DEPARTMENT OF ECONOMICS, MANAGEMENT AND QUANTITATIVE METHODS

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B-74-3-B Time Series Econometics

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Discussion of Exercise Sheet 2

1.

Autocorrelation structure of two AR(2). We discuss parts i. and ii. at the same time.

$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$ where $\varepsilon_t \; wn\,(0, \sigma^2)$.

Discuss stationarity first.

When $\phi_1 = 0.8$, $\phi_2 = -0.8$, the characteristic equation is

$(1 - 0.8z + 0.8z^2) = 0$, Solution is: $0.5 + 1.0i, 0.5 - 1.0i$ and $|0.5 \pm 1| > 1$ so the process is stationary because both the roots are outside the unit circle. From the fact that it has complex roots we can also see it has a cycle.

When $\phi_1 = -0.5$, $\phi_2 = 0.3$, the characteristic equation is

$(1 + 0.5z - 0.3z^2) = 0$, Solution is: $2.840\,3, -1.173\,6$, so the process is stationary because both the roots are outside the unit circle.

i. We are interested in the autocorrelation: for stationary processes, these are

$$\rho_j = \frac{\gamma_j}{\gamma_0}$$

where

$$\mu = E\,(Y_t) \text{ and } \gamma_j = E\left[(Y_t - \mu)\,(Y_{t-j} - \mu)\right]$$

First,

$$\begin{aligned} E\,(Y_t) &= E\,(c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t) \\ &= E\,(c) + \phi_1 E\,(Y_{t-1}) + \phi_2 E\,(Y_{t-2}) + E\,(\varepsilon_t) \end{aligned}$$

and, using stationarity, $E(Y_t) = E(Y_{t-1}) = E(Y_{t-2}) = \mu$; moreover, because ε_t is white noise, $E(\varepsilon_t) = 0$, so

$$\mu = \frac{c}{1 - \phi_1 - \phi_2}.$$

Rewriting $c = \mu(1 - \phi_1 - \phi_2)$, our model becomes

$$(Y_t - \mu) = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + \varepsilon_t \text{ where } \varepsilon_t \ wn\left(0, \sigma^2\right)$$

Of course, we could have skipped all this part if $c = 0$, in which case $\mu = 0$ and we have directly

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \text{ where } \varepsilon_t \ wn\left(0, \sigma^2\right), \text{ and } \gamma_j = E(Y_t Y_{t-j})$$

So,

$$\begin{aligned} \gamma_{j\geq 1} &= E(Y_t \times Y_{t-j}) = E((\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t) Y_{t-j}) \\ &= E(\phi_1 Y_{t-1} Y_{t-j}) + E(\phi_2 Y_{t-2} Y_{t-j}) + E(\varepsilon_t Y_{t-j}) \\ &= \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \end{aligned}$$

where we used

$$E(Y_{t-1} Y_{t-j}) = E\left(Y_t Y_{t-(j-1)}\right) = \gamma_{j-1}, \; E(Y_{t-2} Y_{t-j}) = E\left(Y_t Y_{t-(j-2)}\right) = \gamma_{j-2}$$

because of stationarity, and

$$E(\varepsilon_t Y_{t-j}) = 0 \text{ for } j \geq 1$$

because ε_t is white noise so it does not depend on the past ($E(\varepsilon_t \varepsilon_{t-j}) = 0$ for $j \geq 1$), while Y_{t-j} is a past value (when $j \geq 1$). So

$$\gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \text{ for } j \geq 1$$

(and notice that $\gamma_j = \gamma_{-j}$, so $\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$). Dividing by γ_0,

$$\rho_{j\geq 1} = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2},$$

(Yule Walker equations) which we initialise by setting

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_{-1}$$

(again using stationarity, $\rho_1 = \rho_{-1}$) so

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

and for ρ_2 just notice that $\rho_0 = 1$,

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$

and iterating.

Note. It is worth mentioning here that we did not need to compute γ_0 to derive these autocorrelations.

The plot is very different, a cycle can be observed for the process having complex roots in the characteristic equation, while for the other process the autocorrelations change sign at every step.

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
$if\ \phi_1 = 0.8,\ \phi_2 = -0.8$	0.444	-0.444	-0.711	-0.213	0.398
$if\ \phi_1 = -0.5,\ \phi_2 = 0.3$	-0.714	0.657	-0.542	0.468	-0.397

iii IRF - A plot of $\frac{\partial Y_t}{\partial \varepsilon_{t-j}}$ (against j) is called Impulse Response Function. For a process Y_t that admits

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}$$

for ε_t such that, for any t,

$$\begin{aligned} E(\varepsilon_t) &= 0,\ E(\varepsilon_t^2) = \sigma^2, \\ E(\varepsilon_t \varepsilon_\tau) &= 0 \text{ if } \tau \neq t \end{aligned}$$

notice that

$$\frac{\partial Y_t}{\partial \varepsilon_{t-j}} = \psi_j$$

so ψ_j is the effect on Y_t of a shock that took place $t - j$ periods before.

