$$\nabla g_1(x) = \begin{bmatrix} 3(x_1-1)^2 \\ 1 \end{bmatrix}$$

$$\nabla g_2(x) = \begin{bmatrix} 3(x_1-1)^2 \\ -1 \end{bmatrix}$$

$$\nabla g_3(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$C(1,2)$ non regular

$$\begin{cases} -1 + \mu_1 3(x_1-1)^2 + \mu_2 3(x_1-1)^2 - \mu_3 = 0 & (1) \\ \emptyset + \mu_1 \cdot 1 + \mu_2(-1) + \mu_3 \cdot 0 = 0 \Rightarrow \mu_2 = \mu_1 & (2) \\ \mu_2\left((x_1-1)^3 - (x_2-2)\right) = 0 & (3) \\ \mu_3(-x_1) = 0 & (4) \\ \mu_1\left[(x_1-1)^3 + (x_2-2)\right] = 0 & (5) \end{cases}$$

$(1),(2) \Rightarrow 6\mu_2(x_1-1)^2 = \mu_3 + 1 \quad (*)$

?

$\boxed{\text{pg } 85}$ $\mu_3 \geq 0$

$(*),(2) \Rightarrow \mu_2 = \mu_1 > 0 \quad (x_1 \neq 0)$

$(5),(3) \Rightarrow (x_1-1)^3 = -(x_2-2) \Leftrightarrow (x_1-1)^3 = 0$

$(x_1-1)^3 = x_2 - 2 \Leftrightarrow x_1 = 1$

$\Downarrow$

$x_2 = 2$

$(1) \Rightarrow \mu_3 = 1$

Karush-Kuhn-Tucker conditions

The generalized Lagrangean function is $\ell(x,\mu) = -x_1 + \mu_1 (x_1-1)^3 + \mu_1 (x_2-2) + \mu_2 (x_1-1)^3 - \mu_2 (x_2-2) - \mu_3 x_1$, so that

$$\begin{aligned}
\partial\ell/\partial x_1 &= -1 + 3\mu_1 (x_1-1)^2 + 3\mu_2 (x_1-1)^2 - \mu_3 = 0 \\
\partial\ell/\partial x_2 &= \mu_1 - \mu_2 = 0 \\
\mu_1 g_1 &= \mu_1 \left[(x_1-1)^3 + (x_2-2)\right] = 0 \\
\mu_2 g_2 &= \mu_2 \left[(x_1-1)^3 - (x_2-2)\right] = 0 \\
\mu_3 g_3 &= -\mu_3 x_1 = 0 \\
g_1 \leq 0 &= (x_1-1)^3 + (x_2-2) \leq 0 \\
g_2 \leq 0 &= (x_1-1)^3 - (x_2-2) \leq 0 \\
g_3 \leq 0 &= -x_1 \leq 0 \\
&\left.\begin{aligned} &\mu_1 \geq 0 \\ &\mu_2 \geq 0 \\ &\mu_3 \geq 0 \end{aligned}\right.
\end{aligned}$$

Since $\mu_1 = \mu_2 = \mu$, the first constraint becomes $6\mu(x_1-1)^2 = \mu_3 + 1 \geq 1 > 0$, which implies that $\mu > 0$. Computing the sum and the difference of the third and fourth constraints, one obtains that $x_1 = 1$ and $x_2 = 2$. Therefore, the first constraint would require $\mu_3 = -1$ and the fourth would require $\mu_3 = 0$, a contradiction. Consequently, no point satisfies the Karush-Kuhn-Tucker conditions.

On the other hand, a globally minimum point certainly exists, because the feasible region is close and limited, and the objective function is continuous. The globally optimal point is A, that can be optimal and violate the KKT-conditions because it is nonregular. Notice that in the previous exercise the same nonregular point actually satisfied the KKT-conditions: both cases are possible. This is why it is always necessary to keep into account the nonregular points as candidates.

Exercise 7

Solve the following problem with the KKT-conditions:

$$\min z = (x_1+1)^2 + \left(x_2 + \frac{1}{2}\right)^2$$
$$\begin{aligned} x_1^2 - x_2^2 &\leq 0 \\ x_1 - x_2 &\leq 0 \end{aligned}$$

Solution

This problem has a very peculiar feasible region, that consists in the upper quadrant included between the bisectors of the axes, plus the half-line $x_2 = x_1$ with $x_1 \leq 0$. Figure 4.5 represents the feasible region.

