Irreducible and aperiodic Markov Chains

Irreducibility and aperiodicity are two important properties of MC which allow us to study the asymptotyc behaviour of the process. They are also easy to be checked. We consider, for simplicity, only homogeneous MC's.

Irreducibility is the property that "all states of the MC can be reached from all others".
More precisely:
let $(X_0, X_1, \dots)$ be a MC with state space $S = \{s_1, \dots, s_k\}$ and transition matrix P.

Def: we say that a state s_i communicates with another state s_j (notation: $s_i \rightarrow s_j$) if the chain has positive probability of ever reaching s_j when we start from s_i. In formulas,
$s_i \rightarrow s_j$ if $\exists n$ such that

$$P(X_{m+n} = s_j \mid X_m = s_i) > 0$$

This probability is independent of m (for the homogeneity of the MC) and equals $(P^n)_{ij}$ (A)

If $s_i \rightarrow s_j$ and $s_j \rightarrow s_i$ we write $s_i \leftrightarrow s_j$ and we say that s_i and s_j intercommunicate.

element ij of P^n

Def: A MC $(X_0, X_1, \dots)$ with state space $S = \{s_1, s_2, \dots, s_k\}$ and transition matrix P is called irreducible if $\forall s_i, s_j \in S$ we have $s_i \leftrightarrow s_j$.
Otherwise the chain is called reducible.

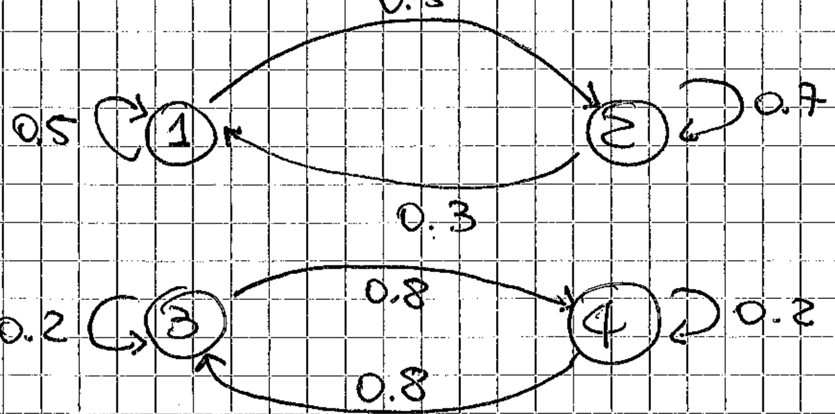
Because of (A), a chain is irreducible if $\forall s_i, s_j \in S$, $\exists n$ such that $(P^n)_{ij} > 0$

An easy way to check if a Markov Chain is irreducible is to look at its transition graph

Example: a reducible MC
Consider the MC $(X_0, X_1, \dots)$ with $S = \{1, 2, 3, 4\}$ and

$$
\begin{array}{c}
n_1 \qquad\qquad\qquad\qquad \\
P = \left[\begin{array}{cc|cc} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ \hline 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.8 & 0.2 \end{array}\right] \\
\qquad\qquad n_2
\end{array}
$$

Transition graph:



If the chain starts in 1 or 2 remains there, and the same if it starts from 3 or 4.
The graph can be splitted into 2 subgraphs and

the MC could be splitted into 2 MC, each with transition matrix the submatrices M_1 and M_2.

Thus this MC is reducible (and we understand the meaning of this term).

Aperiodicity : for a finite or infinite set $\{a_1, a_2, \dots\}$ of positive integers, let

$\gcd\{a_1, a_2, \dots\}$ = greatest common divisor of $a_1, a_2, \dots$

Def : The period $d(s_i)$ of a state $s_i \in S$ is defined as

$$d(s_i) = \gcd\{n \geq 1 : (P^n)_{ii} > 0\}$$

Thus the period of s_i is the gcd of the set of times that a chain can return (= has a positive probability of returning) to s_i, given that we started from $X_0 = s_i$.

Def : if $d(s_i) = 1$ then we say that s_i is aperiodic.

Def : A MC is said to be aperiodic if all its states are aperiodic.
Otherwise it is called periodic.

Example - The Gothenburg weather:

$S = \{s_1, s_2\}$ $s_1 = $ rain $s_2 = $ sun

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(See exercise 2.3 (a) p. 15)

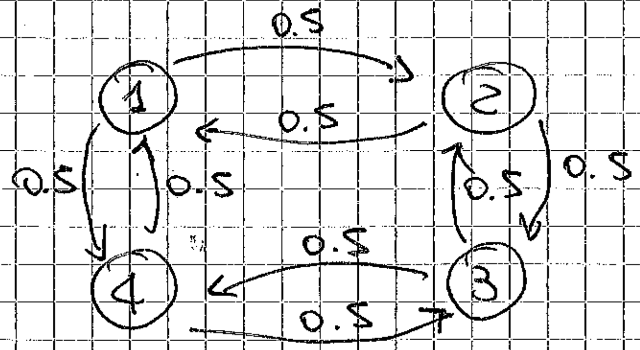
It can be proven, by induction, that the probability to have the same weather of today n days later is

$$(P^n)_{ii} = \frac{1}{2}(1 + 2^{-n}) > 0 \quad \forall n \quad \forall i$$

and the gcd of $\{1, 2, 3, 4, \dots\}$ is 1

$\Rightarrow d(s_1) = d(s_2) = 1 \Rightarrow$ the chain is aperiodic

Example - the random walk



If the random walker starts from ① at time 0, he has to take an even number of steps to come back to ①

$\Rightarrow (P^n)_{11} > 0$ only for $n = 2, 4, 6, \dots$

$\Rightarrow \gcd\{n \geq 1 : (P^n)_{ii} > 0\} = \gcd\{2, 4, 6, \dots\} = 2$

$\Rightarrow$ the chain is periodic

Aperiodicity is important because of the following

<u>THEOREM 1</u>: Suppose that $(X_0, X_1, \dots)$ is an <u>aperiodic</u> MC with state space $S = \{s_1, s_2, \dots, s_k\}$ and transition matrix P. Then there exists an $N < +\infty$ such that $(P^n)_{ii} > 0 \quad \forall i \in \{1, \dots, k\}$ $\forall n \geq N$

<u>Proof</u>: see Häggström Th. 4.1

Theorem 1 states that if a MC is aperiodic, there is a positive prob. of coming back to s_i, if we started from s_i, in a finite number of steps.

Combining aperiodicity with irreducibility we get the following result:

<u>THEOREM 2</u>: Let $(X_0, X_1, \dots)$ be an <u>irreducible</u> and <u>aperiodic</u> MC, with state space $S = \{s_1, \dots, s_k\}$ and transition matrix P. Then there exists an $M < +\infty$ such that $(P^m)_{ij} > 0 \quad \forall i, j \in \{1, \dots, k\}$ and $\forall m \geq M$.

Thus combining the 2 properties we obtain that there is a positive probability to reach any state s_j, starting from any state s_i, in a finite number of steps.

Theor 2 is useful to study the asymptotic behaviour of the MC.