$\to x^2 = \frac{p_y}{p_x} u$$

$$h_x = \left(\frac{p_y}{p_x}\right)^{1/2} u^{1/2} \qquad h_y = \left(\frac{p_x}{p_y}\right)\left(\frac{p_y}{p_x}\right)^{1/2} u^{1/2}$$

$$= \left(\frac{p_x}{p_y}\right)^{1/2} u^{1/2}$$

NOT REPLACING BECAUSE COMPARING THIS SOLUTION WITH HICKSIAN WITH SHEPPARD (THEY ARE THE SAME)

! CAN GET HICKSIAN DEMAND WITHOUT SOLVING UMP

PICK ONE 

Asymmetry of the Gross Substitute and Complement

- Depending on how we check for gross substitutability or complementarities between two goods, there is potential to obtain different results.

- Can we use an alternative approach to check if two goods are complements or substitutes in consumption?
  - Yes. We next present such approach.

Net Substitutes and Net Complements

- The concepts of net substitutes and complements focus solely on SE.
  - Two goods are ***net (or Hicksian) substitutes*** if
$$\frac{\partial h_i}{\partial p_j} > 0$$
  - Two goods are ***net (or Hicksian) complements*** if
$$\frac{\partial h_i}{\partial p_j} < 0$$

  where $h_i(p_i, p_j, u)$ is the Hicksian demand of good i.

Net Substitutes and Net Complements

- This definition looks only at the shape of the indifference curve.
- This definition is unambiguous because the definitions are perfectly symmetric

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

  - This implies that every element above the main diagonal in the Slutsky matrix is symmetric with respect to the corresponding element below the main diagonal.

Net Substitutes and Net Complements

$$S(p,w) = \begin{pmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \frac{\partial h_1(p,u)}{\partial p_2} & \frac{\partial h_1(p,u)}{\partial p_3} \\ \frac{\partial h_2(p,u)}{\partial p_1} & \frac{\partial h_2(p,u)}{\partial p_2} & \frac{\partial h_2(p,u)}{\partial p_3} \\ \frac{\partial h_3(p,u)}{\partial p_1} & \frac{\partial h_3(p,u)}{\partial p_2} & \frac{\partial h_3(p,u)}{\partial p_3} \end{pmatrix}$$

Net Substitutes and Net Complements

- Proof:
  - Recall that, from Shephard's lemma, $h_k(p,u) = \frac{\partial e(p,u)}{\partial p_k}$. Hence,

$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_k \partial p_j}$$

  - Using Young's theorem, we obtain

$$\frac{\partial^2 e(p,u)}{\partial p_k \partial p_j} = \frac{\partial^2 e(p,u)}{\partial p_j \partial p_k}$$

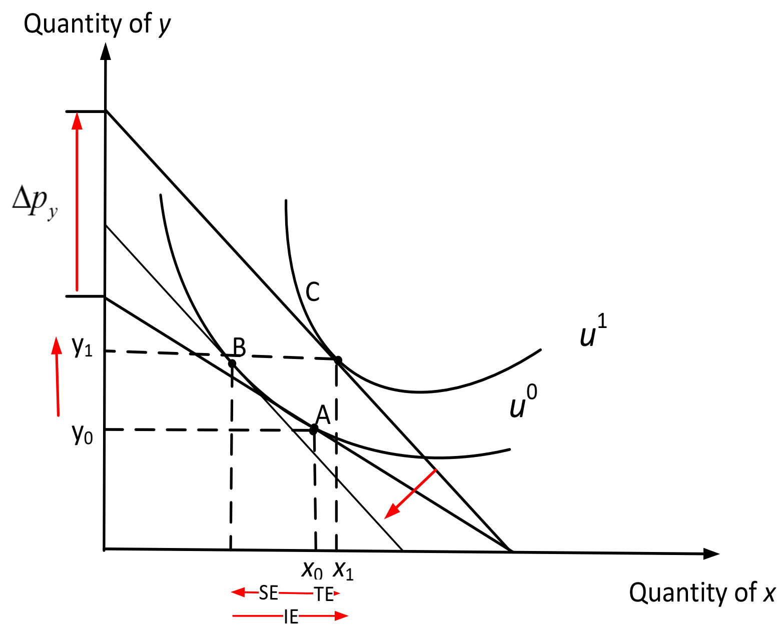
which implies

$$\frac{\partial h_k(p,u)}{\partial p_j} = \frac{\partial h_j(p,u)}{\partial p_k}$$

Net Substitutes and Net Complements

- Even though x and y are gross complements, they are net substitutes.

- Since MRS is diminishing, the own-price SE must be negative $(SE < 0)$ so the cross-price SE must be positive $(TE > 0)$.



A Note on the Euler's Theorem

- We say that a function $f(x_1, x_2)$ is homogeneous of degree k if
$$f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$$
- Differentiating this expression with respect to x_1, we obtain
$$\frac{\partial f(tx_1, tx_2)}{\partial x_1} \cdot t = t^k \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$
or, rearranging,
$$\frac{\partial f(tx_1, tx_2)}{\partial x_1} = t^{k-1} \cdot \frac{\partial f(x_1, x_2)}{\partial x_1}$$

A Note on the Euler's Theorem

- Last, denoting $f_1 \equiv \frac{\partial f}{\partial x_1}$, we obtain

$$f_1(tx_1, tx_2) = t^{k-1} \cdot f_1(x_1, x_2)$$

- Hence, if a function is homogeneous of degree k, its first-order derivative must be homogeneous of degree $k - 1$.

A Note on the Euler's Theorem

- Differentiating the left-hand side of the definition of homogeneity, $f(tx_1, tx_2) = t^k \cdot f(x_1, x_2)$, with respect to t yields

$$\frac{\partial(tx_1, tx_2)}{\partial t} = f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$

- Differentiating the right-hand side produces

$$\frac{\partial(t^k \cdot f(x_1, x_2))}{\partial t} = k \cdot t^{k-1} f(x_1, x_2)$$

A Note on the Euler's Theorem

- Combining the differentiation of LHS and RHS,

$$f_1(tx_1, tx_2)x_1 + f_2(tx_1, tx_2)x_2$$
$$= k \cdot t^{k-1} f(x_1, x_2)$$

- Setting $t = 1$, we obtain

$$f_1(x_1, x_2)x_1 + f_2(x_1, x_2)x_2 = k \cdot f(x_1, x_2)$$

where k is the homogeneity order of the original function $f(x_1, x_2)$.

  – If $k = 0$, the above expression becomes 0.
  – If $k = 1$, the above expression is $f(x_1, x_2)$.

A Note on the Euler's Theorem

- ***Application***:
  - The Hicksian demand is homogeneous of degree zero in prices, that is,

$$h_k(tp_1, tp_2, \ldots, tp_n, u) = h_k(p_1, p_2, \ldots, p_n, u)$$

  - Hence, multiplying all prices by t does not affect the value of the Hicksian demand.
  - By Euler's theorem,

$$\frac{\partial h_i}{\partial p_1} p_1 + \frac{\partial h_i}{\partial p_2} p_2 + \cdots + \frac{\partial h_i}{\partial p_n} p_n$$
$$= 0 \cdot t^{0-1} h_i(p_1, p_2, \ldots, p_n, u) = 0$$

Substitutability with Many Goods

- ***Question:*** Is net substitutability or complementarity more prevalent in real life?
- To answer this question, we can start with the compensated demand function

$$h_k(p_1, p_2, \dots, p_n, u)$$

- Applying Euler's theorem yields

$$\frac{\partial h_k}{\partial p_1}p_1 + \frac{\partial h_k}{\partial p_2}p_2 + \cdots + \frac{\partial h_k}{\partial p_n}p_n = 0$$

- Dividing both sides by h_k, we can alternatively express the above result using compensated elasticities

$$\tilde{\varepsilon}_{i1} + \tilde{\varepsilon}_{i2} + \cdots + \tilde{\varepsilon}_{in} \equiv 0$$

Substitutability with Many Goods

- Since the negative sign of the SE implies that $\tilde{\varepsilon}_{ii} \leq 0$, then the sum of Hicksian cross-price elasticities for all other $j \neq i$ goods should satisfy

$$\sum_{j \neq i} \tilde{\varepsilon}_{ij} \geq 0$$

- Hence, “most” goods must be substitutes.
- This is referred to as ***Hick’s second law of demand***.

$$h_x = \left(\frac{P_y}{P_x}\right)^{1/2} u^{1/2} = \left(\frac{1}{4}\right)^{1/2} (24^2)^{1/2} = \frac{1}{2} \cdot 24 = 12$$

$$h_y = \left(\frac{P_x}{P_y}\right)^{1/2} u^{1/2} = 4^{1/2} (24^2)^{1/2} = 2 \cdot 24 = 48$$

$B(12, 48)$

C?

$$x^* = \frac{w}{2P_x} = \frac{48}{8} = 6 \qquad C(6, 24)$$

$$y^* = \frac{w}{2P_y} = \frac{48}{2} = 24$$

SE => FROM (A) TO (B) -> B - A

$SE\ x = 12 - 24 = -12$

$SE\ y = 48 - 24 = +24$

IE => FROM (B) TO (C) -> C - B

$IE\ x = 6 - 12 = -6$

$IE\ y = 24 - 48 = -24$

TE => SE + IE

$TE_x = -12 - 6 = -18$

$TE\ y = 0$

AFTER BEFORE

C.V. A B $P_x \uparrow$

$$CV = e(P_x^1, u_0) - \boxed{e(P_x^1, u_1)} = P_x^1 \cdot h_x + P_y h_y - 48 =$$

$w = 48$

POSITIVE INCOME TRANSFER

SOLUTION OF EMP -> WE GET BUNDLE CALLED HICKSIAN DEMAND

$$= \overset{\text{NEW PRICE}}{4} \cdot 12 + 1 \cdot 48 - (48) = 96 - 48 = 48$$

TO KEEP THE GUY AT THE SAME LEVEL AFTER PRICE CHANGE WE HAVE TO TRANSFER 48 OF WEALTH

EV BA $\rightarrow$ BEFORE AFTER

$$EV = \underbrace{e(p_x^0, u_0)}_{w = 48} - e(p_x^0, u_1) =$$

$$h_x(p_x^0, p_y, u_1) = h_x^*$$

$$h_y(p_x^0, p_y, u_1) = h_y^*$$

$$e(p_x, p_y, u_1) = h_x^*$$

$$e(p_x, p_y, u_1) = h_y^*$$

HICKSIAN

$$\Rightarrow \left(\frac{p_x}{p_y}\right)^{1/2} u^{1/2}$$

So $u_1 = u(\overset{(6,\,24)}{c}) = u(6, 24) = 144$

$$w - 24 = 48 - 24 = 24$$

BEF PRICE CHANGE, TO GET SAME VALUE WE HAVE TO TAKE AWAY SOME INCOME

IF $p_x \Rightarrow u\downarrow$ $\rightarrow$ SO TAKE INCOME TO IT THAT IS 24

NEGATIVE INCOME TRANSFER

PRICE OF CU SUITE IS W

THE MORE YOU PAY A GUY, MORE HE WILL WORK

IF YOU BECOME A MILLIONAIRE, 1C €

Advanced Microeconomic Theory

Chapter 3: Aggregate demand

Outline

- Welfare evaluation
  - Compensating variation
  - Equivalent variation
- Quasilinear preferences
- Slutsky equation revisited
- Income and substitution effects in labor markets
- Gross and net substitutability
- Aggregate demand

Aggregate Demand

Aggregate Demand

WALRAS

- We now move from individual demand, $x_i(p, w_i)$, to aggregate demand,

$$\boxed{\sum_{i=1}^{I} x_i(p, w_i)}$$

AGG. OF WALRAS OF EACH INDIVIDUAL

which denotes the total demand of a group of I consumers.

- Individual i's demand $x_i(p, w_i)$ still represents a vector of L components, describing his demand for L different goods.

IS CONVENIENT TO HAVE AGGREGATE
DEMAND THAT DEPENDS ON AGGREGATE
WEALTH

Aggregate Demand

- We know individual demand depends on prices and individual's wealth.
  - When can we express aggregate demand as a function of prices and aggregate wealth?
  - In other words, when can we guarantee that aggregate demand defined as

$$x(p, w_1, w_2, \dots, w_I) = \sum_{i=1}^{I} x_i(p, w_i)$$

satisfies

SUM OF IND DEMAND FUNCTION OF P AND

$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

SUM OF WEALTH
(AND NOT HOW WEALTH IS DISTRIBUTED)

Aggregate Demand

- This is satisfied if, for any two distributions of wealth, $(w_1, w_2, \dots, w_I)$ and $(w_1', w_2', \dots, w_I')$ such that $\sum_{i=1}^{I} w_i = \sum_{i=1}^{I} w_i'$, we have

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} x_i(p, w_i')$$

- For such condition to be satisfied, let's start with an initial distribution $(w_1, w_2, \dots, w_I)$ and apply a differential change in wealth $(dw_1, dw_2, \dots, dw_I)$ such that the aggregate wealth is unchanged, $\sum_{i=1}^{I} dw_i = 0$.

Aggregate Demand

- If aggregate demand is just a function of aggregate wealth, then we must have that

$$\sum_{i=1}^{I} \frac{\partial x_i(p,w_i)}{\partial w_i} dw_i = 0 \text{ for every good } k$$

In words, the wealth effects of different individuals are compensated in the aggregate. That is, in the case of two individuals i and j,

$$\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$$

for every good k.

Aggregate Demand

- This result *does not* imply that $IE_i > 0$ while $IE_j < 0$.
- In addition, it indicates that its absolute values coincide, i.e., $|IE_i| = \left|IE_j\right|$, which means that any redistribution of wealth from consumer i to j yields

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} dw_i + \frac{\partial x_{kj}(p, w_j)}{\partial w_j} dw_j = 0$$

which can be rearranged as

$$\frac{\partial x_{ki}(p, w_i)}{\partial w_i} \underbrace{dw_i}_{-} = -\frac{\partial x_{kj}(p, w_j)}{\partial w_j} \underbrace{dw_j}_{+}$$

- Hence, $\frac{\partial x_{ki}(p,w_i)}{\partial w_i} = \frac{\partial x_{kj}(p,w_j)}{\partial w_j}$, since $|dw_i| = \left|dw_j\right|$.

