GTDM – 2019/20

Linear system -
Echelon form and applications

Based on Linear Algebra and Its Applications, David C. Lay, Steven R. Lay, and Judi J. McDonald, PEARSON 5^th ed.

A fundamental problem that surfaces in all mathematical sciences is that of analyzing and solving **m algebraic equations in n unknowns**.
The earliest recorded analysis of simultaneous equations is found in the ancient Chinese book Chiu-chang Suan-shu (Nine Chapters on Arithmetic), estimated to have been written some time around 200 B.C. In the beginning of Chapter VIII, there appears a problem of the following form.

Three sheafs of a good crop, two sheafs of a mediocre crop, and one sheaf of a bad crop are sold for 39 dou. Two sheafs of good, three mediocre, and one bad are sold for 34 dou; and one good, two mediocre, and three bad are sold for 26 dou. What is the price received for each sheaf of a good crop, each sheaf of a mediocre crop, and each sheaf of a bad crop?

Today, this problem would be formulated as three equations in three unknowns by writing….

$$3x + 2y + z = 39,$$
$$2x + 3y + z = 34,$$
$$x + 2y + 3z = 26,$$

where x, y, and z represent the price for one sheaf of a good, mediocre, and bad crop, respectively.

How to solve the problem in general?

A possible answer: using the determinants, Laplace's formula.

But … computational cost for a NxN system: proportional to
N! = Nx(N-1)x(N-2)x…2x1

For example 70! > 10^{100} , in order to solve the system 70x70 (on a fast/parallel computer) we need many years

Equivalent systems

Consider the two systems

(a)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

System (a) is 'easy' to solve because from the last two equations it follows that **$x_3=2$**, and **$x_2=3$**. Using these values in the first equation, we get

$$3x_1 + 2x_2 - x_3 = -2 \;\Leftrightarrow\; 3x_1 + 6 - 2 = -2 \;\Leftrightarrow\; \mathbf{x_1 = -2}$$

Then the solution of the system is the vector (-2 3 2)^T.
System (b) seems to be more 'difficult' to solve but … actually, system (b) has the same solution as system (a).

Definition.
Two systems of linear equations involving the same variables are said to be **equivalent** if they have the **same solution set**.

Elementary Row Operations

The basic method for solving a system of linear equations is to replace the given system by **a new system that has the same solution set** but which is **easier** to solve.

Since the **rows** of an augmented matrix correspond to the **equations** in the associated system, a new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically.
These are called elementary row operations

- Multiply an equation by an nonzero constant
- Interchange two equations
- Add a multiple of one equation to another

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m} = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m} = b_2 \\
&\ldots \quad \ldots \quad \ldots \\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm} = b_n
\end{aligned}$$

n equations
m variables

$$\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1m} & b_1 \\
a_{21} & a_{22} & \ldots & a_{2m} & b_2 \\
\ldots & & \ldots & & \ldots \\
a_{n1} & a_{n2} & \ldots & a_{nm} & b_n
\end{bmatrix}$$

Row Echelon Form and Information From It

What is echelon form?

- "Echelon" is a military term that refers to a specific type of formation
- In echelon formation, each member of the formation is positioned behind and to the right (or left) of the member ahead

