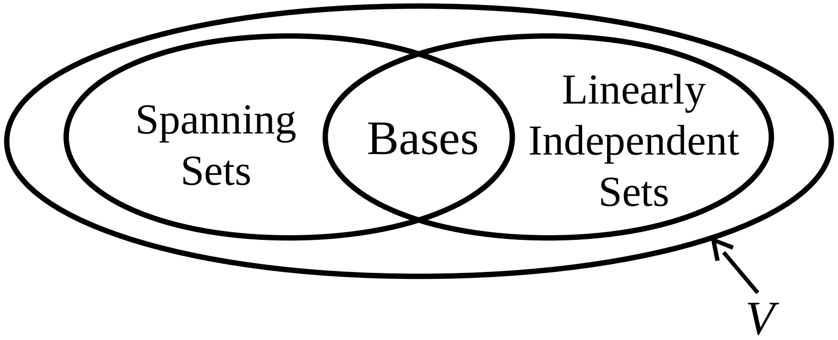
GTDM – 2019/20

Basis

Based on Linear Algebra and Its Applications, David C. Lay, Steven R. Lay, and Judi J. McDonald, PEARSON 5^th ed.

Basis and Dimension

- Basis:



V: a vector space (e.g. $\mathbf{R}^n$) or a subspace (e.g. an hyperplane in $\mathbf{R}^n$)

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset V$

(1) S spans V (i.e., span(S) = V)

(For any $\mathbf{u} \in V$, $\sum x_i \mathbf{v}_i = \mathbf{u}$ has a solution)

(2) S is linearly independent

(For $\sum x_i \mathbf{v}_i = \mathbf{0}$, there is only the trivial solution)

S* is called a basis for *V

- Notes:

A basis S must have enough vectors to span V, but not so many vectors that one of them could be written as a linear combination of the other vectors in S

Example. The **standard basis** for R^n (here vectors as rows):

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0),\ldots$, $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: For R^4, $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

Example. Show that $S=\{\mathbf{v}_1,\mathbf{v}_2\}=\{(1,1),\ (1,-1)\}$ is a basis for R^2

(1) For any $\mathbf{u}=(\mathrm{u}_1,\mathrm{u}_2)\in R^2$, $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{u} \Rightarrow \begin{cases} c_1+c_2=\mathrm{u}_1 \\ c_1-c_2=\mathrm{u}_2 \end{cases}$

The system has a unique solution for each **u**. Thus you can conclude that S spans R^2

(2) For $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{0} \Rightarrow \begin{cases} c_1+c_2=0 \\ c_1-c_2=0 \end{cases}$

The system has only the trivial solution. Thus you can conclude that S is linearly independent.

Theorem [Uniqueness of basis representation for any vectors]

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space *V*, then every vector in *V* can be written in one and only one way as a linear combination of vectors in *S*

Theorem [Bases and linear dependence]

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space *V*, then every set containing more than *n* vectors in *V* is linearly dependent (In other words, every linearly independent set contains at most *n* vectors)

Theorem [Number of vectors in a basis]

If a vector space *V* has one basis with *n* vectors, then every basis for *V* has *n* vectors

- Dimension:

The dimension of a vector space V is defined to be the number of vectors in a basis for V

V: a vector space S: a basis for V

$\dim(V) = \#(S)$ (the number of vectors in a basis S)

Remark. We consider here only Finite dimensional:

A vector space V is finite dimensional if it has a basis consisting of a finite number of elements

Example. Finding the dimension of a subspace of R^3

(a) $W=\{(d, c-d, c)$: c and d are real numbers$\}$

(b) $W=\{(2b, b, 0)$: b is a real number$\}$

Sol: (Hint: find a set of L.I. vectors that spans the subspace, i.e., find a basis for the subspace.)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S spans W)

S is a basis for W

$\dim(W) = \#(S) = 2$

(b) $Q\ (2b, b, 0) = b(2, 1, 0)$

$S = \{(2, 1, 0)\}$ spans W and S is L.I.

S is a basis for W

$\dim(W) = \#(S) = 1$

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LINEAR TRANSFORMATIONS

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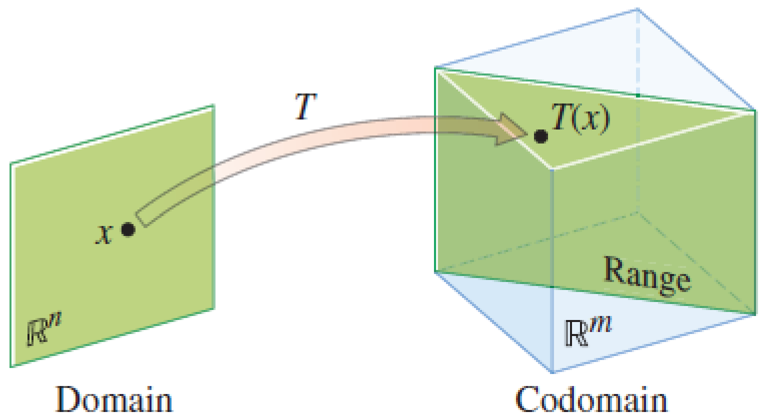
A **transformation** (or **function** or **mapping**) T from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule that assigns to each vector **x** in $\mathbf{R}^n$ a vector T(**x**) in $\mathbf{R}^m$