Aggregate Demand

- In summary, for any
  - fixed price vector p,
  - good k, and
  - wealth level any two individuals i and j

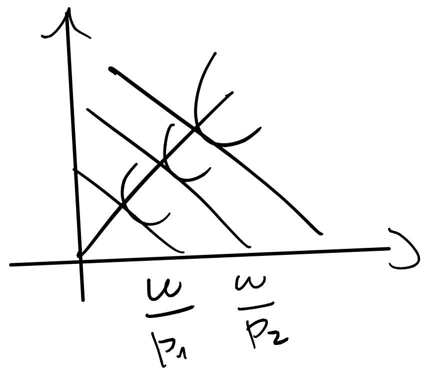
  the wealth effect is the same across individuals.
- In other words, the wealth effects arising from the distribution of wealth across consumers cancel out.
- This means that we can express aggregate demand as a function of aggregate wealth

$$\sum_{i=1}^{I} x_i(p, w_i) = x\left(p, \sum_{i=1}^{I} w_i\right)$$

Aggregate Demand

- Graphically, this condition entails that all consumers exhibit *parallel*, *straight* wealth expansion paths.
  - ***Straight***: wealth effects do not depend on the individuals' wealth level.
  - ***Parallel***: individuals' wealth effects must coincide across individuals.
    - Recall that wealth expansion paths just represent how an individual demand changes as he becomes richer.

$w \not\subset p$

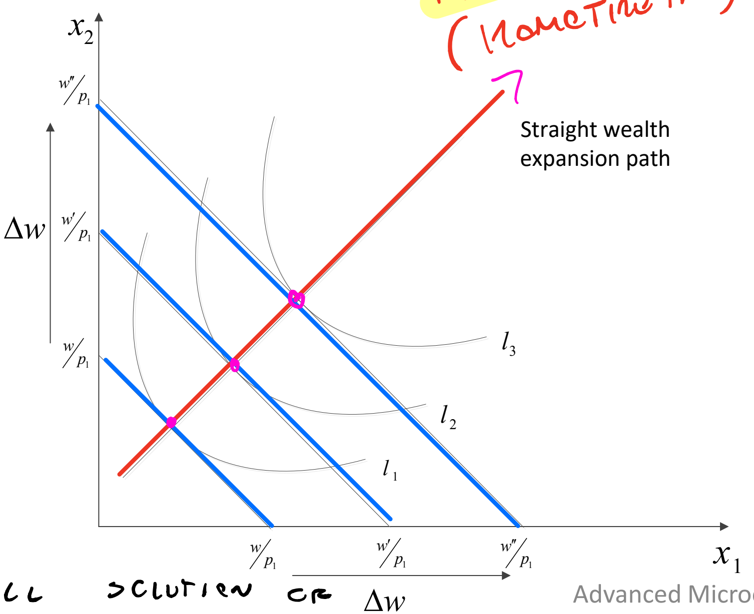


Aggregate Demand

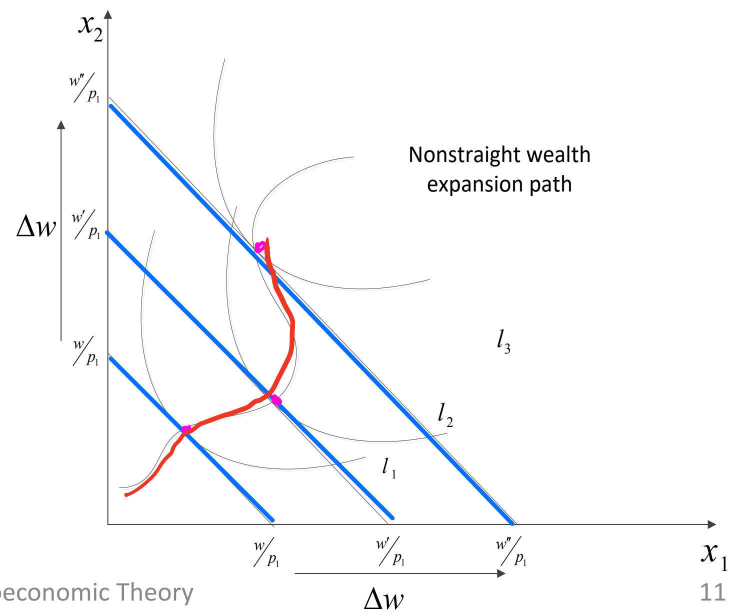
A given increase in wealth leads the same change in the consumption of good x_i, *regardless* of the initial wealth of the individual

HOMOGENEOUS !!

(HOMOTHETIC)



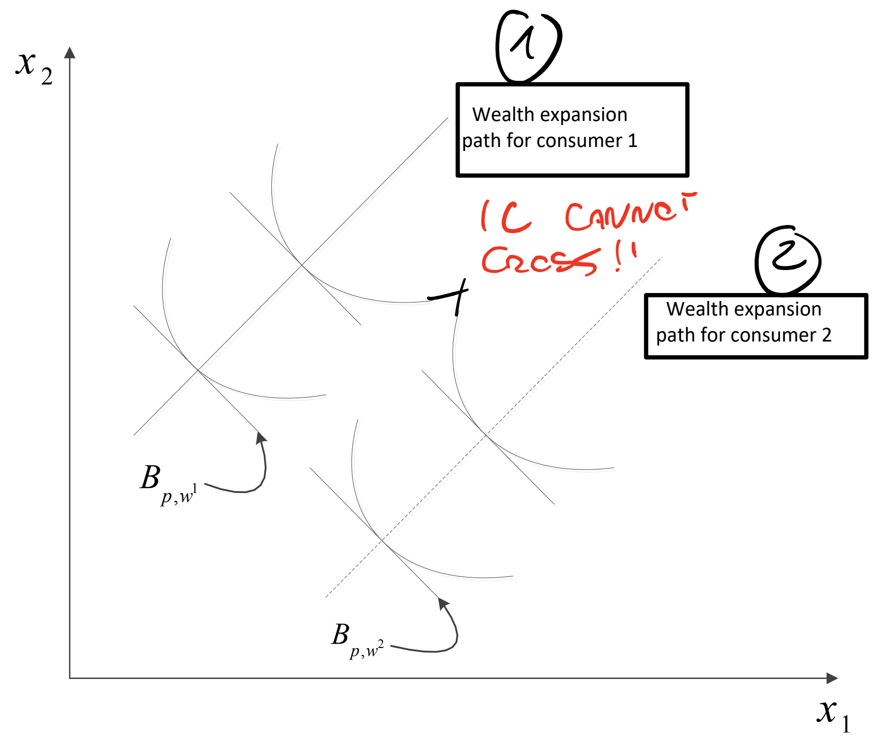
A given increase in wealth leads to changes in the consumption of good x_i that are *dependent* on the individual's wealth level



ALL SOLUTION OF UMP ARE IN THE SAME LINE

Aggregate Demand

- Individuals' wealth effects coincide.
- The wealth expansion path for consumers 1 and 2 are parallel to each other
  - both individuals' demands change similarly as they become richer.



Aggregate Demand

- Preference relations that yield *straight* wealth expansion paths:
  - Homothetic preferences
  - Quasilinear preferences
- Can we embody all these cases as special cases of a particular type of preferences?
  - Yes. We next present such cases.

Aggregate Demand: Gorman Form

- ***Gorman form.*** A necessary and sufficient condition for consumers to exhibit parallel, straight wealth expansion paths is that every consumer's indirect utility function can be expressed as:

  PRICES AND WEALTH

  $$v_i(p, w_i) = a_i(p) + b(p)w_i$$

  EACH INDIVIDUAL

  This indirect utility function is referred to as the *Gorman form*. (IND UTILITY IS LINEAR IN w_i)

- Indeed, in case of quasilinear preferences

  $$v_i(p, w_i) = a_i(p) + \frac{1}{p_k}w_i \quad \text{so that} \quad b(p) = \frac{1}{p_k}$$

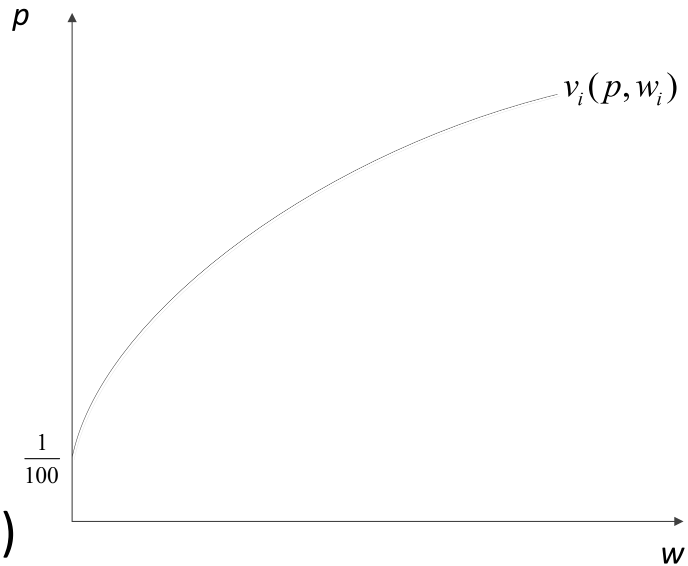
Aggregate Demand: Gorman Form

- *Example* (continued):
  - The vertical intercept of this function is $p(0) = \frac{1}{100}$.
  - The slope of this function is

$$\frac{\partial p(w_i)}{\partial w_i} = \frac{1}{10} + \frac{1}{10\sqrt{1+40w_i}} > 0$$

and it is decreasing in w_i (concavity)

$$\frac{\partial^2 p(w_i)}{\partial w_i^2} = \frac{2}{(1+40w_i)^{3/2}}$$



$x_i = a_i(p) + b(p)\, w_i$ IF LINEAR IN w, WHEN WE TAKE SUMMATION

Aggregate Demand: Gorman Form

$\sum x_i = \sum a_i(p) + \sum b(p)\, w_i$

- Let's show that, for indirect utility functions of the Gorman form, we obtain

$\sum a_i(p) + b(p) \underbrace{\sum w_i}_{\text{TOTAL WEALTH}}$

SINCE b CONSTANT CAN BE OUT OF SUMMATION

$$\sum_{i=1}^{I} x_i(p, w_i) = x(p, \sum_{i=1}^{I} w_i)$$

SO DO NOT DEPEND ON HOW W. IS DISTRIBUTED SINCE IS A LINEAR FUNCTION

- First, use Roy's identity to find the Walrasian demand associated with this indirect utility function

APPLY ROY WE CAN FIND WALRAS DEMAND

$$-\frac{\frac{\partial v_i(p, w_i)}{\partial p}}{\frac{\partial v_i(p, w_i)}{\partial w}} = x_i(p, w_i)$$

NO PROOF

Aggregate Demand: Gorman Form

- In particular, for good j,

$$-\frac{\frac{\partial v_i(p,w_i)}{\partial p_j}}{\frac{\partial v_i(p,w_i)}{\partial w}} = -\frac{\frac{\partial a_i(p)}{\partial p_j}}{b(p)} - \frac{\frac{\partial b(p)}{\partial p_j}}{b(p)} w_i = x_i^j(p,w_i)$$

Walras linear

- In matrix notation,

$$-\frac{\nabla_p v_i(p,w_i)}{\nabla_w v_i(p,w_i)} = -\frac{\nabla_p a_i(p)}{b(p)} - \frac{\nabla_p b(p)}{b(p)} w_i = x_i(p,w_i)$$

for all goods.

Aggregate Demand: Gorman Form

- We can compactly express $x_i(p, w_i)$ as follows

$$-\frac{\nabla_p v_i(p, w_i)}{\nabla_w v_i(p, w_i)} = \alpha_i(p) + \beta(p) w_i = x_i(p, w_i)$$

where $-\frac{\nabla_p a_i(p)}{b(p)} \equiv \alpha_i(p)$ and $-\frac{\nabla_p b(p)}{b(p)} \equiv \beta(p)$.

Aggregate Demand: Gorman Form

- Hence, aggregate demand can be obtained by summing individual demands

$$\alpha_i(p) + \beta(p)w_i = x_i(p, w_i)$$

across all I consumers, which yields

$$\sum_{i=1}^{I} x_i(p, w_i) = \sum_{i=1}^{I} \alpha_i(p) + \beta(p) \sum_{i=1}^{I} w_i$$

$$= \sum_{i=1}^{I} \alpha_i(p) + \beta(p)w = x(p, \sum_{i=1}^{I} w_i)$$

where $\sum_{i=1}^{I} w_i = w$.

QUASI LINEAR UTILITY FUNCTION CAN BE WRITTEN IN GORMAN FORM?