Example of echelon formation

Echelon form of a Matrix

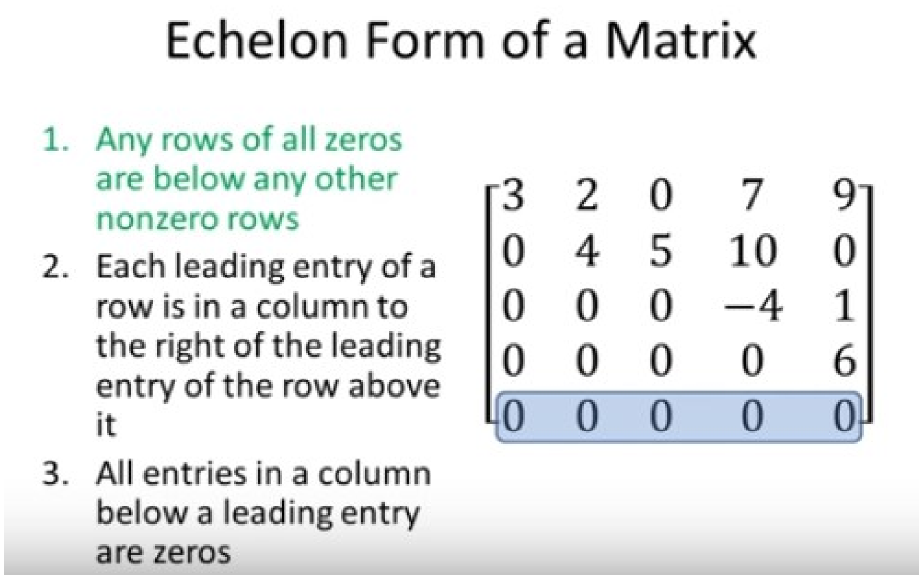
- What does it mean for a matrix to be in echelon form?
- We need to pay attention to the leading entry in each row
- The **leading entry** of a row is the first non zero entry in that row

Echelon form of a Matrix

1. Any rows of all zeros are below any other nonzero rows

2. Each leading entry of a row is in a column to the right of the leading entry of the row above it

3. All entries in a column below a leading entry are zeros



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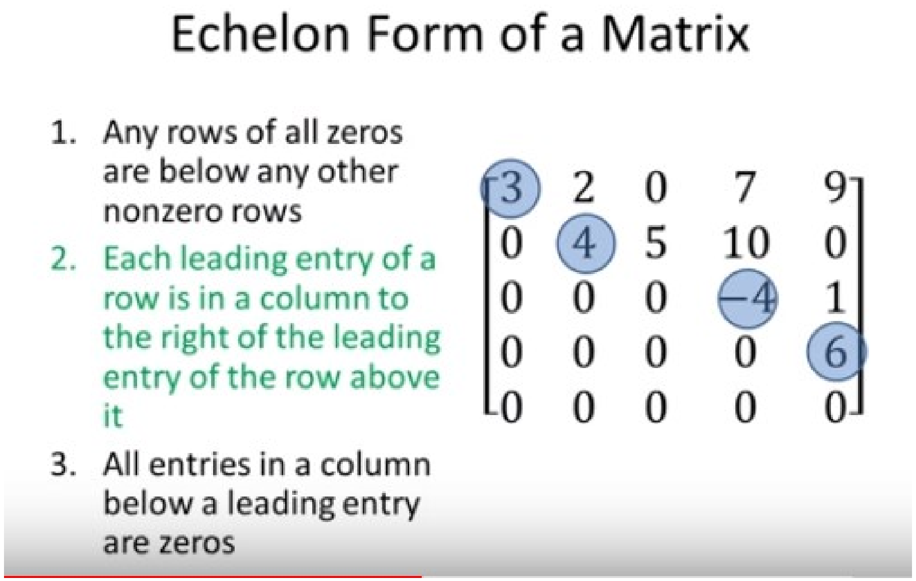
$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



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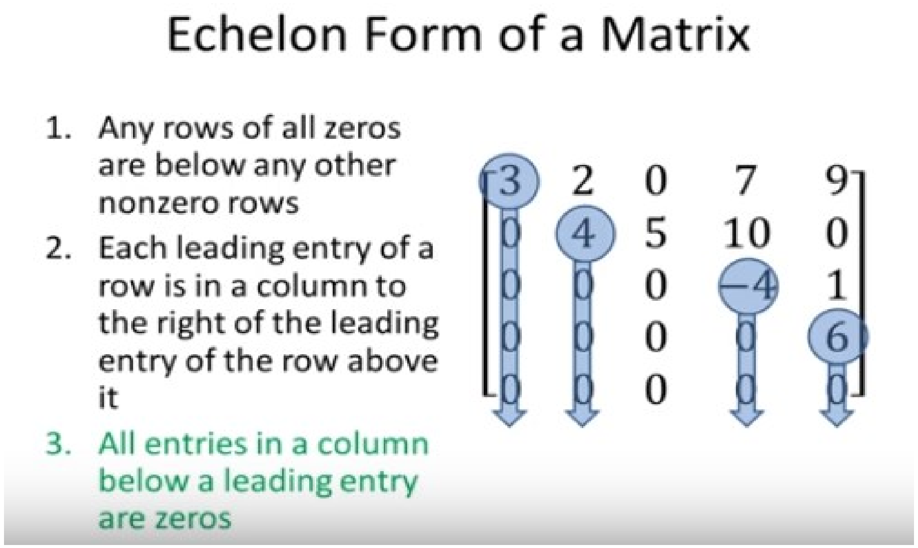