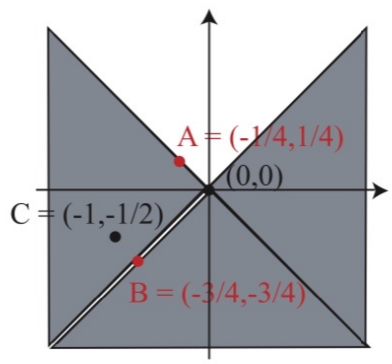


Figure 4.14: Regione ammissibile

Nonregular points

The gradients of the constraints $g_1(x) = x_1^2 - x_2^2 \leq 0$ and $g_2(x) = x_1 - x_2 \leq 0$ are

$$\nabla g_1 = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix} \qquad \nabla g_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The points in which neither constraint is active are regular by definition. Those in which only g_1 is active, that is those of the bisector of the second and fourth quadrant, excluding the origin ($x_2 = -x_1$, with $x_1 \neq 0$) are regular as long as the gradient is nonzero. This would require $x_1 = x_2 = 0$, that is impossible. Then, these points are all regular. There is no point in which only g_2 is active. The points in which both constraints are active, that is those of the bisector of the first and third quadrant ($x_2 = x_1 = \xi$) are all nonregular, because the two gradients are proportional: $\nabla g_1 = [2\xi \; -2\xi]^T$ and $\nabla g_2 = [1 \; -1]^T$.

Karush-Kuhn-Tucker conditions

The generalized Lagrangean function $\ell(x,\mu) = (x_1+1)^2 + \left(x_2+\frac{1}{2}\right)^2 + \mu_1\left(x_1^2 - x_2^2\right) + \mu_2(x_1 - x_2)$ yields the following conditions:

$$\begin{aligned}
\frac{\partial \ell}{\partial x_1} &= 2(x_1+1) + 2\mu_1 x_1 + \mu_2 = 0 \\
\frac{\partial \ell}{\partial x_1} &= 2\left(x_2+\frac{1}{2}\right) - 2\mu_1 x_2 - \mu_2 = 0 \\
\mu_1 g_1(x) &= \mu_1\left(x_1^2 - x_2^2\right) = 0 \\
\mu_2 g_2(x) &= -\mu_2(x_1 - x_2) = 0 \\
\mu_1 &\geq 0 \\
\mu_2 &\geq 0 \\
g_1(x) &\leq 0 \\
g_2(x) &\leq 0
\end{aligned}$$

We set apart all points of the bisector of the first and third quadrant, because they are nonregular, and consequently candidate: $x_1 - x_2 < 0$. As a result, the fourth constraint implies that $\mu_2 = 0$: the multiplier of a nonactive constraint (g_2) is always zero.

$$\begin{aligned}
2(x_1+1) + 2\mu_1 x_1 &= 0 \\
2\left(x_2+\frac{1}{2}\right) - 2\mu_1 x_2 &= 0 \\
\mu_1(x_1 + x_2) &= 0 \\
0 &= 0 \\
\mu_1 &\geq 0 \\
\mu_2 &= 0 \\
x_1 + x_2 &\geq 0 \\
x_1 - x_2 &< 0
\end{aligned}$$

We split the problem into two subproblems based on the third constraint.

Problem $\mu_1 = 0$ The first two constraints yield point $C = (-1, -1/2)$. This violates constraint $g_1 \leq 0$, so that is must be rejected. On the other hand, it is a reasonable result, given that it would be the point of minimum of the objective function without the constraints.