QUASILINEAR SINCE LINEAR IN Y

$$u(x,y) = \ln x + y$$

1 BEFORE WE HAVE TO FIND INDIRECT UTILITY AND 2 THEN CHECK LINEAR IN w_i

S.T. $p_x \cdot x + p_y \cdot y \leq w$

$$L = \ln x + y + \lambda(w - p_x \cdot x - p_y \cdot y)$$

INT. SOLUTION

$$\begin{cases} \dfrac{\partial L}{\partial x} = \dfrac{1}{x} - \lambda p_x = 0 \\ \dfrac{\partial L}{\partial y} = 1 - \lambda p_y = 0 \\ \dfrac{\partial L}{\partial \lambda} = w - p_x x - p_y \cdot y = 0 \end{cases}$$

$x = 0$ $\quad y = \frac{w}{p_y}$ $\qquad x = \frac{w}{p_x}$ $\quad y = 0$ CHECK CORNER SOLUTIONS

$$x^* = \frac{p_y}{p_x}$$

$$y^* = w - \cancel{p_x} \frac{p_y}{\cancel{p_x}} + p_y \cdot y \rightarrow y^* = \frac{w - p_y}{p_y}$$

FIND IND. UTILITY

$$u(x,y) = \ln\left(\frac{p_y}{p_x}\right) + \frac{w - p_y}{p_x} \rightarrow y$$

$$= \underbrace{\ln\left(\frac{p_y}{p_x}\right) - 1}_{a} + \underbrace{\frac{1}{p_y}}_{b} w_i$$

SATISFY GORMAN FORM WHERE INTERCEPT IS

$$a_i(p) = \ln\left(\frac{p_y}{p_x}\right) - 1$$

$$b(p) = \frac{1}{p_y}$$

DISTRIBUTION ON INCOME DOES NOT DEPENDS ON THE TOTAL WEALTH

EXERCISE 3.2 P. 176 GARCIA

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$$

FIND CV, CE, CS

UMP (SOLUTIONS, C'È SVOLGIMENTO SUL LIBRO)

$$x_1(p,w) = \frac{\alpha \cdot w}{(\alpha+\beta)p_1} \qquad x_2(p,w) = \frac{\beta \cdot w}{(\alpha+\beta)p_2}$$

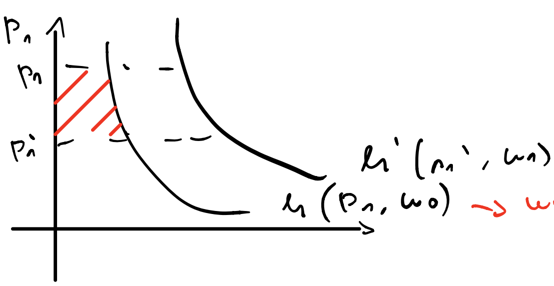
$$h_1(p,w) = \left(\frac{\alpha}{\beta}\frac{p_2}{p_1}\right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

$$h_2(p,w) = \left(\frac{\beta}{\alpha}\frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

CV $\quad p_1 = p_2 = 2 \qquad \alpha = \beta = \frac{1}{2} \quad w = 10$

$p_1' = 1 \Rightarrow p \downarrow$

AB AFTER BEFORE



$p \downarrow \Rightarrow u \uparrow \Rightarrow h_1$ SHIFT RIGHT

2. WAY TO APPLY C.V.

$$CV = \int^{p_1} h_1(p, w_0)\, dp_1 = \int_1^2 \left(\frac{\alpha}{\beta}\frac{p_2}{p_1}\right)^{\frac{\beta}{\alpha+\beta}} u_0^{\frac{1}{\alpha+\beta}}$$

$\cdot \; p_1'$

$p_1' < p_1$ SINCE $p \downarrow$

$$= \int_1^2 \left(\frac{2}{p_1}\right)^{1/2} \frac{\cancel{1}\,1}{\frac{2}{\cancel{1/2}\,1}} \; 2.5 \; dp_1$$

WE MANTAIN p_1 AS VARIABLE

$$u_1^{\frac{1}{\alpha+\beta}} = u_0^{1} \rightarrow \text{OLD LEVEL OF UTILITY}$$

$u = 2.5$

WALRAS

$x_1^* = 2.5$

$x_2^* = 2.5$

$$2^{1/2} \; \frac{1}{2} \, (p_1)^{-1/2} \; 2.5 \; dp_1 = \int p_1^{-1/2} \, dp_1$$

$$\sqrt{2} \;\; 2.5 \qquad = \sqrt{2}\, 2.5 \left(2 \cdot \sqrt{2} - 2\right)$$

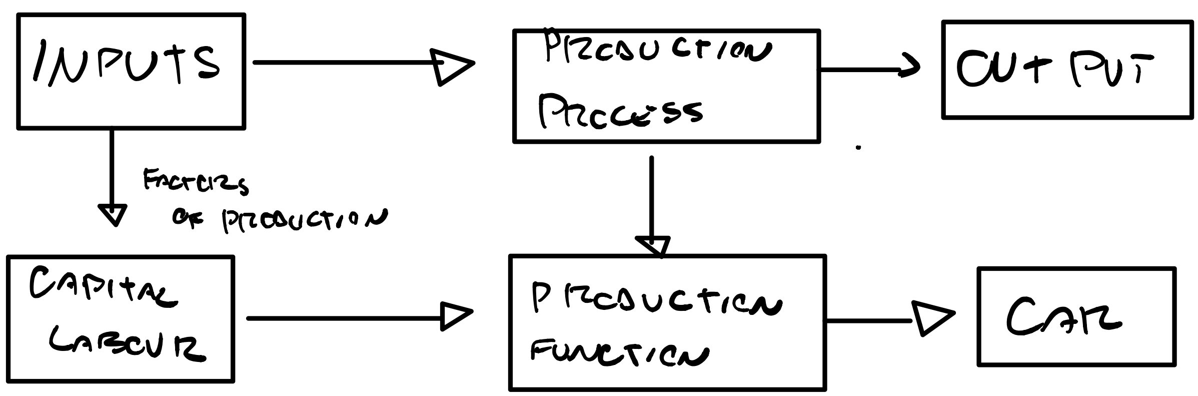
$$\frac{p_1^{-\frac{1}{2}+1}}{\frac{1}{2}} \qquad 2 \, p_1^{1/2}$$

Firm to produce use some technologies.
They use inputs that are factors of production that are combine in a production function (production process). And then after the input are combine in the production process they give and output.
To produce a car we will use capital (machinery) and Labor and then there will be a production process that give an output that is car.
Production process can be approximated by a production function

FIRMS

TECHNOLOGIES

—> process from input to output



PRODUCTION FUNCTION

Maximum amount of output possible from input bundle

Advanced Microeconomic Theory

Chapter 4: Production function and Profit Maximization Problem (PMP)

Outline

- Production sets and production functions
- Profit maximization and cost minimization
- Cost functions
- Aggregate supply
- Efficiency (1st and 2nd FTWE)

Production Functions

Technology

- A technology is a process by which inputs are converted to an output. → Given quantity of output
- *E.g.* labor, a computer, a projector, electricity, and software are being combined to produce this lecture.
- Usually several technologies will produce the same product -- a blackboard and chalk can be used instead of a computer and a projector.
- Which technology is "best"?
- How do we compare technologies?

Inputs

When we have technologies we have inputs bundles. It is similar to consumption bundle but refers to the firm to produce certain output.

- x_i denotes the amount used of input i; *i.e.* the level of input i.
- An input bundle is a vector of the input levels; $(x_1, x_2, \ldots, x_n)$.
- *E.g.* $(x_1, x_2, x_3) = (6, 0, 9)$.

Output

- y denotes the output level.
- The technology's production function states the maximum amount of output possible from an input bundle.

PRODUCTION FUNCTION

$$y = f(x_1, x_2, \ldots\ldots x_n)$$

This is a scalar since we are considering only one good as output.

You will have many technologies and production will give the most efficient way of producing y given x1, x2 ... xn

Technology set

- A production plan is an input bundle and an output level; $(x_1, \ldots, x_n, y)$.
- A production plan is feasible if
$$y \leq f(x_1, x_2, \ldots\ldots x_n)$$
- The collection of all feasible production plans is the technology set.

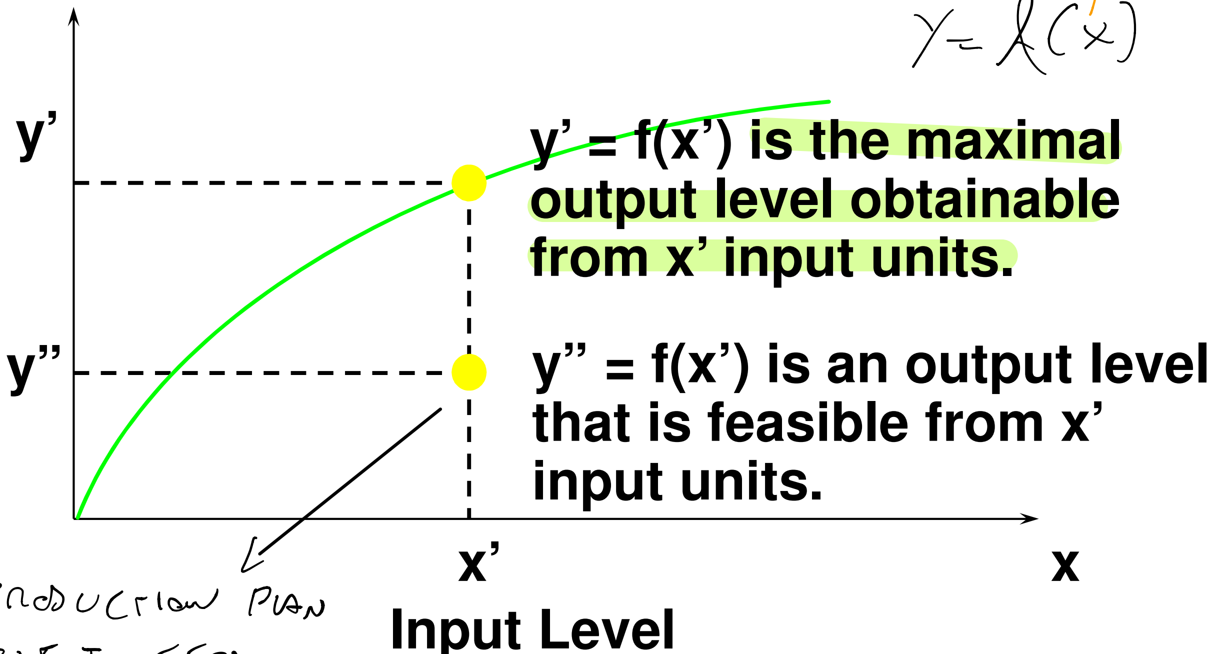
Feasible if this tech logic produce at least y.
So collection of this feasible production plan is called technology set.

Technology set - I

ONLY 1 INPUT

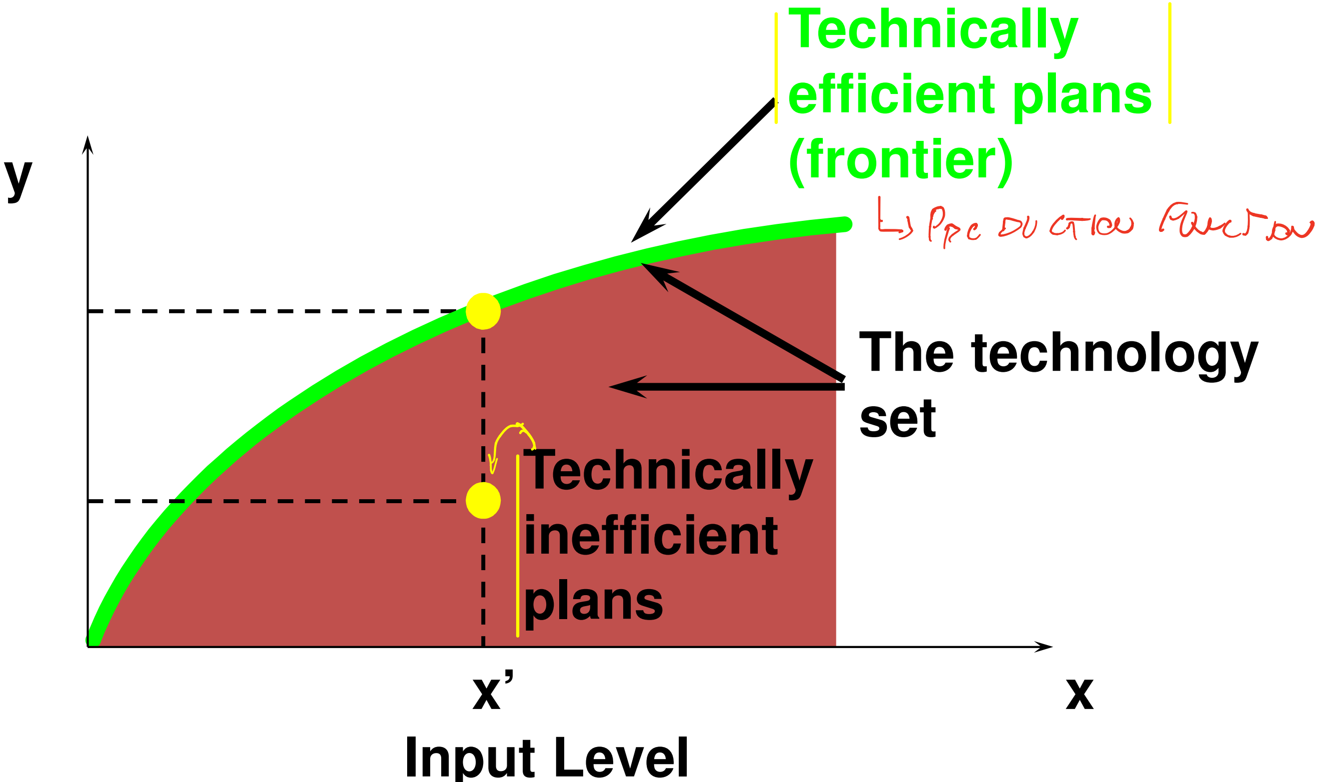
- One input one output (simpler case)