It may also be of interest to compute the ψ_j in the IRF (Wold decomposition):

Any ARMA(p,q) can be represented as $\phi(L) Y_t = \theta(L) \varepsilon_t$, where
$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$, $\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$
and stationarity ensures $Y_t = \phi^{-1}(L) \theta(L) \varepsilon_t$.
We are looking for the parameters ψ_j in the infinite polynomial
$\psi(L) = 1 + \psi_1 L + ...$, such that $Y_t = \psi(L) \varepsilon_t$:
this means that $\phi^{-1}(L) \theta(L) = \psi(L)$, and then $\theta(L) = \phi(L) \psi(L)$
this is

$$\begin{aligned} & 1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3 + ... + \theta_q L^q \\ = & \left(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3 - ... - \phi_p L^p\right)\left(1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + ...\right) \end{aligned}$$

$$\begin{aligned} & 1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3 ... + \theta_q L^q \\ = & 1 - \phi_1 L + \psi_1 L - \psi_1 \phi_1 L^2 + \psi_2 L^2 - \phi_2 L^2 - \phi_3 L^3 - \phi_2 \psi_1 L^3 - \phi_1 \psi_2 L^3 + \psi_3 L^3 + ... \end{aligned}$$

since this is an identity the elements of the same order must be equal,

so,

for the terms of order L, $\theta_1 = \psi_1 - \phi_1$, which means $\psi_1 = \theta_1 + \phi_1$,

for the terms of order L^2, $\theta_2 = \psi_2 - \phi_2 - \psi_1 \phi_1$, which means $\psi_2 = \theta_2 + \phi_2 + \psi_1 \phi_1$ (notice that ψ_1 is known at this point, because it was determined in the previous step)

for the terms of order L^3, $\theta_3 = \psi_3 - \phi_3 - \psi_1 \phi_2 - \phi_2 \psi_1$, which means $\psi_3 = \theta_3 + \phi_3 + \psi_1 \phi_2 + \phi_2 \psi_1$

....

In this case $\theta_{j \geq 1} = 0$, so we have
$1 = 1 - \phi_1 L + \psi_1 L - \psi_1 \phi_1 L^2 + \psi_2 L^2 - \phi_2 L^2 - \phi_3 L^3 - \phi_2 \psi_1 L^3 - \phi_1 \psi_2 L^3 + \psi_3 L^3 ...$
and then
$\psi_1 = \phi_1$,
$\psi_2 = \phi_2 + \psi_1 \phi_1$
$\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1$
i.e.
$\psi_{j \geq 1} = \phi_2 \psi_{j-2} + \phi_1 \psi_{j-1}$ (recall $\psi_0 = 1$).
For the given parameters, these weights are
When $\phi_1 = 0.8$, $\phi_2 = -0.8$:
$\psi_1 = 0.8$, $\psi_2 = -0.16$, $\psi_3 = -0.768$, $\psi_4 = -0.4864$, $\psi_5 = 0.2253$
(notice again the cyclical component);
When $\phi_1 = -0.5$, $\phi_2 = 0.3$:

$\psi_1 = -0.5$, $\psi_2 = 0.55$, $\psi_3 = -0.425$, $\psi_4 = 0.377\,5$, $\psi_5 = -0.316\,25$.

2.

i. The process is stationary ($|-0.5| < 1$) and invertible ($|0.7| < 1$).

ii. Rewriting the ARMA(p,q) as $\phi(L)Y_t = \theta(L)\varepsilon_t$, where

$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p$, $\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$

and the polynomial of the MA(∞) given by the Wold decomposition as

$Y_t = \psi(L)\varepsilon_t$ where $\psi(L) = 1 + \psi_1 L + ...$,

then $\phi^{-1}(L)\theta(L) = \psi(L)$, and then $\theta(L) = \phi(L)\psi(L)$

this is $1 + \theta_1 L + ... + \theta_q L^q = (1 - \phi_1 L - \phi_2 L^2 - ... - \phi_p L^p)(1 + \psi_1 L + ...)$

$1 + \theta_1 L + ... + \theta_q L^q = 1 - \phi_1 L + \psi_1 L - \psi_1\phi_1 L^2 + \psi_2 L^2 - \phi_2 L^2 ...$

since this is an identity the elements of the same order must be equal, so,

for the terms of order L, $\theta_1 = \psi_1 - \phi_1$, which means $\psi_1 = \theta_1 + \phi_1$,

for the terms of order L^2, $\theta_2 = \psi_2 - \phi_2 - \psi_1\phi_1$, which mean $\psi_2 = \theta_2 + \phi_2 + \psi_1\phi_1$ (notice that ψ_1 is known at this point, because it was determined in the previous step)

....