Problem $\mu_1 > 0$ In this subproblem $x_2 = -x_1$, so that

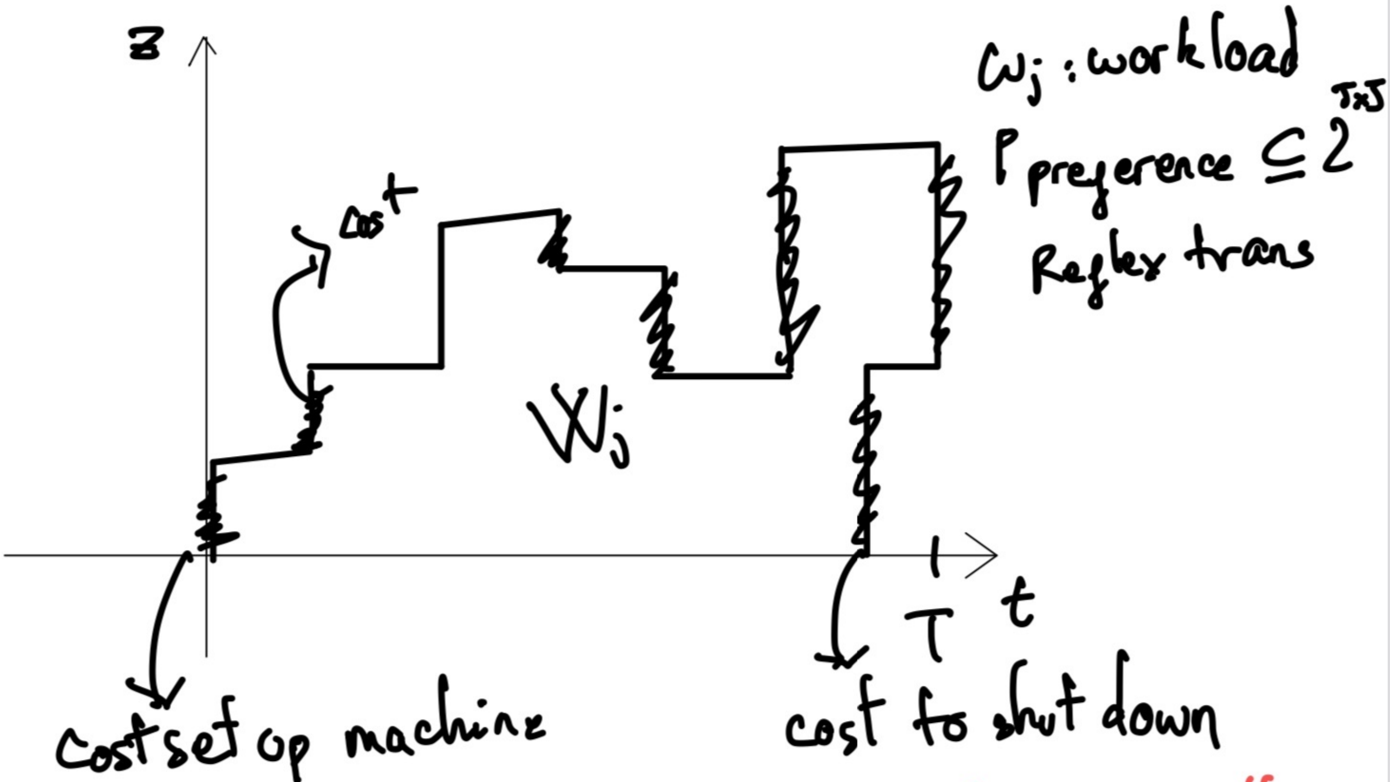
$$\begin{aligned}
2(x_1+1) + 2\mu_1 x_1 &= 0 \\
2\left(-x_1+\frac{1}{2}\right) + 2\mu_1 x_1 &= 0
\end{aligned}$$

The difference of the two constraints yields $x_1 = -1/4$ and $x_2 = 1/4$, while $\mu_1 = 3$. This is the only points suggested by the Karush-Kuhn-Tucker conditions.

Now, it is necessary to compare point $A = (-1/4, 1/4)$ and the nonregular points of line $x_2 = x_1 = \alpha$. In order to do that, first we determine the point of minimum among the latter, and then we compare it to A.