$Y = f(x)$

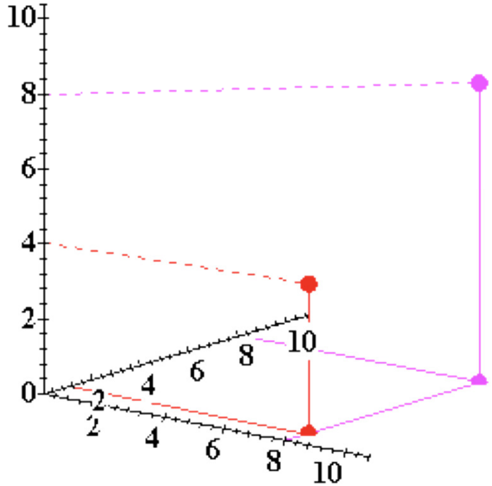


THIS PRODUCTION PLAN IS FEASIBLE TO FIRM BUT INEFFICIENT SINCE Y' GIVE MORE EFFICIENT OUTPUT

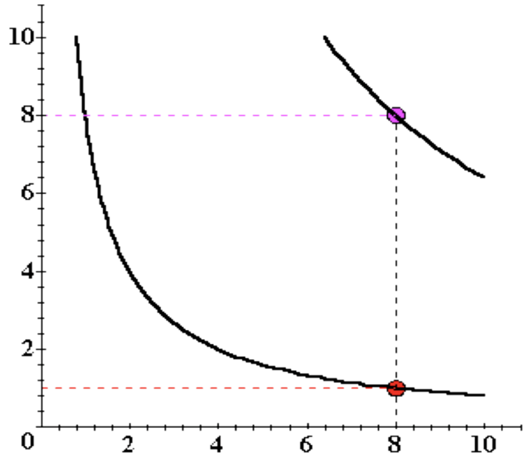
Technology set - II



Multiple inputs, one output



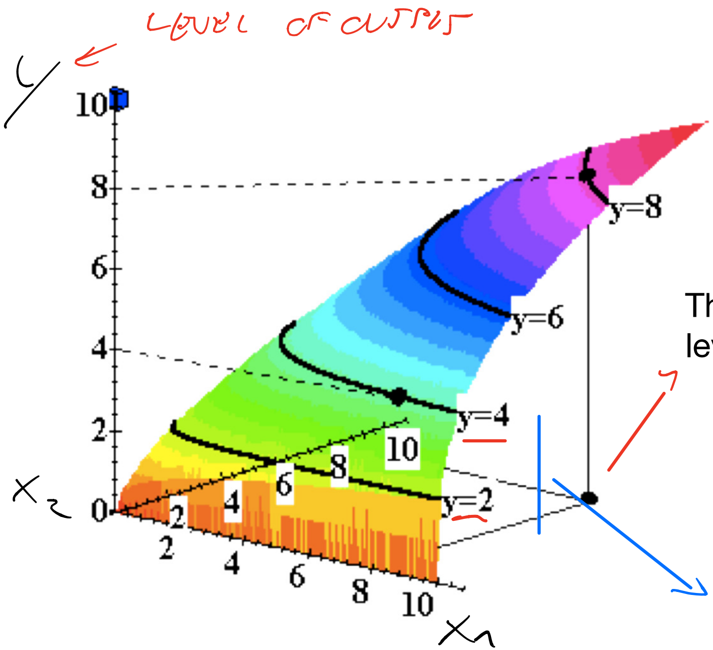
Multiple inputs, one output



Isoquant: the set of all input bundles that yield at most the same output level y.

EXAMPLE

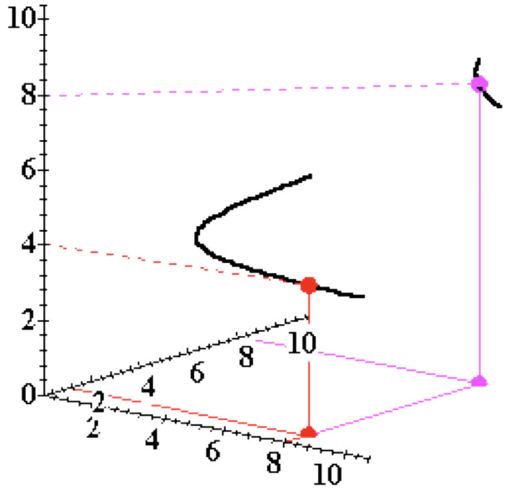
Multiple inputs, one output



This production plan gives this level of production.

This sections give the same level of production output.

Multiple inputs, one output

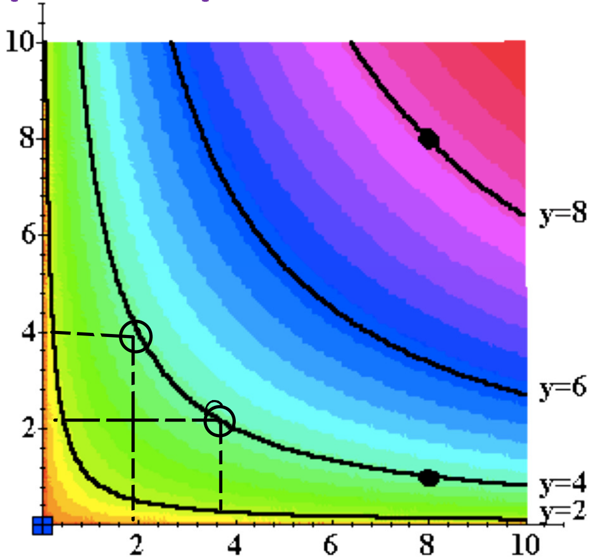


Isoquant: How is it obtained?

Multiple inputs, one output

Sort of

substitution!

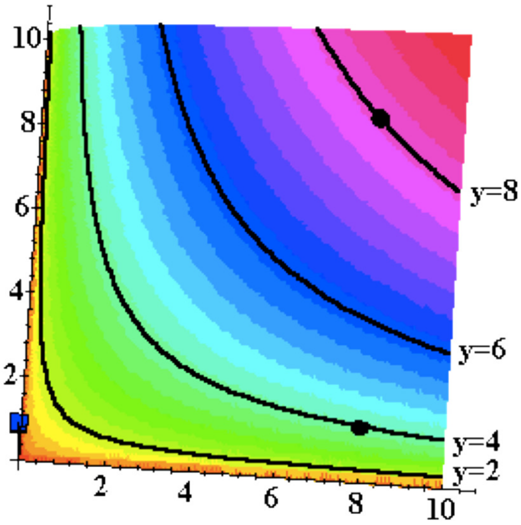


Combination of factors that give the same level of output. We can notice that as the IC for the consumer were representing combination of good that gave the same level of utility.

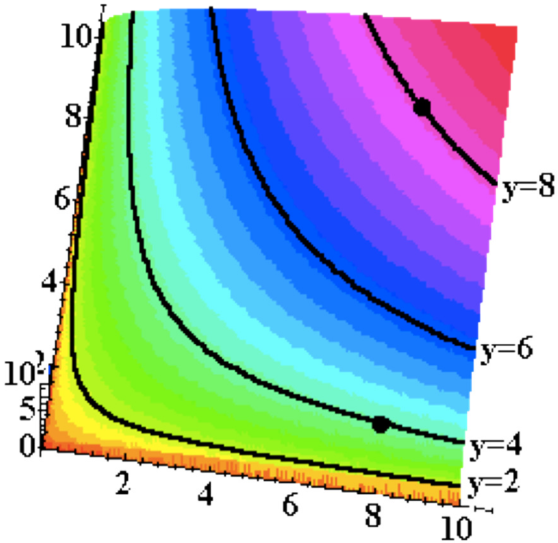
Isoquant represent the combination of inputs that give the same level of output(or production)

Isoquant: level map (like indifference curve for utility) – combination of inputs that give same output level

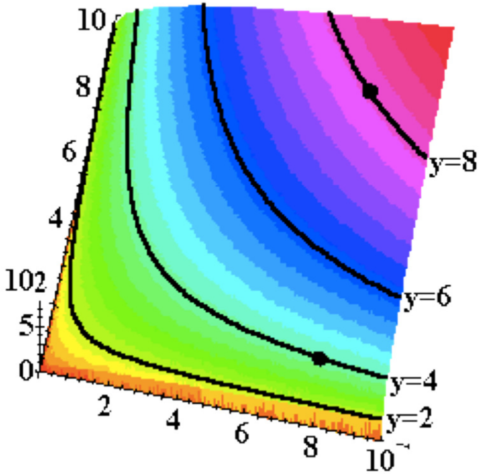
Multiple inputs, one output



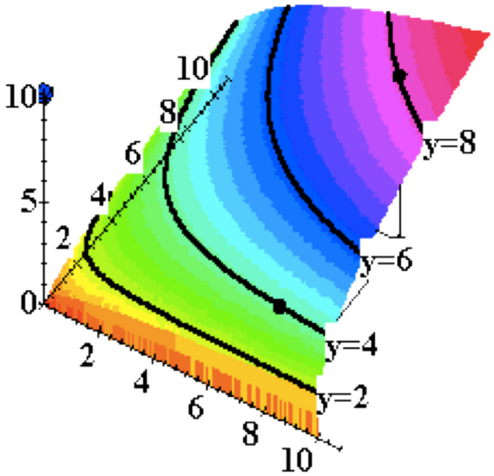
Multiple inputs, one output



Multiple inputs, one output



Multiple inputs, one output



Slope of Isoquant is called the Marginal rate of technical substitution

EXAMPLE WITH 2 INPUTS

A simple production function

- We consider the following production function in which output depends on physical capital (k), such as machinery, and labour (l)

N. Of workers or hours of works, It depends on the model

Inputs cannot be negative: positive capital and labour

$$\boxed{y = f(k, l)}$$

with $k, l \geq 0$, $\frac{\partial f(y)}{\partial x} > 0$ and decreasing, i.e. $\frac{\partial^2 f(y)}{\partial x \partial x} < 0$, where x is the generic input.

$\partial x \longrightarrow$ GENERIC INPUT

The first derivative is called the marginal productivity of input x (= k, l).

Derivative: If i increase small amount of capital how much production will increase?

0 1 2

Half worker is consider like a part-time worker. So we are not considering discrete case but continuous

$$0 \quad \underbrace{1 \quad 2}_{\Delta Y} \quad 3 \; \ldots \; \underbrace{100 \quad 101}_{\Delta Y'}$$

Same increase of the two firms but delta y' < delta y. ==> marginal productivity is decreasing.

In agriculture you have an amount of land: initially production will increase if i put 2 worker instead of 1 but if i put more worker in the same instance of land then worker will get a decreasing production since there is a lot of persons.

"A firm uses intermediate goods before reach the production in reality"

Now define the MRTS.

Marginal rate of technical substitution (MRTS)

Is the slope of the isoquant —> isoquant is the combination of inputs giving the same output level.

To find the MRTS we compute the total differential of the production function.

$$y = f(k, l)$$

This is a production function in two variables. The total differential now is:

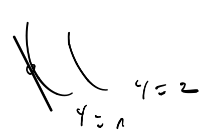
VARIATION OF Y

$$dy = \frac{\partial f}{\partial k} \cdot dk + \frac{\partial f}{\partial l} dl = 0$$

VARIATION SHOULD BE 0!

TOTAL VARIATION → MARG PRODUCTIVITY OF K how MANY UNIT OF CAPITAL + MARG PRODUCTIVITY OF L ∘ how MANY UNIT OF l

$$dy = 0$$



↳ SLOPE OF THIS ISOQUANT

SLOPE OF ISOQUANT = $\dfrac{\delta K}{\partial \ell}$

Y IS NO MORE A VARIABLE THEN WE ONLY CONSIDER ONE VARIABLES TO COMPUTE THE SLOPE SO

$$\frac{\delta K}{\partial \ell}$$

$$\frac{\delta f}{\delta k} dk = -\frac{\delta f}{\delta \ell} d\ell \rightarrow \frac{dk}{d\ell} = -\frac{\frac{\delta f}{\partial \ell}}{\frac{\delta f}{\delta k}} \begin{array}{l} MP_\ell \\ MP_k \end{array}$$

ISOQUANT = RATIO MARG PRODUCTIVITY OF ℓ AND K

M. PROD ARE BOTH POSITIVE SO SLOPE IS NEGATIVE (DECREASING SLOPE)

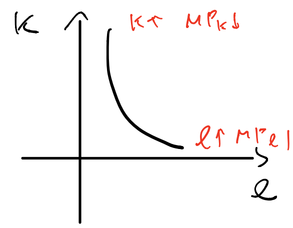
MRTS is given by the ratio of the Marginal productivity. The MRTS is how much you have to substitute the two good to maintain the same level of production.

