(1) Determine the rank of the following matrices,

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & \alpha \end{bmatrix} \quad (\alpha \in \mathbb{R}).$$

(2) Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ -h & h & h \end{pmatrix},$$

where h is a real parameter, the matrix of a linear transformation T. Which of the following statements is true?

(1.a) for $h = 1$ the dimension d of the subspace $\{\ \mathbf{x}\ :\ A\mathbf{x} = \mathbf{0}\}$ is greater than or equal to 1.

(1.b) For $h = -1$, T is one to one and the range of T is $\mathbb{R}^3$.

(1.c) $rank(A) = 2$ for any $h \in \mathbb{R}$.

(3) Let $T : \mathbb{R}^p \to \mathbb{R}^q$ the function defined as

$$T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} -3a + 6b - c + d \\ a - 2b + 2c + 3d \\ 2a - 4b + 5c + 8d \end{bmatrix},$$

a linear transformation. Find the values of p and q and the matrix A of the linear transformation. Find the dimension and a basis for the range of T.

(4) Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ a linear transformation defined as

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 + \alpha x_3 + x_4 \\ 2x_1 + \alpha x_2 + \alpha x_4 \\ x_1 + 4x_2 + (2\alpha)x_3 \end{bmatrix}$$

where α is a real parameter. Find the matrix A of T. Find, varying the parameter α, the range space of the linear transformation T, and its basis. Find, if any, for which values of α the vector $\mathbf{v} = [0\,1\,0]^T$ belongs to the range of T.

(5) Find the dimension of the subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{V}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{V}_2 = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 2 \end{bmatrix}, \mathbf{V}_3 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \mathbf{V}_4 = \begin{bmatrix} 3 \\ 8 \\ -5 \\ 4 \end{bmatrix},$$

(6) Project $\mathbf{X}$ onto $\mathbf{Y}$ where

$$\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

If P is the projection verify that $(\mathbf{X} - \mathrm{P}) \cdot \mathbf{Y} = 0$.

(7) Let $\mathbf{v}_1 = (1,1,1)^T$, $\mathbf{v}_2 = (2,1,-3)^T$, $\mathbf{v}_3 = (4,-5,1)^T$, verify that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$. Then form an orthonormal set.

(8) Verify that the following matrix A is an orthogonal matrix,

$$A = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}.$$

Find the solution of the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (1,1,1)^T$.

(9) Find the solution of the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 1 & 3 \end{bmatrix}, \ \mathbf{b} \in \mathbb{R}^3,$$

and deduce the matrix A^{-1}.