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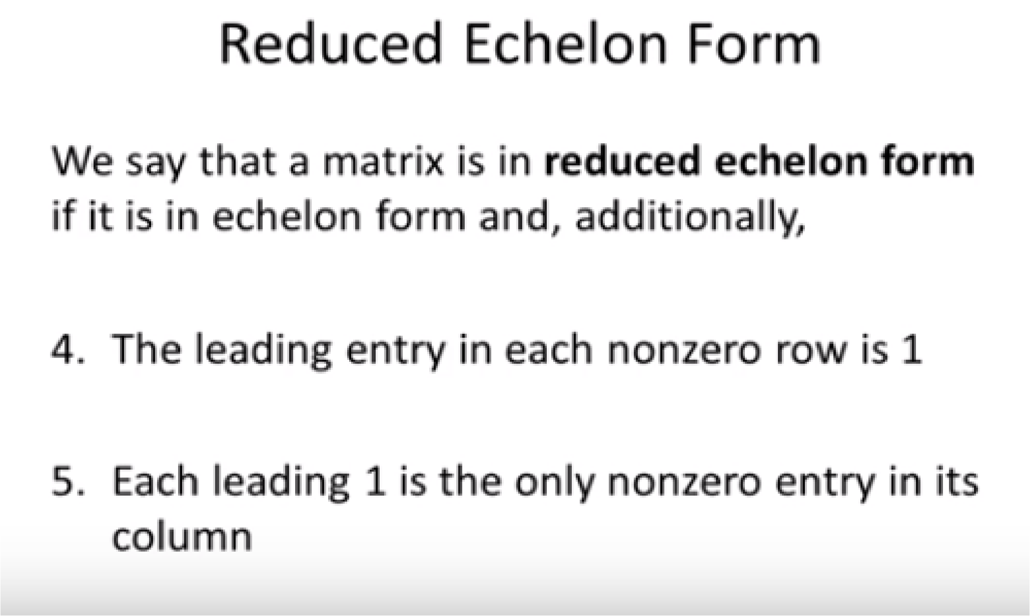
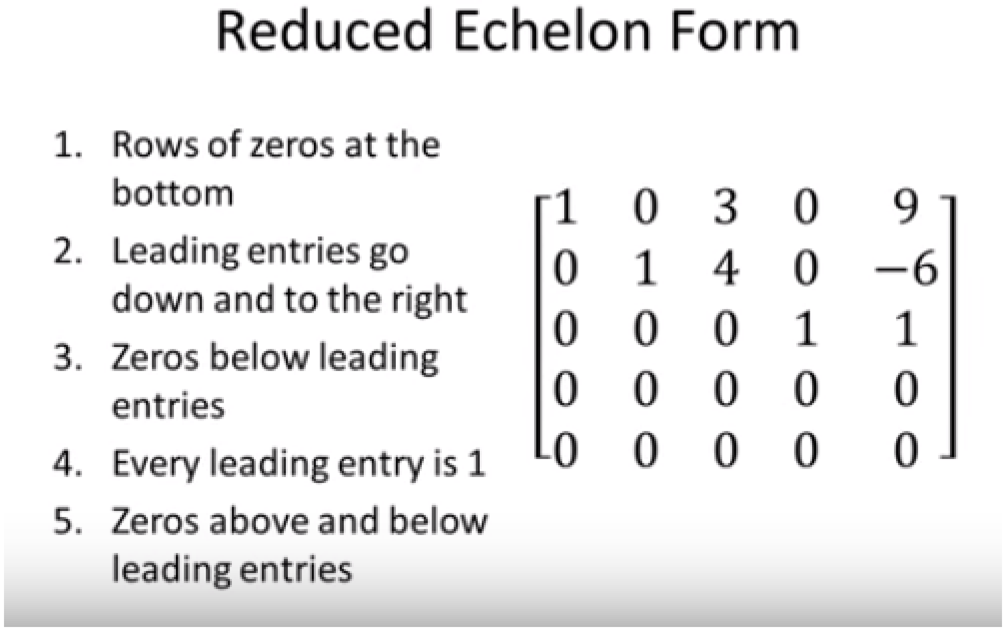