According to the example the MRTS is increasing or decreasing moving to the right?
Increasing I the MRTS is decreasing.

$$\frac{\frac{\delta f}{\partial \ell}}{\frac{\delta f}{\delta k}}$$

1° DER IS POSITIVE BUT 2° DER IF IS NEGATIVE IS DECREASING

IF $\ell \uparrow \Rightarrow MP_\ell \downarrow \rightarrow K \downarrow \Rightarrow MP_K \uparrow \Rightarrow$ MRTS $\downarrow$ SINCE DENOMINATOR $\uparrow$ AND NUMERATOR $\downarrow$



K
K↑ MPk↓
l↑ MPl↓
l

Production function

- Along an **isoquant** y is constant, therefore totally differentiating the production function

$$(dy =) \quad \frac{\partial f(\bar{y})}{\partial k} dk + \frac{\partial f(\bar{y})}{\partial l} dl = 0$$

solving

$$\frac{dl}{dk} = -\frac{\frac{\partial f(\bar{y})}{\partial k}}{\frac{\partial f(\bar{y})}{\partial l}}, \text{ where } -\frac{\frac{\partial f(\bar{y})}{\partial k}}{\frac{\partial f(\bar{y})}{\partial l}} = MRTS_{l,k}(\bar{y})$$

  - $MRTS_{l,k}(\bar{y})$ is the **Marginal Rate of Technical Substitution** measures how much k must decrease (increase) if l increases (decreases) so as the maintain the same output [the book defines MRTS without the minus sign]

Diminishing MRTS

- The slope of the firm's isoquants is

$$MRTS_{l,k} = \frac{dk}{dl}, \text{ where } MRTS_{l,k} = -\frac{f_l}{f_k}$$

(NB. K is in the vertical axes ointhe isoquant graph)

- Where $f_l = \frac{\partial f(y)}{\partial l}$ is the marginal productivity of labour and $f_l = \frac{\partial f(y)}{\partial l}$ is the marginal productivity of capital
- Differentiating $MRTS_{l,k}$ with respect to labor and taking into account that along an isoquant $k = k(l)$ i.e. capital is a function $k(.)$ of labour yields

$$\boxed{\frac{\partial |MRTS_{l,k}|}{\partial l} = \frac{f_k\left(f_{ll}+f_{lk}\cdot\frac{dk}{dl}\right)-f_l\left(f_{kl}+f_{kk}\cdot\frac{dk}{dl}\right)}{(f_k)^2}}$$

(we apply the rule of a composite function)

y = f(x) / g(x)

allora

y' = [f '(x) · g(x) - f(x) · g'(x)] / [g(x)]²

SOSTITUISCO

WE WANT TO CHECK HOW THE SLOPE CHANGE

Diminishing MRTS

- Using the fact that $\boxed{\frac{dk}{dl} = -\frac{f_l}{f_k}}$ (slope an isoquant) along an isoquant and Young's theorem $f_{lk} = f_{kl}$ (if f double differentiable than cross derivatives are symmetric),

$$\boxed{\begin{aligned}\frac{\partial |MRTS_{l,k}|}{\partial l} &= \frac{f_k\left(f_{ll} - f_{lk} \cdot \frac{f_l}{f_k}\right) - f_l\left(f_{kl} - f_{kk} \cdot \frac{f_l}{f_k}\right)}{(f_k)^2} \\ &= \frac{f_k f_{ll} - f_{lk} f_l - f_l f_{kl} + f_{kk} \cdot \frac{f_l^2}{f_k}}{(f_k)^2}\end{aligned}}$$

Along and isoquant k is a function of l. There is a relationship between k and l. So computing derivative we have to keep in mind that k is function of l

$$|MRTS| = \left|\frac{dk}{dl}\right| = \frac{\boxed{f_l(l, k(l))}}{f_k(l, k(l))}$$

DERIVE

K AND l ARE NOT FREE TO MOVE ALONG THE ISOQUANT

DER OF A RELATION

MRTS → I HAVE TO PUT '−' SIGN

$$\frac{\partial |MRTS|}{\partial l} = \frac{f_k\left(f_{ll} - \underline{f_{lk}\left(\frac{\partial k}{\partial l}\right)}\right) - f_l\left(f_{kl} - f_{kk}\left(\frac{\partial k}{\partial l}\right)\right)}{f_k^2} =$$

COMPOSITE FUNCTION DERIVATIVE

$$l \to k(l) \to f(k(l))$$

f_l WITH RESPECT TO K · DER OF TERMS

$f_l(l) \to$ DER IS f_{ll}

$f_l(k_l) \to$ DER $f_{lk} \cdot \frac{\partial k}{\partial l}$

$$= \frac{f_k\left(f_{ll} - f_{lk} \cdot \frac{f_l}{f_k}\right) - f_l\left(f_{kl} - f_{kk} \cdot \frac{f_l}{f_k}\right)}{f_k^2} =$$

× PRODUCTS

THIS TWO DERIVATIVE ARE THE SAME

$$= \frac{f_k \cdot f_{ll} - \overline{f_{lk} \cdot f_l - f_l f_{kl}} + f_{kk} f_l^2 / f_k}{f_k^2} =$$

NEG NEG

$$\simeq \frac{\left| f_k f_{\ell\ell} \right| - 2 f_{\ell k} f_\ell + \left| f_{kk} \frac{f_\ell^2}{f_k} \right|}{f_k^2} = \frac{|MRS|}{\delta \ell}$$

POS

SIGN OF THIS ?? DEPENDS ON SIGN OF 2° DER OF PRODUCTION FUNCTION

SUFFICIENT CONDITION FOR THE DERIVATIVE TO BE NEGATIVE ?

IF $f_{\ell k} > 0 \Rightarrow$ DERIVATIVE < 0

BUT NOT NECESSARY!

IF $f_{\ell k} < 0$ NOT FOR SURE DERIVATIVE > 0

Given a fix amount of workers if you increase capital the Marginal productivity of the worker will increase!

Diminishing MRTS

- Multiplying numerator and denominator by f_k

$$\frac{\partial MRTS_{l,k}}{\partial l} = \frac{\overbrace{f_k^2}^{+}\,\overbrace{f_{ll}}^{-} + \overbrace{f_{kk}}^{-}\,\overbrace{f_l^2}^{+} - \overbrace{2 f_l f_k}^{+}\ \overbrace{f_{lk}}^{-\text{ or}+}}{(f_k)^3}$$

NO

(I have used f_{lk}=f_{kl} by Young's theorem, if f twice differentiable, i.e. second derivatives exist.)

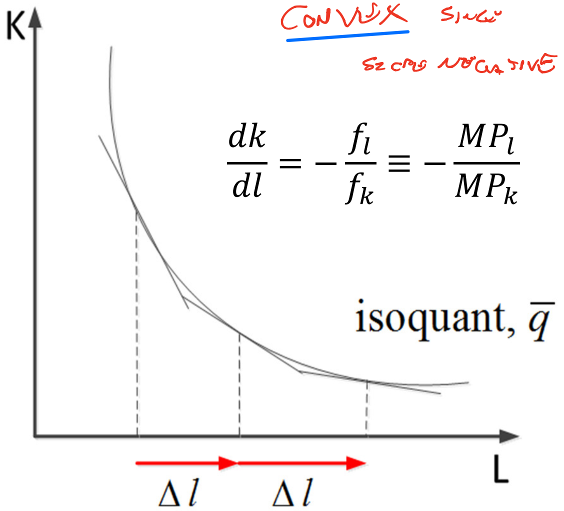
- Thus,
  - If $\boxed{f_{lk} > 0}$ (i.e., $\uparrow k \Longrightarrow \uparrow MP_l$), then $\boxed{\frac{\partial MRTS_{l,k}}{\partial l} < 0}$
  - If $\boxed{f_{lk} < 0}$, then we have

$$\left|f_k^2 f_{ll} + f_{kk} f_l^2\right| \begin{Bmatrix} > \\ < \end{Bmatrix} |2 f_l f_k f_{lk}| \Longrightarrow \boxed{\frac{\partial MRTS_{l,k}}{\partial l} \begin{Bmatrix} < \\ > \end{Bmatrix} 0}$$

If folk < 0 is like stundyting by your self give you a greater grade than also follow lectures.
If folk >0 following lectures and studying by your self gives you a greater grade

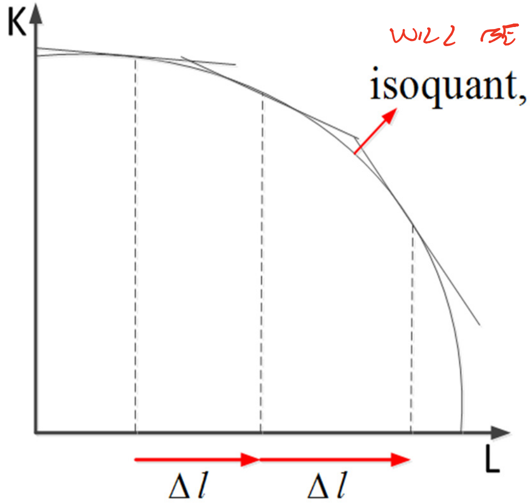
Diminishing MRTS

ISOQUANT CONVEX SINCE SLOPE NEGATIVE



$$\frac{dk}{dl} = -\frac{f_l}{f_k} \equiv -\frac{MP_l}{MP_k}$$

SLOPE OF ISOQUANT WILL BE CONCAVE



$f_{lk} > 0$ ($\uparrow k \Longrightarrow \uparrow MP_l$), or
$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow MP_l$) but small $\downarrow$ in MP_l

$f_{lk} < 0$ ($\uparrow k \Longrightarrow \downarrow\downarrow MP_l$)

We will use convex to be able to use the maximisation problem

Diminishing MRTS

- ***Example***: Let us check if the production function $f(k,l) = kl$ yields convex isoquants (i.e. decreasing MRTS).
- Use the generic equation of an isoquant, i.e.

$$kl = \bar{q}; \text{ i.e. } k = \frac{\bar{q}}{l}$$

- $MRTS_{l,k} = \frac{\partial k}{\partial l} = -\frac{\bar{q}}{l^2} = -\bar{q}l^{-2}$, to check is if convex I compute the

second derivative of the MRTS, i.e.

- $\frac{\partial MRTS_{l,k}}{\partial l} = \frac{\partial^2 k}{\partial l \partial l} = \frac{2\bar{q}}{l^3} > 0$

Thus isoquant is convex. MRTS $\Downarrow$

Constant Returns to Scale

- If production function $f(k,l)$ exhibits CRS, then increasing all inputs by a common factor t yields

$$\boxed{f(tk,tl) = tf(k,l)}$$

I can exactly replicate a technology. Double amount of capital and labour i also duplicate the production.

- Hence, $f(k,l)$ is homogenous of degree 1, thus implying that its first-order derivatives

$$f_k(k,l) \text{ and } f_l(k,l)$$

are homogenous of degree zero.

So this is like homogeneous of degree 1 when production function exhibit constant return to scale

Constant Returns to Scale

NO

- Therefore,

$$MP_l = \frac{\partial f(k,l)}{\partial l} = \frac{\partial f(tk,tl)}{\partial l}$$
$$= f_l(k,l) = f_l(tk,tl)$$

- Setting $t = \frac{1}{l}$, we obtain

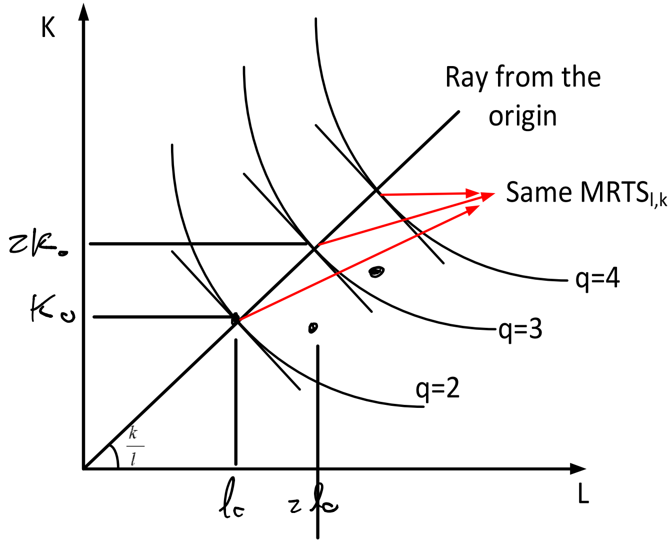
$$MP_l = f_l(k,l) = f_l\left(\frac{1}{l}k, \frac{l}{l}\right) = f_l\left(\frac{k}{l}, 1\right)$$

- Hence, $\boldsymbol{MP_l}$ **only depends on the ratio** $\boldsymbol{\frac{k}{l}}$**, but not on the absolute levels of** $\boldsymbol{k}$ **and** $\boldsymbol{l}$ **that firm uses.**
- A similar argument applies to MP_k.

Constant Returns to Scale

- Thus, $MRTS = -\frac{MP_l}{MP_k}$ only depends on the ratio of capital to labor.
- The slope of a firm's isoquants coincides at any point along a ray from the origin.
- Firm's production function is, hence, **homothetic**.

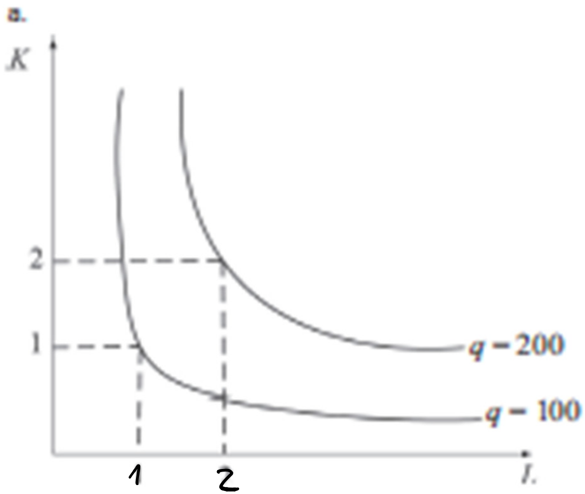
Fl fk do not depends on q (scale of production) so MRTS does not depend on q



Doubling the input also doubling the production

Constant Returns to Scale

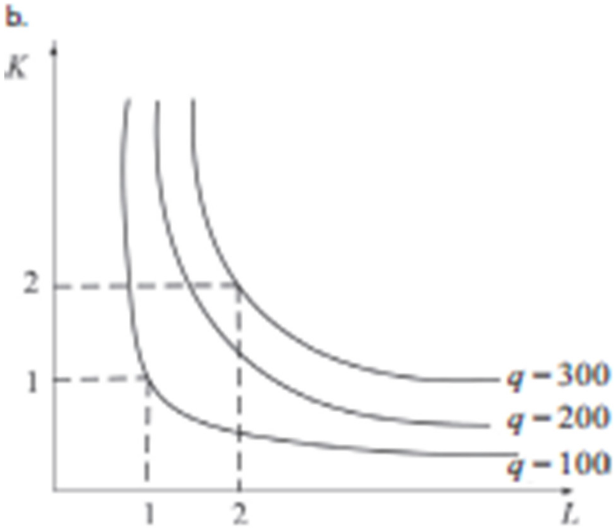
- $f(tk, tl) = tf(k, l)$



Increasing Returns to Scale

- $f(tk, tl) > tf(k, l)$

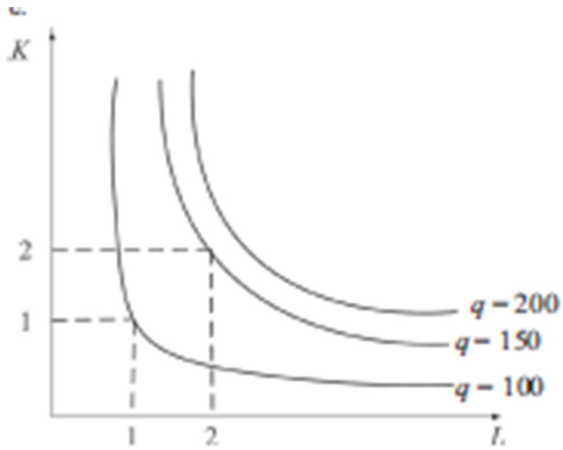
Increasing inputs by same proportion the amount of production increase more than proportion



Decreasing Returns to Scale

- $f(tk, tl) < tf(k, l)$

Increasing input by same proportion the amount of production increase less than proportion



Buying inputs is costly so we have some cost to achieve a certain amount of production.
Increasing return to scale: doubling the size of your plan you will receive a larger production than splitting the plan in half and double them by the same proportion.

