Exercises - Combinatorics

(1) We draw, with replacement, 2 balls from an urn containing 4 white balls and 3 black balls.

a) What is the probability that the two drawn balls are of the same colour?

b) What is the probability that at least one of the drawn balls is black?

Solution: Let us label the balls with numbers:

$$A = \{W_1, W_2, W_3, W_4, B_1, B_2, B_3\}$$

Sample space: $\Omega = \{(b_1, b_2) : b_1 \in A, b_2 \in A\}$

Possible cases: n. of elements of $\Omega = 49$

$$\begin{matrix}(b_1, & b_2)\\ \uparrow & \uparrow \\ 7 \times & 7\end{matrix}$$

a) Favourable cases: 25

$$\begin{matrix}(W, & W) \\ \uparrow & \uparrow \\ 4 \times & 4 = 16\end{matrix} \quad + \quad \begin{matrix}(B, & B) \\ \uparrow & \uparrow \\ 3 \times & 3 = 9\end{matrix}$$

$\Rightarrow P(\text{two balls of the same colour}) = \dfrac{25}{49}$

b) In this case it is easier to compute the probability of the complement:

P("at least one ball is black") =

$= 1 - P($"both balls are white"$)$

n. favourable cases: (w, w)

$4 \times 4 = 16$

$= 1 - \frac{16}{49} = \frac{33}{49}$

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(2) Repeat the previous exercise but assuming now that the balls are drawn <u>without replacement</u>.
Answer to a) and b) of (1)

Now the sample space is

$$\Omega = \{ (b_1, b_2) : b_1 \in A, \ b_2 \in A \setminus \{b_1\} \}$$

possible cases: (b_1, b_2)

$7 \times 6 = 42$

a) Favourable cases:

$(w, w) + (B, B)$

$4 \times 3 + 3 \times 2 = 18$

$\Rightarrow \mathbb{P}(\text{two balls of same colour}) = \frac{18}{42} = \frac{3}{7}$

b) $\mathbb{P}(\text{"at least one ball is black"}) =$

$= 1 - \mathbb{P}(\text{"both balls are white"})$

n. of favourable cases: (w, w)
$4 \times 3 = 12$

$= 1 - \frac{12}{42} = \frac{30}{42} = \frac{5}{7}$

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③ The door of a computer center has a lock which has 5 buttons, numbered from 1 to 5. The combination of number which opens the lock is a sequence of 5 numbers and is changed every week.

a) How many combinations are possible if every button must be used only once?

b) Assume that the lock can also have combinations that require you to push two buttons simultaneously, and then the other 3 one at a time. How many more combinations does this permit?

Solution:

(a) The lock combinations are the permutations of 5 elements: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

(b) The 2 simultaneous buttons can be chosen into $\binom{5}{2}$ different ways, while the other 3 can be pressed in $3!$ different orders. Thus the additional combinations are

$$\binom{5}{2} \cdot 3! = \frac{5!}{\cancel{(5-2)!}\, 2!} \cdot \cancel{3!} = \frac{5!}{2!} = \frac{120}{2} = 60$$