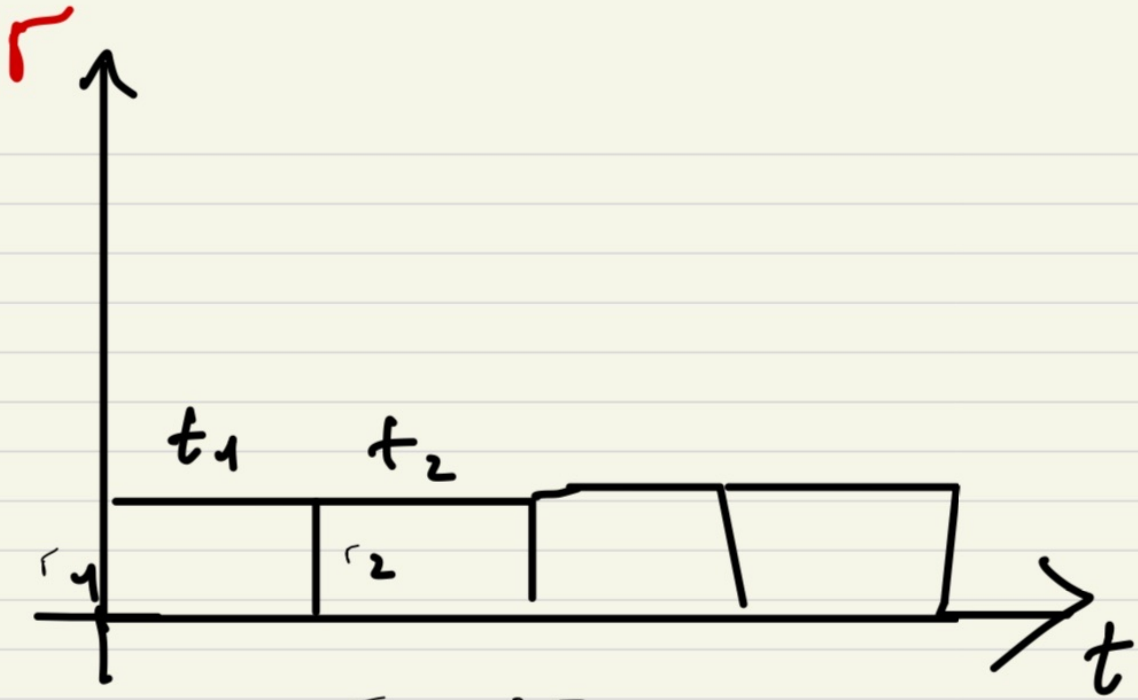
$$\min f(\alpha) = (\alpha + 1)^2 + \left(\alpha + \frac{1}{2}\right)^2$$

The solution is obtained setting to zero the derivative of f with respect to parameter α: $f'(\alpha) = 2(\alpha + 1) + 2\left(\alpha + \frac{1}{2}\right) = 0$, which implies $\alpha = -3/4$, and therefore $B = (-3/4, -3/4)$, where the objective function value is $f(B) = 1/8$. The value of the objective function in A, on the other hand, is $f(A) = 9/8$. Hence, the globally optimal point is the nonregular point $B = (-3/4, -3/4)$.



$\Rightarrow$ The lowestcost is doing follow retangal model

largecost $\rightarrow$ $\downarrow$ time



$$\begin{cases} \min\limits_{z,t} \sum\limits_{j\in J} 2z_j \\ \sum\limits_{j\in J} t_j \le T \\ r_j t_j = w_j \quad \forall j \in J \\ r_j\, t_j \text{ have a max value} \end{cases}$$

(w: work
z: cost
t: time)

$$\Rightarrow \begin{cases} \min\limits_{z,t} \sum\limits_{j\in J} 2z_j \\ \sum\limits_{j\in J} t_j - T \le 0 \\ r_j t_j - w_j = 0 \\ (\gamma_j) \quad -r_j + r_j^{min} \le 0 \end{cases}$$

r: cost (resource)

$$\begin{cases} \min_{r,t} \sum_{j\in J} 2 r_j \\ \sum_{j\in J} t_j \le T \\ r_j t_j = w_j \\ r_j \geq r_J^{min} \\ t_j \geq t_j^{min} \end{cases} \Rightarrow \begin{cases} \min \sum_{j\in J} 2\frac{w_j}{t_j} \\ \sum_{j\in J} t_j \le T & (1)\ (\mu) \\ r_j = \frac{w_j}{t_j} \\ t_j \le \frac{w_j}{r_j^{min}} & (2)\ (\alpha_j) \\ t_j \geq t_j^{min} & (3)\ (\beta_j) \end{cases}$$