100 K

50 K 50 K TOTAL COST SAME BUT ACHIEVE MORE PRODUCTION

IN MONOPOLY THERE ARE SOME INDUSTRIES IN WHICH IS CONVENIENT TO INCREASE PLAN (LIKE TRANSPORT)

INCREASING SCALE YOU WILL DECREASE SOME COST

Elasticity of Substitution

Elasticity of Substitution

- ***Elasticity of substitution*** (σ) measures the proportionate change in the k/l ratio relative to the proportionate change in the $MRTS_{l,k}$ along an isoquant:

$$\boxed{\sigma = \frac{\%\Delta(k/l)}{\%|\Delta MRTS|}} = \frac{d(k/l)}{d|MRTS|} \cdot \frac{|MRTS|}{k/l} = \frac{\partial \ln(k/l)}{\partial \ln(|MRTS|)}$$

where $\sigma > 0$ since ratio k/l and $|MRTS|$ move in the same direction.

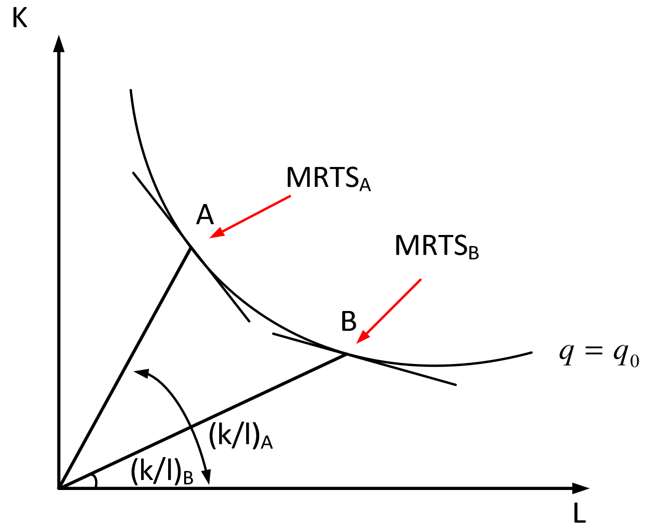
$$\frac{\frac{d\, k/l}{k/l}}{d\,|MRTS|\,/}$$

$$\frac{\frac{d\, k/l}{k/l}}{\frac{d\,|MRTS|}{|MRTS|}}$$

/ |MRTS|

Elasticity of Substitution

- Both $MRTS$ and k/l will change as we move from point A to point B.
- σ is the ratio of these changes.
- σ measures the **curvature of the isoquant**.



Elasticity of Substitution

Elasticity of Substitution

$$\frac{\delta \ln\left(\frac{K}{\ell}\right)}{\delta \frac{K}{\ell}} \quad d\,\frac{K}{\ell}$$

- ***Elasticity of substitution*** ***(σ)*** measures the proportionate change in the k/l ratio relative to the proportionate change in the $MRTS_{l,k}$ along an isoquant:

$$1/\tfrac{K}{\ell}$$

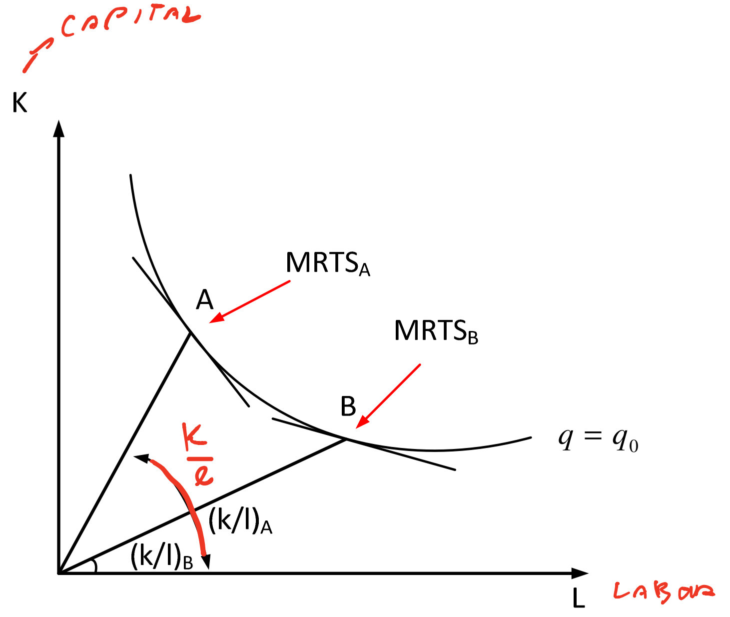
$$\sigma = \frac{\%\Delta(k/l)}{\%|\Delta MRTS|} = \frac{d(k/l)}{d|MRTS|} \cdot \frac{|MRTS|}{k/l} = \frac{\boxed{\partial \ln(k/l)}}{\partial \ln(|MRTS|)}$$

where $\sigma > 0$ since ratio k/l and $|MRTS|$ move in the same direction.

CURVATURE OF THE ISOQUANT

Elasticity of Substitution

- Both $MRTS$ and k/l will change as we move from point A to point B.
- σ is the ratio of these changes.
- σ measures the **curvature of the isoquant**.



BIG CHANGES MEANS THAT THE RATIO WILL BE LOW

ELASTICITY OF SUBSTITUTION IS LOW

Elasticity of Substitution

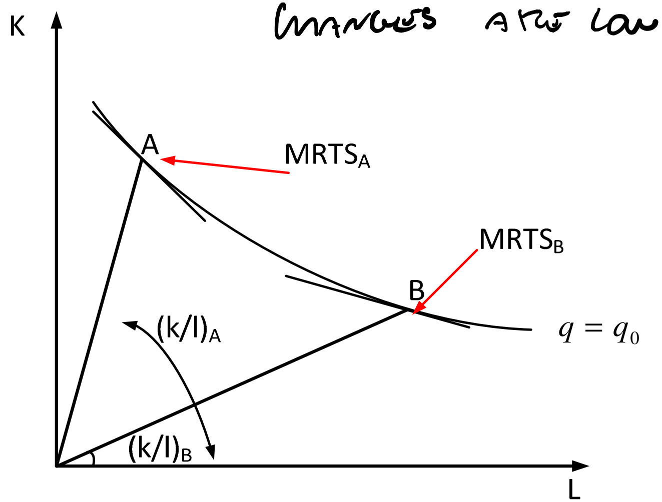
- If we define the elasticity of substitution between two inputs to be proportionate change in the ratio of the two inputs to the proportionate change in $MRTS$, we need to hold:
  - output constant (so we move along the same isoquant), and
  - the levels of other inputs constant (in case we have more than two inputs). For instance, we fix the amount of other inputs, such as land.

Elasticity of Substitution

- **High elasticity of substitution (σ):**
  - $MRTS$ does not change substantially relative to k/l.
  - **Isoquant is relatively flat.**

IMPLY AN HIGH LEVEL OF ELASTICITY OF SUBSTITUTION

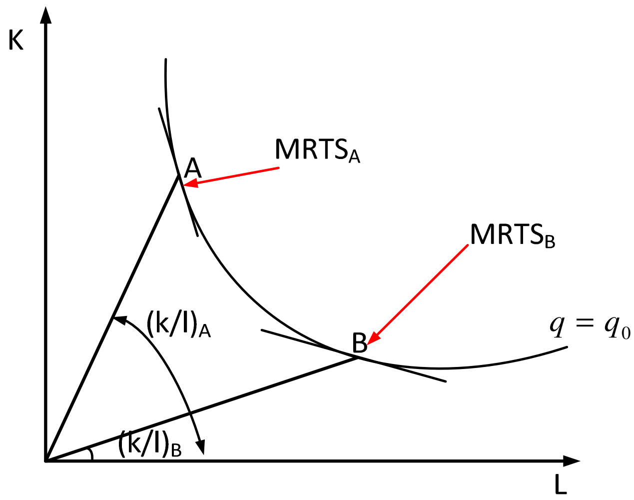
CHANGES ARE LOW



THE EXTREME CASE IS WHEN THE ISOQUANT IS A STRAIGHT LINE

Elasticity of Substitution

- **Low elasticity of substitution (σ):**
  - $MRTS$ changes substantially relative to k/l.
  - **Isoquant is relatively sharply curved.**



Elasticity of Substitution: Linear Production Function

WE WANT MARGINAL PRODUCTION POSITIVE!

$a, b > 0$

- Suppose that the production function is
$$q = f\,(k, l) = \underline{ak + bl}$$
LINEAR FUNCTION
- This production function exhibits constant returns to scale
$$f\,(tk, tl) = atk + btl = t(ak + bl)$$
$$= tf(k, l)$$
- Solving for k in q, we get $k = \frac{f\,(k,l)}{a} - \frac{b}{a}l$.
  - All isoquants are straight lines
  - k and l are perfect substitutes

ELASTICITY OF SUBSTITUTION IS HIGH

CHECK CONSTANT RETURN TO SCALE

$$f(tk, tl) = a(tk) + b(tl) =$$

$$= t(ak + bl) = t f(k,l) \implies$$ HOMOGENEOUS OF DEGREE 1

SO CONSTANT RETURN TO SCALE

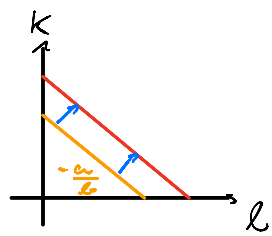
$t > 1$

$> t f(k,l) \rightarrow$ INCREASE

$< t f(k,l) \rightarrow$ DECREASE

$$\bar{q} = ak + bl$$

$$k = \frac{\bar{q}}{a} - \frac{bl}{a}$$



CURVE IS NEGATIVELY SLOPE

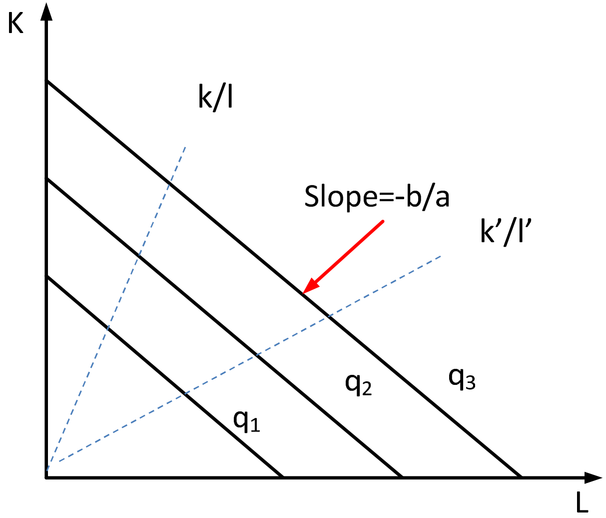
Elasticity of Substitution: Linear Production Function

- $MRTS$ (slope of the isoquant) is constant as k/l changes.

$$\sigma = \underbrace{\frac{\%\Delta(k/l)}{\%\Delta MRTS}}_{0} = \underline{\infty}$$

SLOPE IS CONSTANT

- **Perfect substitutes**
- This production function satisfies homotheticity.



Elasticity of Substitution: Fixed Proportions Production Function

- Suppose that the production function is
$$q = \min(ak, bl) \quad a, b > 0$$
- Capital and labor must always be used in a fixed ratio (**perfect complements**)
  - No substitution between k and l
  - The firm will always operate along a ray where k/l is constant (i.e., at the kink!).
- Because k/l is constant (b/a),
$$\sigma = \frac{\%\Delta(k/l)}{\underbrace{\%\Delta MRTS}_{\infty}} = 0$$

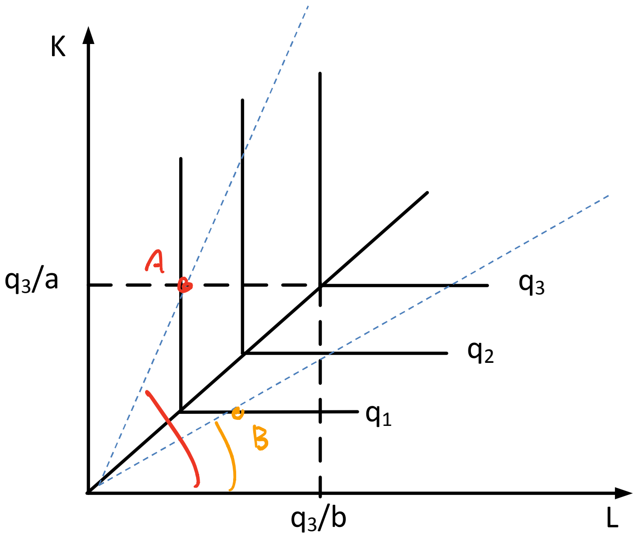
$\boxed{ak = bl}$

THIS HOLD AT THE OPTIMAL

SO, $\frac{k}{l} = \frac{b}{a}$

Elasticity of Substitution: Fixed Proportions Production Function

- $MRTS = \infty$ for l before the kink of the isoquant.
- $MRTS = 0$ for l after the kink.
- The change in MRTS is infinite (**perfect complements**)
- This production function also satisfies homotheticity.