Reduced Echelon Form

We say that a matrix is in **reduced echelon form** if it is in echelon form and, additionally,

4. The leading entry in each nonzero row is 1

5. Each leading 1 is the only nonzero entry in its column



Reduced Echelon Form

1. Rows of zeros at the bottom
2. Leading entries go down and to the right
3. Zeros below leading entries
4. Every leading entry is 1
5. Zeros above and below leading entries

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basic method in order to obtain the echelon form:

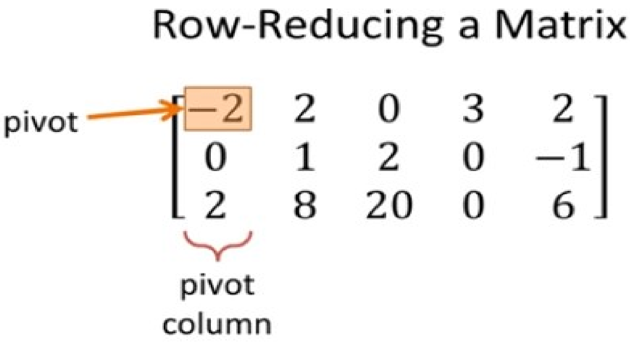
Row-reducing a Matrix

1. Begin with the first nonzero column. This is a pivot column and the pivot position is at the top.

2. If necessary, swap rows to get a nonzero entry into the pivot position.

3. Use the replacement row operation to create zeros in all positions below the pivot.

4. Move to the next pivot column and repeat this process.

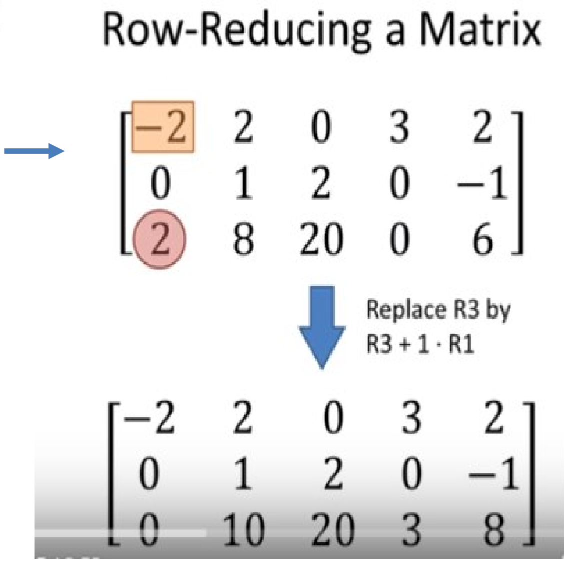


Row-Reducing a Matrix

pivot

$$\begin{bmatrix} \boxed{-2} & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix}$$

pivot column

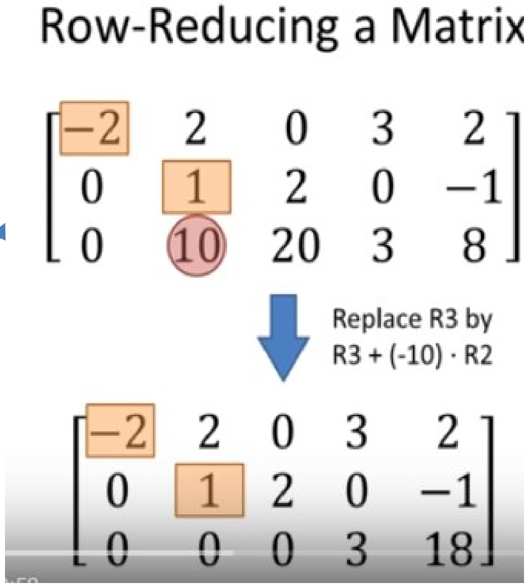


Row-Reducing a Matrix

$$\begin{bmatrix} \boxed{-2} & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ \textcircled{2} & 8 & 20 & 0 & 6 \end{bmatrix}$$

