LLN AND CLT

$$Y_t + Y_{t-1} + Y_{t-2} =$$
$$3\mu + \varepsilon_t + (1+\theta)\varepsilon_{t-1} + (1+\theta)\varepsilon_{t-2} + \theta\varepsilon_{t-3}$$

$$\sum_{t=1}^{T} Y_t = T\mu + \varepsilon_T + (1+\theta)\sum_{t=1}^{T-1}\varepsilon_t + \theta\varepsilon_0$$

For the LLN,

$$\frac{\sum_{t=1}^{T} Y_t}{T} = \mu + (1+\theta)\frac{\sum_{t=1}^{T-1}\varepsilon_t}{T} + \frac{\varepsilon_T}{T} + \frac{\theta\varepsilon_0}{T}$$

$$\frac{\sum_{t=1}^{T-1}\varepsilon_t}{T} \to_p 0$$, since ε_t is stochastically bounded, it also follov

$$\frac{\varepsilon_0}{T} \to_p 0, \ \frac{\varepsilon_T}{T} \to_p 0$$

$$\frac{\sum_{t=1}^{T} Y_t}{T} \to_p \mu$$

Proceeding in the same way, for the CLT,

$$\sqrt{T}\frac{\sum_{t=1}^{T}(Y_t-\mu)}{T} = (1+\theta)\sqrt{T}\frac{\sum_{t=1}^{T-1}\varepsilon_t}{T} + \frac{\varepsilon_T}{\sqrt{T}} + \frac{\theta\varepsilon_0}{\sqrt{T}}$$

and notice that $\{\varepsilon_t\}_{t=-\infty}^{\infty}$ meets the conditions for the CLT, therefore

$$\sqrt{T}\frac{\sum_{t=1}^{T-1}\varepsilon_t}{T} \to_d N(0,\sigma^2)$$

Moreover, since ε_t is stochastically bounded, it also follows that

$$\frac{\varepsilon_0}{\sqrt{T}} \to_p 0, \ \frac{\varepsilon_T}{\sqrt{T}} \to_p 0$$

and therefore

$$\sqrt{T}\frac{\sum_{t=1}^{T}(Y_t-\mu)}{T} \to_d N(0,\sigma^2(1+\theta)^2)$$