Elasticity of Substitution: Cobb-Douglas Production Function

- Suppose that the production function is

$$q = f(k,l) = \underbrace{Ak^a l^b} \quad \text{where } A, a, b > 0$$

(*A* is sometimes called the "efficiency" parameter)

WE CAN USE THIS TO MEASURE THE EFFICIENT PRODUCTIVITY

- This production function can exhibit any returns to scale

$$f(tk,tl) = A(tk)^a(tl)^b = At^{a+b}k^a l^b = t^{a+b}f(k,l)$$

$$\frac{\partial q}{\partial k} = At \cdot a k^{a-1} l^b$$

$A t\uparrow$

  - If $a + b = 1 \Rightarrow$ **constant returns to scale**,
  $$f(tk,tl) = tf(k,l)$$
  - If $a + b > 1 \Rightarrow$ **increasing returns to scale**
  $$f(tk,tl) > tf(k,l)$$
  - If $a + b < 1 \Rightarrow$ **decreasing returns to scale**
  $$f(tk,tl) < tf(k,l)$$

CHECK

$$q = f(tk, tl) = A(kt)^a (lt)^b = A k^a l^b (t^{a+b})$$

HOMOGENEOUS OF DEGREE $a+b$

RETURN TO SCALE

- CONSTANT : $a+b=1$ $\left(q = f(tk, tl) = A k^a l^b t\right)$
- INCREASING : $a+b>1$ $\left(q = f(tk, tl) > A k^a l^b t\right)$
- DECREASING : $a+b<1$ $\left(q = f(tk, tl) < A k^a l^b t\right)$

$$q = A k^a l^b$$

$$\underbrace{\ln}_{\tilde{q}} = \ln A + a \underbrace{\ln k}_{\tilde{k}} + b \underbrace{\ln l}_{\tilde{l}}$$

→ IS LINEAR IN THE NEW TRANSFORMED VARIABLE

$$\varepsilon_{q,k} = \frac{\frac{dq}{q}}{\frac{dk}{k}} \rightsquigarrow \frac{d \ln q}{d \ln k} \rightsquigarrow \frac{\partial \ln q}{\partial q} \cdot dq = \frac{1}{q} dq$$

$$\rightsquigarrow \frac{d\hat{q}}{d\tilde{k}} = a$$

$$\varepsilon_{q,l} : \frac{\frac{dq}{q}}{\frac{dl}{l}} \rightsquigarrow \frac{d \ln q}{d \ln l} = b$$

Elasticity of Substitution: Cobb-Douglas Production Function

- The Cobb-Douglass production function is linear in logarithms

$$\ln(q) = \ln(A) + a\ln(k) + b\ln(l)$$

  - a is the elasticity of output with respect to k

$$\varepsilon_{q,k} = \frac{\partial \ln(q)}{\partial \ln(k)}$$

  - b is the elasticity of output with respect to l

$$\varepsilon_{q,l} = \frac{\partial \ln(q)}{\partial \ln(l)}$$

Elasticity of Substitution: Cobb-Douglas Production Function

- The elasticity of substitution (σ) for the Cobb-Douglas production function:
  - First,

$$MRTS = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{bAk^a l^{b-1}}{aAk^{a-1} l^b} = \frac{b}{a} \cdot \frac{k}{l}$$

  - Hence,

$$\ln(|MRTS|) = \ln\left(\frac{b}{a}\right) + \ln\left(\frac{k}{l}\right)$$

MRTS

$$q = A k^a l^b$$

$$|MRTS| = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{A k^a \cdot b l^{b-1}}{A l^b a k^{a-1}} =$$

$$= \underline{\frac{b}{a} \frac{K}{l}}$$

$$\frac{\partial q}{\partial l} = A k^a b l^{b-1} \qquad \frac{\partial q}{\partial k} = A l^b a k^{a-1}$$

$$\sigma = \frac{\Delta \% \frac{k}{l}}{\Delta \% \, MRTS} = \frac{d \ln\left(\frac{k}{l}\right)}{d \ln |MRTS|} = 1$$

ELASTICITY OF SUBSTITUTION IS 1

$$\downarrow$$

$$\frac{\frac{\frac{\partial l}{l}}{\frac{K}{l}}}{\frac{\partial |MRTS|}{|MRTS|}} = \frac{\frac{\frac{\partial k}{l}}{l}}{(\partial |MRTS|)} \cdot \frac{|MRTS|}{\frac{k}{l}}$$

$$\ln \underbrace{|MRTS|}_{\widetilde{MRTS}} = \ln \frac{b}{a} + \ln \underbrace{\frac{k}{l}}_{\widetilde{\frac{k}{l}}} \rightsquigarrow \ln \underbrace{\left(\frac{k}{l}\right)}_{\widetilde{\frac{k}{l}}} =$$

$$= -\ln \frac{b}{a} + \ln |MRTS|$$

Elasticity of Substitution: Cobb-Douglas Production Function

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\ln\left(\frac{k}{l}\right) = \ln(|MRTS|) - \ln\left(\frac{b}{a}\right)$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(|MRTS|)} = 1$$

Transformations of a degree 1 homogenous Function

- Assume $y = f(k, l)$ is homogeneous of degree one, i.e. $f(tk, tl) = tf(k, l)$ i.e CRS. $\longrightarrow$ CONSTANT RETURN to SCALE
- Then define the new production function

$$F(k, l) = [f(k, l)]^{\gamma}$$

Then the Returns to Scale (RTS) of this new function depend on γ. Indeed,

$$F(tk, tl) = [f(tk, tl)]^{\gamma} = [tf(k, l)]^{\gamma} = t^{\gamma}[f(k, l)]^{\gamma}$$
$$= t^{\gamma} F(k, l)$$

That is the new function is homogenous of degree γ, which also determines the RTS. *If* $\gamma > 1$ *IRS; if* $\gamma = 1$ *CRS; if* $\gamma < 1$ *DRS.*

Elasticity of Substitution: CES Production Function

- Suppose that the production function is

$$q = f(k, l) = (k^{\rho} + l^{\rho})^{\gamma/\rho}$$

$\rightsquigarrow f(f(k, \ell))^{\gamma} =$

where $\rho \leq 1, \rho \neq 0, \gamma > 0$. Applying what we just said:
  - $\gamma = 1 \Longrightarrow$ constant returns to scale
  - $\gamma > 1 \Longrightarrow$ increasing returns to scale
  - $\gamma < 1 \Longrightarrow$ decreasing returns to scale

  $= \left[(k^{\rho} + \ell^{\rho})^{\frac{1}{\rho}} \right]^{\gamma}$

  HOMOGENEOUS OF DEGREE 1

  This happens because $f(k, l) = (k^{\rho} + l^{\rho})^{1/\rho}$ is homogeneous of degree 1, i.e. $f(tk, tl) = t(k^{\rho} + l^{\rho})^{1/\rho}$ [prove it!]

  $f(k, \ell)^{\gamma}$

  EX $(k^{1/3} + \ell^{1/3})^{3}$

- Alternative representation of the CES function

$$\boxed{f(k, l) = \left(k^{\frac{\sigma-1}{\sigma}} + l^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma-1}{\sigma}}}$$

$\sigma = \frac{1}{1 - 1/3}$

where σ is the elasticity of substitution.

$\rho = \frac{\sigma - 1}{\sigma}$

$\sigma = \frac{1}{1-\rho}$

$\sigma \longrightarrow$ ELASTICITY OF SUBSTITUTION

$$q = f(k, \ell) = \left(k^{\rho} + \ell^{\rho}\right)^{\frac{1}{\rho}}$$

$$f(tk, t\ell) = \left((tk)^{\rho} + (t\ell)^{\rho}\right)^{\frac{1}{\rho}} = \left[t^{\rho} \cdot \left(k^{\rho} + \ell^{\rho}\right)\right]^{\frac{1}{\rho}} =$$

$$= t \cdot \left(k^{\rho} + \ell^{\rho}\right)^{\frac{1}{\rho}} = t \cdot f(k, \ell)$$

Elasticity of Substitution: CES Production Function

- The elasticity of substitution (σ) for the CES production function:

$\rightsquigarrow (k^{\rho} + \ell^{\rho})^{\gamma/\rho}$

  – First,

$$|MRTS| = \frac{MP_l}{MP_k} = \frac{\frac{\partial q}{\partial l}}{\frac{\partial q}{\partial k}} = \frac{\cancel{\frac{\gamma}{\rho}}\cancel{[k^{\rho}+l^{\rho}]^{\frac{\gamma}{\rho}-1}}(\cancel{\rho}\boxed{l^{\rho-1}})}{\cancel{\frac{\gamma}{\rho}}\cancel{[k^{\rho}+l^{\rho}]^{\frac{\gamma}{\rho}-1}}(\cancel{\rho}\boxed{k^{\rho-1}})}$$

$$= \left(\frac{l}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{\underline{1-\rho}}$$

Elasticity of Substitution: CES Production Function

$$\frac{d \ln \frac{k}{l}}{d \ln |MRTS|}$$

– Hence,

$$\ln(|MRTS|) = (1-\rho)\ln\left(\frac{k}{l}\right)$$

$$y = \frac{1}{1-\rho}\, x$$

– Solving for $\ln\left(\frac{k}{l}\right)$,

$$\underbrace{\ln\left(\frac{k}{l}\right)}_{y} = \frac{1}{1-\rho}\overbrace{\ln(|MRTS|)}^{x}$$

– Therefore, the elasticity of substitution between k and l is

$$\sigma = \frac{d\ln\left(\frac{k}{l}\right)}{d\ln(|MRTS|)} = \frac{1}{1-\rho}$$

Elasticity of Substitution: CES Production Function

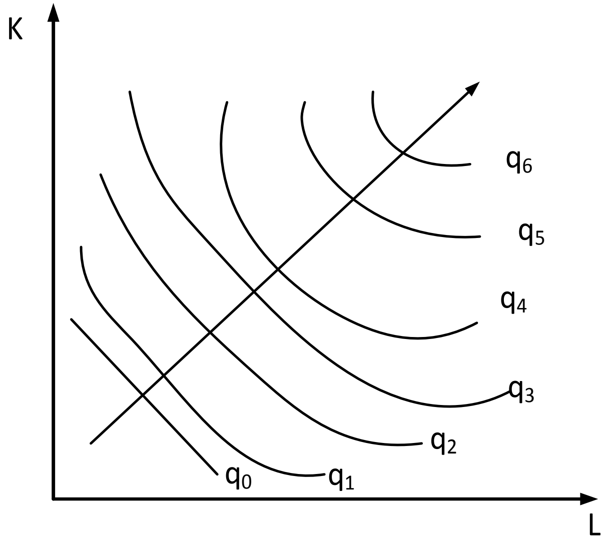
- Elasticity of Substitution in German Industries (Source: Kemfert, 1998):

Industry	σ
Food	0.66
Iron	0.50
Chemicals	0.37
Motor Vehicles	0.10

Elasticity of Substitution

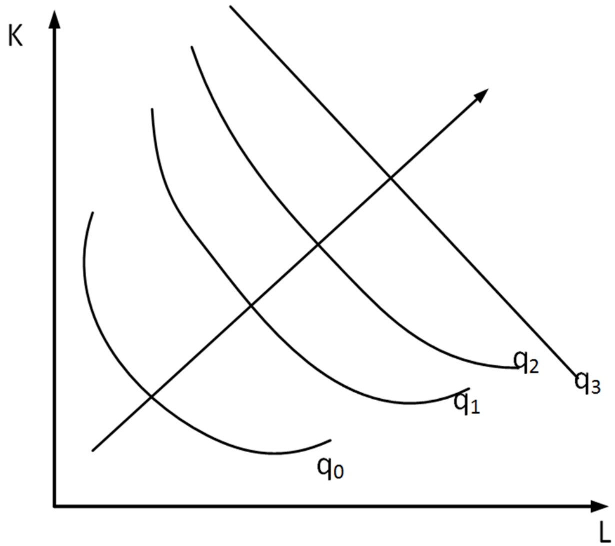
$q \uparrow \quad \sigma \downarrow$

- The elasticity of substitution σ between k and l is *decreasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very high σ
  - q_5 and q_6 have very low σ



Elasticity of Substitution

- The elasticity of substitution σ between k and l is *increasing* in scale (i.e., as q increases).
  - q_0 and q_1 have very low σ
  - q_2 and q_3 have very high σ



Elasticity of scale

ELASTICITY OF OUTPUT IN RESPECT TO FACTOR

- The elasticity of scale is the elasticity of output q to increasing the scale of production (λ), i.e.