$(1) \Rightarrow \sum_{j\in J} - T \le 0$ variable: t_j

$(2) \Rightarrow t_j - \frac{w_j}{r_j^{min}} \le 0$

$(3) \Rightarrow -t_j + t_j^{min} \le 0$

Choose μ, α_j, β_j are constant that

$$\begin{cases} -\frac{w_j}{t_j^2} + \mu 1 + \alpha - \beta = 0 & j \in J \\ \mu \left(\sum_{j\in J} t_j - T\right) = 0 \\ \alpha_j \left(t_j - \frac{w_j}{r_j^{min}}\right) = 0 \\ \beta_j\left(-t_j + t_j^{min}\right) = 0 \end{cases}$$

$\tilde{P}$

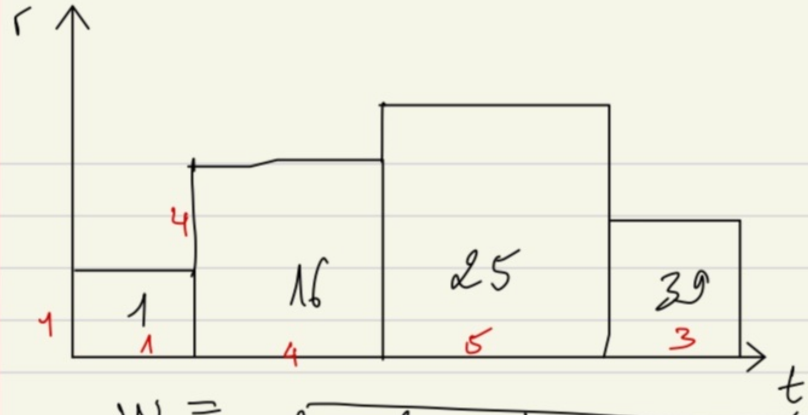
If $\alpha_j = \beta_j = 0 \ \forall j \in J$

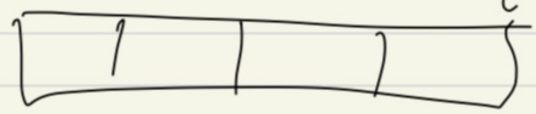
$$\begin{cases} -\frac{w_j}{t_j} + \mu = 0 & (j \in J) \\ \mu\left(\sum_{j\in J} t_j - T\right) = 0 \end{cases}$$

???! Parag 80 Condition or ..

Consider: $\mu = \frac{w_j}{t_j} \quad \forall j \geq 0$

$$\Rightarrow \sum_{j\in J} t_j = T \Rightarrow \sum_{j\in J} \frac{\sqrt{w_j}}{\sqrt{\mu}} = T \Rightarrow \sqrt{\mu} = \sum_{j\in J} \frac{\sqrt{w_j}}{T} \Rightarrow \boxed{t_j = \frac{\sqrt{w_j}}{\sum_{j\in J}\sqrt{w_j}} \cdot T}$$



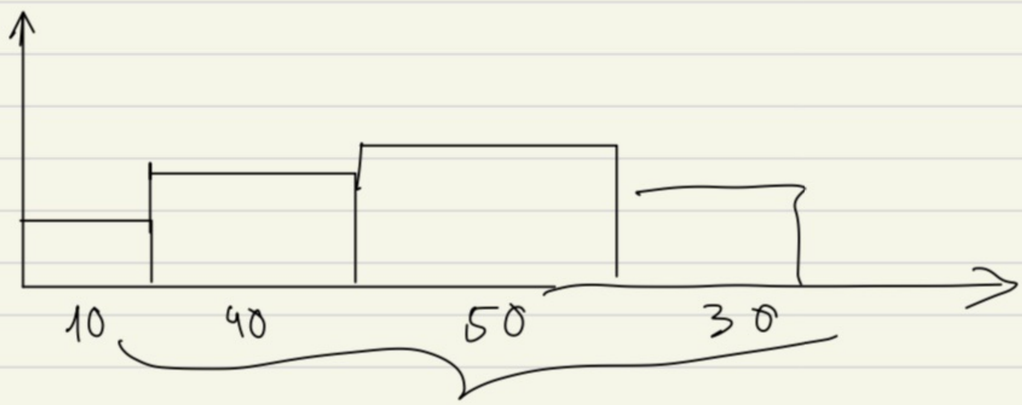
$w =$ 

$A \to B \to C \to D$

$\sqrt{w}$	1	4	5	3
t	10	40	50	30
r	$1/10$	$4/10$	$5/10$	$3/10$

$\sum_j \sqrt{w_j} = 13$

$\tau = 130$



$\tau = 130$