Replace R3 by R3 + 1 · R1

$$\begin{bmatrix} -2 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 10 & 20 & 3 & 8 \end{bmatrix}$$



Row-Reducing a Matrix

$$\begin{bmatrix} \boxed{-2} & 2 & 0 & 3 & 2 \\ 0 & \boxed{1} & 2 & 0 & -1 \\ 0 & \textcircled{10} & 20 & 3 & 8 \end{bmatrix}$$

Replace R3 by R3 + (-10) · R2

$$\begin{bmatrix} \boxed{-2} & 2 & 0 & 3 & 2 \\ 0 & \boxed{1} & 2 & 0 & -1 \\ 0 & 0 & 0 & 3 & 18 \end{bmatrix}$$

Example – row reduction (from the book of Lay-Lay-McDonald)

Apply elementary row operations to transform the following matrix into echelon form

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

STEP 1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Pivot column

STEP 2. Select a nonzero entry in the pivot column as a pivot. If necessary, Interchange rows to move this entry into the pivot position.

$$\begin{bmatrix} \overset{\text{Pivot}}{3} & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 3. Use row replacement operations to create zeros in all positions below the pivot.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

STEP 4. Ignore the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

Pivot

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

New pivot column

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & \underset{\uparrow\ \text{Pivot}}{1} & 4 \end{bmatrix}$$

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. The equivalent system is the following:

$$\begin{array}{rrrrrl} 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \\ & 2x_2 & -4x_3 & +4x_4 & +2x_5 & = -6 \\ & & & & x_5 & = 4 \end{array}$$

with 3 equations and 5 variables. Pivot columns: 1,2, and 5.

Example – row reduction

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

- Step1. Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

- Step2. Interchange the top row with another row, to bring a nonzero entry to top of the column found in Step1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1th and 2th rows in the preceding matrix were interchanged.

Example – row reduction

Step3. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-1st row of the preceding matrix was added to the 3rd row.

Step4. Now cover the top row in the matrix and begin again with Step1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row-echelon form

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

Example – row reduction

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

+5/2 times the 2st row of the matrix was added to the 3nd row of the matrix to introduce a zero below the leading -2.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Leftmost nonzero column in the new submatrix

We have only one row in the new submatrix, then stop.

From the echelon form to the solution

Example 1. Suppose, for example, that the augmented matrix of a linear system has been changed into the **equivalent *reduced* echelon form**

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -1/2 & -3/2 \end{bmatrix}$$

or the system , 3 equations and 3 unknowns

$$\begin{aligned} x_1 \quad +x_2 \quad +2x_3 \quad &= 9 \\ +2x_2 \quad -7x_3 \quad &= -17 \\ -x_3/2 \quad &= -3/2 \end{aligned}$$

From the last equation: x_3 =3, then (second equation) x_2 =2, and (first equation) x_1 =1.

From the echelon form to the solution

Example 2. Suppose, for example, that the augmented matrix of a linear system has been changed into the **equivalent *reduced* echelon form**

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

or the system , 3 equations and 5 unknowns

$$\begin{array}{rrrrrl} x_1 & +6x_2 & +2x_3 & -5x_4 & -2x_5 & =-4 \\ & & +2x_3 & -8x_4 & -x_5 & =3 \\ & & & & x_5 & =7 \end{array}$$

We have 3 basic variables and 2 free variables but: how to choose them? The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1, x_3, and x_5. The remaining variables, x_2 and x_4, must be free. Solve for the basic variables to obtain the general solution:

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Each different choice of x_2 and x_4 *determines a (different) solution of the system, and every solution of the system is determined by a choice of* x_2 and x_4