% CHANGE IN PRODUCTION

$$\epsilon_{q,\lambda} \equiv \frac{\frac{\partial f(\lambda k, \lambda l)}{f(k,l)}}{\left|\frac{\partial \lambda}{\lambda}\right|} = \frac{\partial f(\lambda k, \lambda l)}{\partial \lambda} \frac{\lambda}{f(k,l)}$$

% CHANGE

SCALE

$\varepsilon_{q,k}$

$\varepsilon_{q,l}$

Relation btw returns to scale and elasticity of scale

- We have the production function $q = f(l, k)$ and assume that is homogeneous of degree α.
- We take the total differential

$$dq = f_l dl + f_k dk$$

- Divide both sides by q

$$\frac{dq}{q} = \frac{f_l}{q} dl + \frac{f_k}{q} dk$$

- Then multiply the first term of the RHS by $\frac{l}{l}$ and the second term by $\frac{k}{k}$

$$\frac{dq}{q} = \frac{f_l l}{q}\frac{dl}{l} + \frac{f_k k}{q}\frac{dk}{k}$$

$$\frac{dl}{l} = \frac{dk}{k} = \frac{d\lambda}{\lambda}$$

Relation btw returns to scale and elasticity of scale - II

- Since we are considering a change in scale, all inputs increase by the same proportion, i.e. $\frac{dl}{l} = \frac{dk}{k} = \frac{d\lambda}{\lambda}$ and substituting in the previous equation

$$\frac{dq}{q} = \left(\frac{f_l l}{q} + \frac{f_k k}{q}\right)\frac{d\lambda}{\lambda} = \frac{(f_l l + f_k k)}{q}\frac{d\lambda}{\lambda}$$

$\alpha q \to \alpha f(k, l)$

- But by the Euler's theorem, if f homogeneous of degree α, then $f_l l + f_k k = \alpha q$

- Thus $\frac{dq}{q} = \frac{\alpha\, q}{q}\frac{d\lambda}{\lambda} = \alpha\frac{d\lambda}{\lambda}$, or

$$\frac{dq}{q} = \alpha \frac{d\lambda}{\lambda} \rightarrow \frac{\frac{dq}{q}}{\frac{d\lambda}{\lambda}} = \alpha$$

$$\epsilon_{q,\lambda} \equiv \frac{\frac{dq}{q}}{\frac{d\lambda}{\lambda}} = \alpha$$

- **NB. Scale elasticity coincides with the production function degree of homogeneity.**

$K^a l^b \rightarrow$ elasticity of scale if $a+b$

EULER THEOREM

X, Y FACTOR

f HOMOGENEOUS OF DEGREE α

$$f(tx, ty) = t^{\alpha} f(x,y)$$ ← WE CAN DIFFERENTIATE

t IS THE SCALE OF PRODUCTION

$$\frac{\partial f}{\partial tx} \cdot x + \frac{\partial f}{\partial ty} \cdot y = \alpha \cdot t^{\alpha - 1} f(x,y)$$

EVALUATE THIS EQUALITY WITH $t = 1$

$$\frac{\partial f}{\partial tx} \cdot x + \frac{\partial f}{\partial ty} \cdot y = \alpha f(x,y)$$

$$\underline{PM_x \cdot X + PM_y \cdot y = \alpha f(x,y)}$$

THIS RESULT IS USEFULL IN THIS CASE ⊛

$$\pi = TR - TC = P \cdot Q - TC(Q)$$

PROFIT

TOTAL REVENUE

TOTAL COST

FUNTION OF QUANTITY

QUANTITY OF PRODUCTION FACTOR THAT YOU BUY TO PRODUCE Q

K, L

$$Q = f(K, L)$$

$$\pi = \underbrace{P \cdot f(K, L)}_{TR} - (rK + wL)$$

Q

GIVEN

PERFECT COMPETITION

r = INTEREST RATE UNIT PRICE OF K

w = WAGE RATE UNIT PRICE OF L

- MANY CONSUMER AND FIRMS
- PERFECT INFORMATION
- HOMOGENEOUS GOODS

two producers $P_A > P_C$

p is exogenous (not a choice)

NO CAPITAL

$\max \pi = p \cdot f(L) - wL$

$L \geq 0$

opt quantity of labour that maximise the profit

FOC $\frac{\partial \pi}{\partial L} = 0 : p f'_L - w = 0$ $\quad f'_L = \frac{\partial f}{\partial L}$

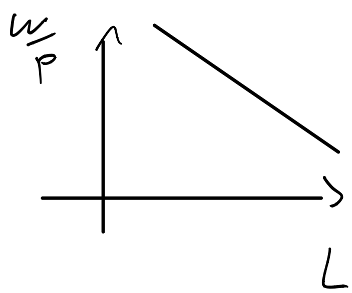
at $L^* \rightarrow f'_L = \frac{w}{p}$

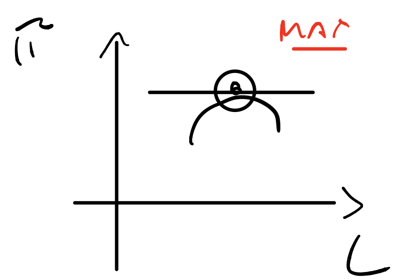
w nominal wage

p price level

$\frac{w}{p} \rightarrow$ real wage

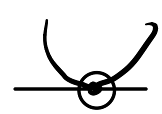
$f'_L \rightarrow$ decrease





MAX

CONCAVE



MIN

CONVEX

We have to check concavity $\rightarrow$ 2° order condition

SCC $\rightarrow \frac{\partial^2 \pi(L^*)}{\partial L \partial L} < 0$ is concave in L^*

$$L = f_L'^{-1}\left(\frac{w}{p}\right)$$

IN MOST CASE WE WORK WITH FUNCTIONS ALWAYS NEGATIVE IN 2° DERIVATIVE

$\boxed{\text{GLOBALLY CONCAVE FUNCTION}}$

$$\frac{\partial \pi}{\partial L} = p f_L' - w$$

$$\frac{\partial^2 \pi}{\partial L \partial L} = p \cdot f_{LL}'' \longrightarrow f_{LL}'' < 0$$

PRODUCTION FUNCTION IS GLOBALLY CONCAVE

$$\max_{K,L} \pi = \underbrace{p \cdot f(K,L)}_{T.R} - \underbrace{wL - rK}_{T.\ COST}$$

FOR ONLY FUNCTION OF K, BOTH VARIABLE

$f'_K(K,L)$

FOC $$\frac{\partial \pi}{\partial K} = p \cdot f'_K - r = 0 \longrightarrow f'_K = \frac{r}{p}$$

$$\frac{\partial \pi}{\partial L} = p \cdot f'_L - w = 0 \longrightarrow f'_L = \frac{w}{p}$$

$\downarrow$

$f'_K(K,L)$

i => NOMINAL INTEREST RATE

$\frac{r}{p}$ REAL INTEREST RATE

NOMINAL INTEREST RATE

$f'_L > \frac{w}{p}$ $\qquad f'_L$ DECREASING $\Rightarrow$ $f''_{KK} < 0$

$f''_{LL} < 0$

INCREASE LABOUR -> IF $f'_L \downarrow \rightarrow \frac{w}{p}$

RATIO BETWEEN f'_K AND f'_L

$$-\frac{f'_L}{f'_K} = \frac{w}{r} \rightarrow \frac{f'_L}{w} = \frac{f'_K}{r}$$

PRICE RATIO

$\boxed{|MRTS|}$

USUALLY WE WORK WITH GLOBALLY CONCAVE FUNCTIONS

SOC

HESSIAN → NEGATIVE DEFINITE

$$H = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\downarrow$$

$$\overline{H}$$

$-wL - rK$ DERIVE WITH RESPECT TO L

WILL BE $-w$

TO K WILL BE $\emptyset$

(CROSS DERIVATIVE)

SO WE DON'T CARE ABOUT THIS

SO TAKE FIRST PART $p \cdot f(K, L)$

$$H = \begin{bmatrix} f''_{LL} & f''_{LK} \\ f''_{KL} & f''_{KK} \end{bmatrix}$$

<0

→ DEFINITE NEGATIVE ?

CHECK

$f''_{LL} < 0$

DETERMINANT POSITIVE ← $|H| > 0$

$$|H| = f_{LL} f_{KK} - (f_{KL})^2 > 0$$

WE CAN CHECK IF GLOBALLY CONCAVE

IF $f''_{KK} > 0 \rightarrow f_{LL} f_{KK} - (f_{KL})^2$

IF $f''_{KK} < 0 \rightarrow$ NEGATIVE $-$

SO MUST BE NEGATIVE

EXERCISES

$$\max_{z_1,\, z_2 \geq 0} \quad p \cdot \underbrace{f(z_1, z_2)}_{q} - w_1 z_1 - w_2 z_2$$

$q = f(z_1, z_2) \rightarrow$ PRODUCTION FUNCTION

$$p\left(A z_1^{a} z_2^{b}\right) - w_1 z_1 - w_2 z_2 \qquad \frac{\text{COBB-DOUGLAS}}{q = A z^{a} z^{b}}$$

FOC

$$\begin{cases} \dfrac{\partial \pi}{\partial z_1} = p \cdot A a z_1^{a-1} z_2^{b} - w_1 = 0 \\[2ex] \dfrac{\partial \pi}{\partial z_2} = p \cdot A b z_1^{a} z_2^{b-1} - w_2 = 0 \\[2ex] \dfrac{\partial \pi}{\partial \lambda} \end{cases}$$

$$\overbrace{A a z_1^{a-1}}^{f'_{z_1}} z_2^{b} = \frac{w_1}{p} \quad \text{REAL PRICE OF } z_1 \qquad (1)$$

$$\underbrace{A b z_2^{b-1}}_{f'_{z_2}} z_1^{a} = \frac{w_2}{p} \quad \text{REAL PRICE OF } z_2 \qquad (2)$$

OPT CONDITION AS RATIO BETWEEN THE TWO

$$\frac{A a z_1^{a-1} z_2^{b} = \frac{w_1}{p}}{A b z_2^{b-1} z_1^{a} = \frac{w_2}{p}} = \underbrace{\frac{a}{b} \frac{z_2}{z_1}}_{|MRTS|} = \underbrace{\frac{w_1}{w_2}}_{\text{PRICE RATIO}}$$

$$z_2^* = \frac{b}{a} \frac{w_1}{w_2} z_1^* \qquad \frac{z_2}{z_1} = \frac{b}{a} \frac{w_1}{w_2}$$

I REPLACE IN (1) $\rightarrow A a z_1^{a-1} z_2^{b} = \frac{w_1}{p}$

$$A a z_1^{a-1} \left(\frac{\beta}{a}\right)^{\beta} \left(\frac{w_1}{w_2}\right)^{\beta} z_1^{\beta} = w_1$$

$$A a z_1^{a-1+\beta} = w_1 \left(\frac{b}{a}\right)^{-\beta} \left(\frac{w_1}{w_2}\right)^{-\beta} \frac{1}{A a}$$

IN THE END WE FIND

PRODUCTION IN THE BOOK IS $y \rightarrow y = q$

$$\left.\begin{aligned} z_1^* &= \frac{a p}{w_1} \cdot y \\ z_2^* &= \frac{\beta p}{w_2} \cdot y \end{aligned}\right\} \text{CONDITIONAL DEMAND FACTOR}$$

(WHICH DEPEND ON PRODUCTION)

$$\left.\begin{aligned} z_1^* &= A^{\frac{1}{1-a-\beta}} \left(\frac{a p}{w_1}\right)^{\frac{1-b}{a-b}} \left(\frac{b p}{w_2}\right)^{\frac{b}{1-a-b}} \\ z_2^* &= A^{\frac{1}{1-a-b}} \left(\frac{a p}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b p}{w_2}\right)^{\frac{1-a}{1-a-b}} \end{aligned}\right\} \text{UNCONDITIONAL DEMAND FACTORS}$$

IF PRICE $\uparrow$ ($\uparrow w_1$), FIRM WILL BUY MORE OR LESS? OF THAT FACTOR

COMPUTE DERIVATIVE

$$\frac{\partial z_1^*}{\partial w_1} = \underbrace{\theta_1 \theta_2}_{>0} \underbrace{\left(\frac{1-b}{1-a-b}\right)}_{X} \underbrace{\left(\frac{ap}{w_1}\right)^{\frac{1-b}{1-a-b}-1}}_{>0} \cdot \underbrace{\left(-\frac{ap}{w_1^2}\right)}_{<0}$$

$A>0 \quad b>0 \quad p>0 \quad w_2>0 \quad \theta_2>0$

SO $\theta_1>0$

$+ \quad + \quad -$ BUT WE WILL RETURN TO SIGN OF a, b

COBB DOUGLAS WAS DECREASING RETURN TO SCALE?

$X>0$

WITH DRS $\frac{\partial z_1^*}{\partial w_1} < 0$

HOW DO WE FIND THE SUPPLY OF THE FIRM?

(QUANTITY THAT MAXIMISE PROFITS)

q

RELATION BETWEEN z_1, z_2 TO q?

THE PRODUCTION FUNCTION!

$$q^* = f(z_1^*, z_2^*) \to q = A(z_1^*)^a (z_2^*)^b$$

$$q^* = \underbrace{A^{\frac{1}{1-a-b}}}_{\substack{>0 \\ \theta_1}} p^{\frac{a+b}{1-a-b}} \underbrace{\left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}}}_{\substack{>0 \\ \theta_2}} \underbrace{\left(\frac{b}{w_2}\right)^{\frac{\beta}{1-a-b}}}_{\substack{>0 \\ \theta_3}}$$

FIRM SUPPLY FUNCTION

HOW QUANTITY DEPENDS ON PRICE?

$$\frac{\partial q^*}{\partial p^*} = \theta_1 \theta_2 \theta_3 \cdot \frac{a+b}{1-a-b} p^{\frac{a+b}{1-a-b}-1}$$

IF DRS SIGN WILL BE POSITIVE

LAW OF SUPPLY HOLDS!!

CONCAVE? CHECK SOC AND VERIFY IF HOLDS

$$f(z) = 2^{3/4} z_1^{1/4} z_2^{1/4} \qquad z_1, z_2 \geq 0$$

$\downarrow$

VECTOR

$$t > 1$$

CHECK IF NONINCR. R.T.S., NON DECR, CONSTANT ?

$$f(t z_1, t z_2) = 2^{3/4} (t z_1)^{1/4} (t z_2)^{1/4} = 2^{3/4} t^{2/4} z_1^{1/4} z_2^{1/4} =$$

$$= t^{1/2} (2^{3/4} z_1^{1/4} z_2^{1/4}) = t^{1/2} f(z)$$

IF $> t f(z)$ $\rightarrow$ THIS WILL BE INCREASING R.T.S.

ELSE $\leq t f(z)$

$(t^{1/2} - t) \leq 0 \quad t^{1/2} \leq t \quad \sqrt{t} \leq t$? YES

SO NON INCREASING

$\Downarrow$

NOT CONSTANT R.T.S

CHECK NON DECREASING R.T.S.

TRY EXERCISE SET N. 1 (EX SECTION CH. 4)