The set $\mathbf{R}^n$ is called the **domain** of T, and $\mathbf{R}^m$ is called the **codomain** of T. For **x** in $\mathbf{R}^n$, the vector T(**x**) in $\mathbf{R}^m$ is called the **image** of **x** (under the action of T). The set of all images T(**x**) is called the **range** of T.

Linear Transformations

Linear Transformations satisfy:

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

- **u** and **v** are vectors
- *a* and *b* are scalars

Linear mapping T from R^n to R^m can be expressed by using a *m x n* matrix A.

Example. The linear transformation T from R^3 to R^2 is defined as,

$$T\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 + 2\,u_2 \\ 3u_2 + 4u_3 \end{pmatrix}$$

Can be written as $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

For each **x** in R^n, T(**x**) is computed as A**x**, where A is an m x n matrix. For simplicity, we sometimes denote such a *matrix transformation* by $\mathbf{x} \rightarrow A\mathbf{x}$.

Example.

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and

define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

Solve T (**x**) = **b** for **x**, that is, solve A**x** =**b**, means to find an **x** whose image under T is **b**.

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$$

Remark. The question of a *uniqueness* problem for a system of linear equations, translated here into the language of matrix transformations: Is **b** the image of a *unique* **x** in $\mathbf{R}^n$. Similarly, for the *existence* problem: does there *exist* an **x** whose image is **b?**

Identity matrix $I_n = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$, for example $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Usually want a "formula" for T(**x**), Every linear transformation from R^n to R^m is actually a matrix transformation $\mathbf{x} \to A\mathbf{x}$ and that important properties of T are intimately related to familiar properties of A.

The key to finding A is to observe that T is completely determined by what it does to the columns of the n x n identity matrix I_n.

Example. The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^3$ such that

$$T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

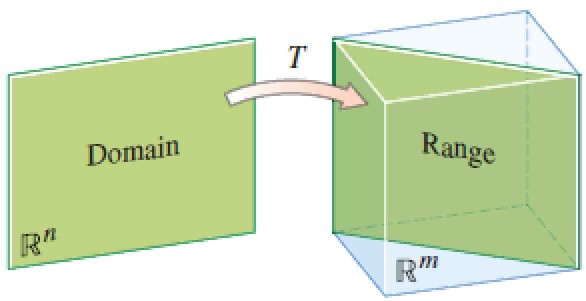
With no additional information, find a formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

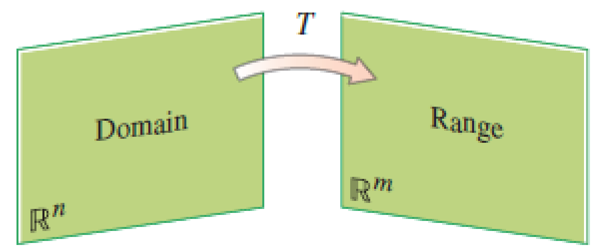
Since T is a *linear* transformation,

$$\begin{aligned} T(\mathbf{x}) &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 + 0 \end{bmatrix} \end{aligned}$$

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at least one* $\mathbf{x}$ in $\mathbb{R}^n$.

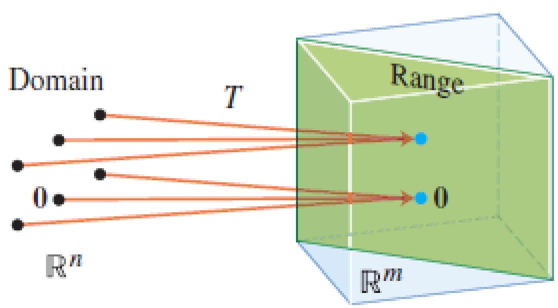


T is *not* onto $\mathbb{R}^m$

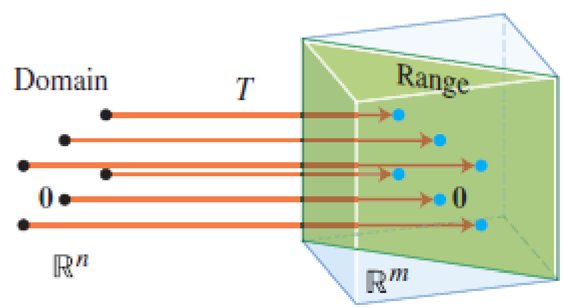


T is onto $\mathbb{R}^m$

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of *at most one* $\mathbf{x}$ in $\mathbb{R}^n$.