In this case

$1 + \theta L = 1 - \phi L + \psi_1 L - \psi_1\phi L^2 + \psi_2 L^2 - \phi\psi_2 L^3 + \psi_3 L^3 ...$

and then

$\psi_1 = \phi + \theta = 1.2$

$\psi_2 = \phi\psi_1 = 0.6$

$\psi_3 = \phi\psi_2 = 0.3$

and, in general,

$\psi_{j\geq 2} = \phi\psi_{j-1}$.

You can also prove it by looking at $Y_t = \phi Y_{t-1} + \xi_t$ where $\xi_t = \varepsilon_t + \theta\varepsilon_{t-1}$. Then,

$Y_t = \sum_{j=0}^{\infty}\phi^j\xi_{t-j} = \sum_{j=0}^{\infty}\phi^j(\varepsilon_{t-j} + \theta\varepsilon_{t-j-1}) = \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j} + \theta\sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j-1}$ $= \sum_{j=0}^{\infty}\phi^j\varepsilon_{t-j} + \theta\sum_{l=1}^{\infty}\phi^{l-1}\varepsilon_{t-l} = \varepsilon_t + \sum_{j=1}^{\infty}\phi\phi^{j-1}\varepsilon_{t-j} + \theta\sum_{j=1}^{\infty}\phi^{j-1}\varepsilon_{t-j} = \varepsilon_t + (\theta + \phi)\sum_{j=1}^{\infty}\phi^{j-1}\varepsilon_{t-j}$

iii. Before computing the autocorrelations, notice that

$Cov(Y_t, \varepsilon_t) = Cov(\phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_t) =$

$= Cov(\phi Y_{t-1}, \varepsilon_t) + Cov(\varepsilon_t, \varepsilon_t) + Cov(\theta\varepsilon_{t-1}, \varepsilon_t) = 0 + \sigma^2 + 0.$

Next,

$\gamma_0 = Var(Y_t) = Var(\phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}) =$

$= Var(\phi Y_{t-1}) + Var(\varepsilon_t) + Var(\theta\varepsilon_{t-1}) + 2Cov(\phi Y_{t-1}, \theta\varepsilon_{t-1})$ (the other covariances are 0) so

$\gamma_0 = \phi^2 Var(Y_{t-1}) + Var(\varepsilon_t) + \theta^2 Var(\varepsilon_{t-1}) + 2\phi\theta Cov(Y_{t-1}, \varepsilon_{t-1}) =$

$\gamma_0 = \phi^2\gamma_0 + \sigma^2 + \theta^2\sigma^2 + 2\phi\theta\sigma^2$ using stationarity,

$\gamma_0 = \frac{1+\theta^2+2\phi\theta}{1-\phi^2}\sigma^2$

and

$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(\phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}, Y_{t-1}) = \phi\gamma_0 + 0 + \theta\sigma^2$

so $\gamma_1 = \left(\frac{1+\theta^2+2\phi\theta}{1-\phi^2}\phi + \theta\right)\sigma^2 = \frac{\phi+\phi\theta^2+2\phi^2\theta+\theta-\phi^2\theta}{1-\phi^2}\sigma^2 = \frac{\phi+\phi\theta^2+\phi^2\theta+\theta}{1-\phi^2}\sigma^2 =$

$= \frac{\phi+\theta+\phi\theta(\theta+\phi)}{1-\phi^2}\sigma^2 = \frac{(\theta+\phi)(1+\phi\theta)}{1-\phi^2}\sigma^2$

and

$\gamma_2 = Cov(Y_t, Y_{t-2}) = Cov(\phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}, Y_{t-2}) = Cov(\phi Y_{t-1}, Y_{t-2}) = \phi\gamma_1$,

and in general

$\gamma_{j\geq 2} = Cov(Y_t, Y_{t-j\geq 2}) = Cov(\phi Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}, Y_{t-j\geq 2}) = Cov(\phi Y_{t-1}, Y_{t-j\geq 2}) = \phi\gamma_{j-1}$,

So

$\rho_1 = \frac{(\theta+\phi)(1+\phi\theta)}{1+\theta^2+2\phi\theta}$

$\rho_{j\geq 2} = \phi\rho_j$

so the autocorrelation function has a bump at the first lag, but behaves like an AR(1) otherwise (notice that this argument could be generalised in order to recognise any ARMA(p,q) model).