From the echelon form to the solution

Example 3. Suppose, for example, that the augmented matrix of a linear system has been changed into the **equivalent *reduced* echelon form**

$$\begin{bmatrix} 1 & -1 & 3 & 7 \\ 0 & 2 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which represents a linear system (3 equations and 3 unknowns)

$$\begin{aligned} x_1 \quad -x_2 \quad +3x_3 &= 7 \\ 2x_2 \quad +4x_3 &= -2 \\ 0 &= 0 \end{aligned}$$

From the last equation, 0=0, we deduce that there is a free variable. Columns 1 and 2 are pivot columns, then x_1 and x_2 are basic variables while x_3 is a free variable. The solutions (infinitly many Solutions) are:

$$\begin{aligned} x_1 &= 6 - 5x_3 \\ x_2 &= -1 - 2x_3 \\ x_3 & \qquad free \end{aligned}$$

From the echelon form to the solution

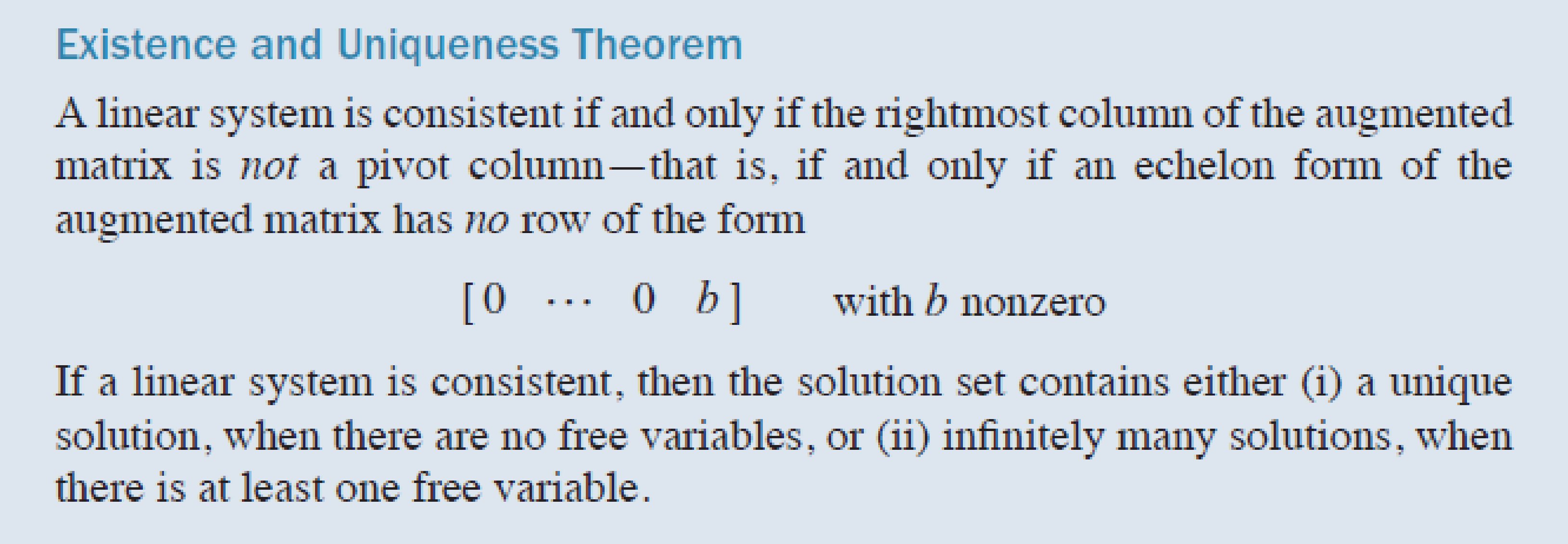
Example 4. Suppose, for example, that the augmented matrix of a linear system has been changed into the **equivalent *reduced* echelon form**

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

which represents a linear system
(3 equations and 3 unknowns)

$$\begin{array}{cccl} x_1 & +x_2 & +2x_3 & = 6 \\ & 2x_2 & +x_3 & = 2 \\ & & 0 & = 3 \end{array}$$

From the last equation, 0=3,
we deduce that there can be no solutions.
Then: No solution.