T is *not* one-to-one



T is one-to-one

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Theorem

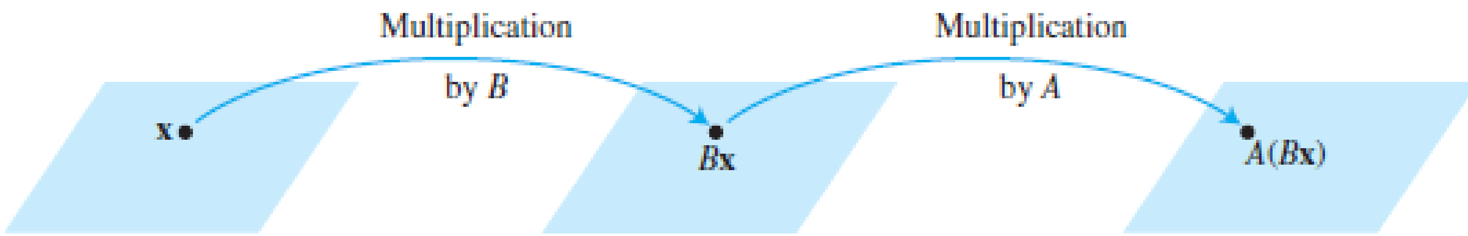
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

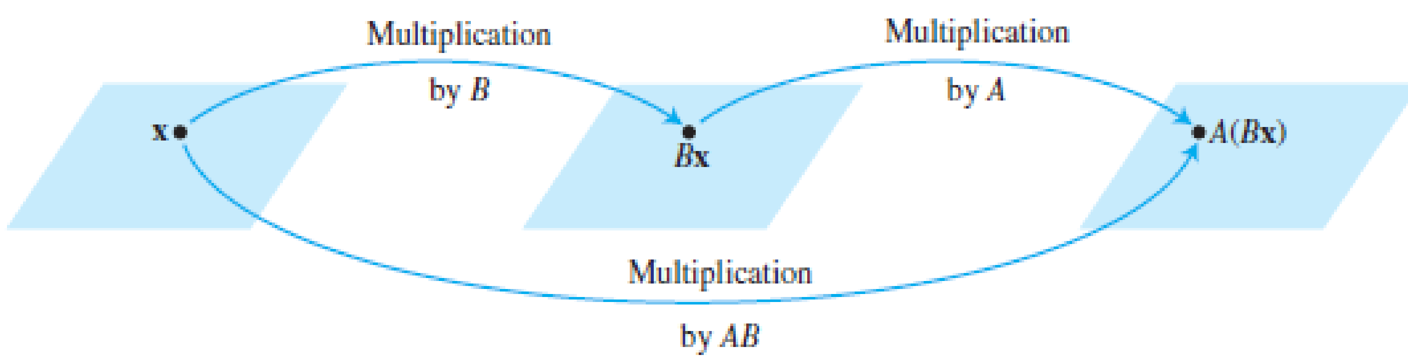
Matrix Multiplication

When a matrix B multiplies a vector **x**, it transforms **x** into the vector B**x**. If this vector is then multiplied in turn by a matrix A, the resulting vector is A(B**x**)



Thus A(B**x**) is produced from **x** by a *composition* of mappings. Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by *AB*, so that

$$A(B\mathbf{x}) = (AB)\,\mathbf{x}$$



Example. Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.

Write $B = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,]$, and compute:

$$A\mathbf{b}_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}, \quad A\mathbf{b}_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}, \quad A\mathbf{b}_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

Then

$$AB = A[\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\,] = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

$A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3$

ROW–COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example.

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square \\ -4+21-12 & 6+3-8 \\ \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ 5 & 1 \\ \square & \square \\ \square & \square \end{bmatrix}$$

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -27 & -17 \\ 5 & 1 \\ 15 & 36 \end{bmatrix}$$

Powers of a Matrix

If A is an nxn matrix and if k is a positive integer, then A^k denotes the product of k copies of A

$$A^k = \underbrace{A \cdots A}_{k}$$

The Transpose of a Matrix

Given an m x n matrix A, the **transpose** of A is the n x m matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

THE INVERSE OF A MATRIX

An n x n matrix A is said to be **invertible** if there is an n x n matrix C such that

CA =I and AC = I

where I is the n x n identity matrix. In this case, C is an **inverse** of A. In fact, C is uniquely determined by A, because if B were another inverse of A, then B =C : this unique inverse is denoted by A^{-1}

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

Example.

If $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$, then

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $C = A^{-1}$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The quantity ad-bc is called the **determinant** of A, and we write det A = ad-bc

Example. $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. $\Longrightarrow$ $A^{-1} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.