Lezione 4 - 01/10/2019

venerdì 1 novembre 2019 19:43

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Yesterday we saw how to answer most important questions : impulse and forecasting

Invertibility is a very important feature. We now represent a class of model that represent these models.

We introduce white noise: a process without memory and we build block with something bigger and we saw that the first object is the process we see is the average 1 MA(1). As we check the moment we where also able to verify feature. If we want to make forecast using MA(1) for the whole forecast matrix we only have to characterized teta.

Instead of calculating hundred autocovariance, we just have to calculate the value of teta. The MA1 is a easy process and the last thing we saw is that for each MA1 two different value of teta will fit alpa in the same way. If teta is <1 we know we get a linear function. There is another way that is using the Lag operator. It can be written as a polynomial using proprieties.

MA(q)

MA(q)

Let ε_t w.n.$(0,\sigma^2)$, then

$$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}$$

is MA(q).

Again, it is easy to verify that this MA(q) is stationary, as

$$\sum_{j=0}^{\infty}\psi_j^2 = 1 + \theta_1^2 + \ldots + \theta_q^2 < \infty.$$

Mean:

$$E(Y_t) = E(\mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}) = \mu$$

Autocovariances:

$$\begin{aligned}\gamma_0 &= E\left[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q})^2\right]\\ &= (\sigma^2 + \theta_1^2\sigma^2 + \ldots + \theta_q^2\sigma^2)\\ &= \left(1 + \theta_1^2 \quad + \ldots + \theta_q^2\right)\sigma^2\end{aligned}$$

(using $E(\varepsilon_{t-j}\varepsilon_{t-k}) = 0$ for $k \neq j$).

$$\begin{aligned}\gamma_{j\le q} &= E[(\varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q})\\ &\quad \times (\varepsilon_{t-j} + \theta_1\varepsilon_{t-1-j} + \ldots + \theta_q\varepsilon_{t-q-j})]\\ &= E(\theta_j\varepsilon_{t-j}^2 + \theta_{j+1}\theta_1\varepsilon_{t-j-1}^2 + \theta_{j+2}\theta_2\varepsilon_{t-j-2}^2 + \ldots + \theta_q\theta_{q-j}\varepsilon_{t-q}^2\\ &= \left(\theta_j \quad + \theta_{j+1}\theta_1 \quad + \theta_{j+2}\theta_2 \quad + \ldots \quad + \theta_{q-j}\theta_q\right)\sigma^2\\ \gamma_{j>q} &= 0\end{aligned}$$

Autocorrelations:
the autocorrelations drop to 0 after q lags

Impulse Response Function:
the impulse response are $\psi_j = \theta_j, j \le q$, and drop to 0 after q lags.

If I only have memory for today and yesterday is very poor. How can I estimate the value for 3 days ago and even back? We use Lag value of q. We have moving average of q. We can do the same action for MA(1) for MA(q) but it's longer. We can get mean, autocovariances, gamma1, gamma2. Autocovariance for MA(1) drop to 0 after the second autocovariance (gamma $_{j>=2}$ = 0) Epsilon depend on what happen today and yesterday. With MA(q) the autocovariance will be non-zero up to the q lag. After the q LAG the autocovariance will be 0. We can stretch thing up to q. In MA(1) we saw how to get invertibility: just calculate the invert of teta.

MA(1) = Yt = eps t + teta*eps t-1

Let's use the Lag operator

(1 + teta L) eps t → this will be the polynomial psi of L [Phi(L)]

I want to invert it.

If |teta| <1 so (1+ teta L) ^-1 EXIST

(1+ tetaL) ^-1 Yt = eps t

This is another polynomial that is Pi of L.