So,

$\rho_1 = \frac{(\theta+\phi)(1+\phi\theta)}{1+\theta^2+2\phi\theta} = \frac{(0.7+0.5)(1+0.5*0.7)}{1+0.7^2+2*0.5*0.7} = 0.739\,73$

$\rho_2 = 0.5 * \rho_1 = 0.369\,87$

$\rho_3 = 0.5 * \rho_2 = 0.5 * 0.369\,87 = 0.184\,94$

An alternative way to compute ρ_1 is to use the $MA(\infty)$ representation:

then $\gamma_0 = \sum_{k=0}^{\infty}\psi_k^2\sigma^2$ and $\sum_{k=0}^{\infty}\psi_k^2 = 1^2 + (\phi+\theta)^2\sum_{k=1}^{\infty}\phi^{(k-1)2} = 1^2 + (\phi+\theta)^2\sum_{l=0}^{\infty}\phi^{2l} = 1^2 + \frac{(\phi+\theta)^2}{1-\phi^2} =$

$1^2 + \frac{(0.5+0.7)^2}{1-0.5^2} = 2.\,92$

and $\gamma_j = \sum_{k=0}^{\infty}\psi_k\psi_{k+j}\sigma^2$ and, for $j = 1$, $\sum_{k=0}^{\infty}\psi_k\psi_{k+j} = 1*(\theta+\phi) + (\theta+\phi)^2\sum_{k=1}^{\infty}\phi^{(k-1)+(k-1+1)} =$

$1*(\theta+\phi)+(\theta+\phi)^2\phi\sum_{k=1}^{\infty}\phi^{2(k-1)} = (\theta+\phi)+\frac{(\theta+\phi)^2\phi}{1-\phi^2} = 0.7+0.5+0.5\frac{(0.7+0.5)^2}{1-0.5^2} = 2.\,16$

$\rho_1 = \frac{2.\,16\sigma^2}{2.\,92\sigma^2} = 0.739\,73$

(could use this procedure for higher lags as well)

3.

Just factorise $Y_t = 0.7Y_{t-1} - 0.1Y_{t-2} + \varepsilon_t + 0.5\varepsilon_{t-1} - 0.14\varepsilon_{t-2}$

$\left(1 - 0.7L + 0.1L^2\right) Y_t = \left(1 + 0.5L - 0.14L^2\right) \varepsilon_t,$

$\left[(1 - 0.5L)(1 - 0.2L)\right] Y_t = \left[(1 + 0.7L)(1 - 0.2L)\right] \varepsilon_t,$

(verify that the model is stationary, then, because $|0.5| < 1$ and $|0.2| < 1$)

so the factor $(1 - 0.2L)$ is common, and the model can be reparametrised as

$(1 - 0.5L) Y_t = (1 + 0.7L) \varepsilon_t,$

$Y_t = 0.5Y_{t-1} + \varepsilon_t + 0.7\varepsilon_{t-1}$

for which we already computed the autocorrelation function.

4.

i. Assuming that $E(Y_t) = 0$,

$$\widehat{Y}_{t+1|t,t-1,\ldots,1} = \alpha_1^{(t)} Y_t + \alpha_2^{(t)} Y_{t-1} + \ldots + \alpha_t^{(t)} Y_1$$

where, letting $\gamma_j = E(Y_t Y_{t+j})$,

$$\begin{pmatrix} \alpha_1^{(t)} \\ \alpha_2^{(t)} \\ \ldots \\ \alpha_1^{(t)} \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{t-1} \\ \gamma_1 & \gamma_0 & \cdots & \gamma_{t-2} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{t-1} & \gamma_{t-2} & \cdots & \gamma_0 \end{pmatrix}^{-1} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_t \end{pmatrix}$$

ii. Inverting this matrix is computationally intensive, when t is large. As an alternative, setting $\widehat{\varepsilon}_1 = 0$, we may compute

$$\begin{aligned} \widehat{\varepsilon}_2 &= Y_2 - 0.2Y_1 \\ \widehat{\varepsilon}_s &= Y_s - 0.2Y_{s-1} - 0.6\widehat{\varepsilon}_{s-1} \text{ for } s \geq 1 \end{aligned}$$

and finally,

$$\widehat{Y}_{t+1|t,t-1,\ldots,1,\widehat{\varepsilon}_1=0} = 0.2Y_t + 0.6\widehat{\varepsilon}_t$$