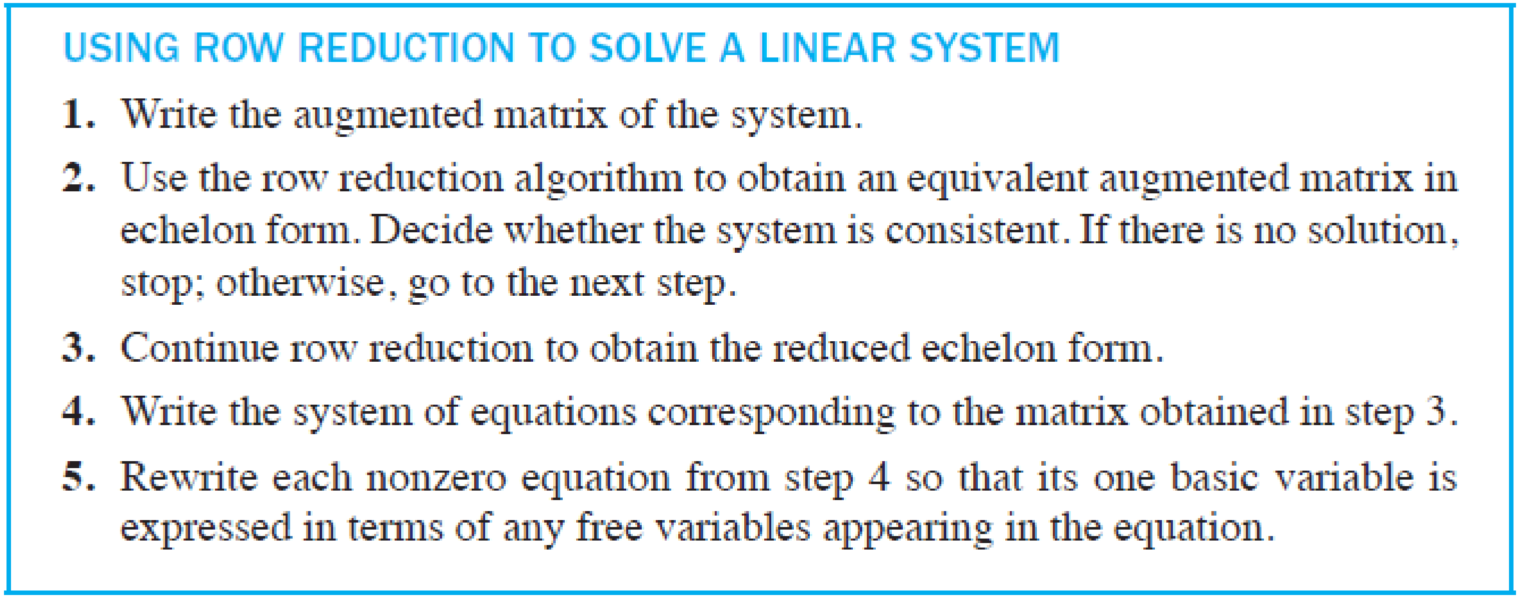
Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[\,0 \quad \cdots \quad 0 \quad b\,] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.



USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Some Applications of (reduced) echelon form:

- Linear system (Existence, Uniqueness, and Computations), use the augmented matrix

- Linear independence of a set of a vectors $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m\}$, forming the matrix whose columns are the vectors:
  $\mathrm{A} = [\,\mathbf{V}_1 \mid \mathbf{V}_2 \mid \ \dots \mid \mathbf{V}_m\,]$, computing its echelon form: no zero rows implies that the vectors are linearly independent (otherwise linearly dependent).

- Rank of a matrix A (= number of linearly independent columns
  = number of linearly independent rows)
  compute the echelon form of A, rank(A)= number of non-zero rows in the echelon form of the matrix A.

- Components of a vector **b** with respect to a basis $\{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m\}$. We have to find the solution $\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_m$ of the following vector equation (a linear system) $\mathrm{x}_1\mathbf{V}_1 + \mathrm{x}_2\mathbf{V}_2 + \dots + \mathrm{x}_m\mathbf{V}_m = \mathbf{b}$ (as a linear combination of the columns of a matrix)

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