In other particular case, Pi of L was sum (-teta L) ^ j

Invertibility
Set $\mu = 0$; recall

$$Y_t = (1 + \theta_1 L + \ldots + \theta_q L^q)\varepsilon_t$$

and factor

$$\begin{aligned}&(1 + \theta_1 L + \ldots + \theta_q L^q)\\ &= (1 - \lambda_1 L)(1 - \lambda_2 L)\ldots(1 - \lambda_q L)\end{aligned}$$

in the MA(1) we asked that $|\lambda_1| < 1$: in the same way here we have to ask that $|\lambda_1| < 1$, $|\lambda_2| < 1, \ldots, |\lambda_q| < 1$.
This is sometimes stated as asking that the roots of the equation in z

$$(1 + \theta_1 z + \ldots + \theta_q z^q) = 0$$

lie outside the unit circle.

If the MA(q) process is invertible, we can write

$$\varepsilon_t = (1 + \theta_1 L + \ldots + \theta_q L^q)^{-1} Y_t$$

and then derive $\pi_0, \pi_1, \pi_2, \ldots$ such that

$$\varepsilon_t = \sum_{j=0}^{\infty} -\pi_j Y_{t-j}, \text{ i.e. } Y_t = \sum_{j=1}^{\infty}\pi_j Y_{t-j} + \varepsilon_t$$

AR(p) $Y_t = c + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} + \varepsilon_t$

MA(q) $Y_t = \mu_t + \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}$

μ_t

→ $Y_t = c + \phi_1 Y_{t-1} + \ldots \phi_p Y_{t-p} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}$

→ $(1 + \phi_1 L - \phi_2 L^2 \ldots \phi_p L^p) Y_t = c + (1 + \theta_1 L + \ldots + \theta_q L^q)\varepsilon_t$

→ $\Phi(L) Y_t = c + \Theta(L)\varepsilon_t$

$\Phi(L) Y_t = c + \Theta(L)\varepsilon_t$ Θ MAIUSC

TO HAVE STATIONARITY I NEED $\Phi^{-1}(L)$ EXISTS, SO THAT I CAN WRITE

$Y_t = \Phi(1)^{-1}c + \underbrace{\Phi(L)^{-1}\Theta(L)}_{\psi(L)}\varepsilon_t$

TO HAVE STATIONARITY I NEED $\boxed{Y_t = \mu + \sum \psi_s\varepsilon_{t-s}}$ WITH $\sum \psi_s^2 < \infty$

THIS IS $\boxed{Y_t = \mu + \Psi(L)\varepsilon_t}$ SO I NEED THAT $\Phi(L)^{-1}$ EXIST

$Y_t = \Phi(1)^{-1}c + \underbrace{\Phi(L)^{-1}\Theta(L)}_{\Psi(L)}\varepsilon_t$

How to invert these guys with q element?

I will break q order polynomial in q polynomial of order 1. I will able to invert them with the properties that we used before. How do I break these in q polynomial? It's a equation of q polynomial, we find the solution of the q order equation. We want the various lambda that will be less than 1. We will just look at the equation here where lambda is the inverse of the solution. If I can do these I can break in to more polynomial. Lambda we will the inverse of the solution. We want the solution (1 + teta1 z …) bigger than 1. Outside the unit circle, equation with grade q will have more then q solutions. Not all the solution have to be real and some can be complex. How can we find absolute value of complex number? We will look the unit circle. Drawing a circle and check if we are inside or outside the circle.

We will see an example.

MA(q) works in the same way of MA(1) but everything became heavier in computational.

Can we stretch q up to infinity?

Yes and NO. We know that we are interested in model of these kind u + sum j= 0 to infinite phi j eps t -j.

We are interested in reducing the parameter that we want to estimate. We have an infinity of parameter here. It's interesting to characterized the moment, mean and autocovariance using the same absolute rule of M(q).

Instead doing that we will do something like AR(1) → autoregression 1. These was the first model used with success in time series.

Y depend on its self the day before. If we compare it to the MA1. **MA1 is not an auto regression? It's not because epsilon is not observable!**

In the regression model Yt and Yt -1 are dependent over time. How can I discuss the propriety hat I like in this model (stationarity , ergodicity)?

If Phi <1 let's do the following : (slide)

I will get (1 + phi + phi^2) c + phi …

If I do this n times, I get sum of all to the power of j and the sum ..

If phi is less than 1 these value goes to 0. Y it's a random variable and has a probability.

Is these guy stationary? YES it is. We look at model of the 1° slide with additional condition where the sum of phi j ^ 2 is finite.

Sum of phi j ^2 = sum (phi ^j) ^2 = sum phi (^2J) = 1 / 1 – phi^2.

It's also ergotic, using absolute value instead of squared.

|Phi| < 1 what does mean? 0.5 0.3 … after few step it will not be 0 but closer to 0.

MA(∞)

Let ε_t w.n.$(0,\sigma^2)$, then

$$Y_t = \mu + \varepsilon_t + \psi_1\varepsilon_{t-1} + \ldots = \mu + \sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}$$

is MA(∞).
Under the additional assumption that

$$\sum_{j=0}^{\infty}|\psi_j| < \infty,$$

we can derive the moments replacing θ_j by ψ_j in a MA(q) and taking the limit for $q \to \infty$.

Mean:

$$E(Y_t) = E(Y_t = \mu + \varepsilon_t + \psi_1\varepsilon_{t-1} + \ldots) = \mu$$

Autocovariances:

$$\gamma_0 = \sum_{k=0}^{\infty}\psi_k^2\sigma^2$$

$$\gamma_j = \sum_{k=0}^{\infty}\psi_k\psi_{k+j}\sigma^2$$

We want to be cool and the cool way to proceed is to use the Lag operator.

And the inversion is the exactly of what we study yesterday.

MA(1) has memory of the past up to one period before (before we cannot, we have autocorrelation of 0). MA(q) Event that occur q time before are not correlated. In these case what happened 1000 days ago change the today (in a really small amount). These is cool model it's basically have correlation with large days before.

Large number .. of the CLT still hold on these guys

What are the moment of these guys?

Expected value is the formula Yt with c. and then we can get the gammas value.

Partial autocorrelation? If I tell you Yt-1 what we have to know about the past? Nothing because the other will be 0. So alpha1 (1) = phi, alpha j>=2 (j) = 0.

Impulse response

Phi j = ∅ ^ j

He give us a simpler method:

If I know that's stationary I can use these guys. How we do it? C is not random and so I have c + phi and E (Yt) = mu, E(Yt-1) = mu because it's stationary and E(eps t) = 0 and we get mu = c + phi mean which is mean = c / 1 – phi

Because is stationary if I'm speaking with Yt or Yt-1 it's the same. Eps is only correlated with thing that happened on today and so not correlated with Yt-1 so these final correlations will be 0 and at the end I have the equation phi ^2 sigma ..

And then I can get gamma0.

If I use the formula for

Gamma1 = Phi * gamma0 = Phi * sigma^2 / 1- Phi^2

Gamma2 = Phi * gamma1 = Phi * Phi * sigma^2 / 1 – Phi ^2

Every time I farther away I multiplicate for PHI.

This model is a sort of innovational we can remember even farther away. But the farther away is very, very small.

MA(q) so I can AR(p). I can push the dependence of past of Y more.

We know the AR(1) is stationary when I can write:

Yt = phi Yt-1 + eps t

(1 – Phi L) Yt = eps t.

If |Phi| < 1 then (1 – phi L)^-1 EXIST and I can derive a finite PSIj

Yt = Sum PSI j eps j with sum of PSI j ^2 < infinite

Each individual lambda will be less than 1. I will need the solution of the equation (1 – phi1 z - …) and I want these more than 1. The solution for z is the inverse for Lambda.

I will ha ve to find out the factor of lambda first checking the solution of the equation with z.

When I have stationarity I can compute the moment → I can compute mean, autocovariances.

These is interesting and establish a very nice properties. Epsilon t and Yt give me sigma square.

And these is like the previous one, solving for gamma0. When we know sigma^2 and Phi^2 we can calculate everything. All we need to know it's Phi. And here is the same thing and instead of 1 previous autocovariance this depends on p autocovariance. We will have a SYSTEM of p equation..

We can get also autocorrelation and partial correlation. After point p partial autocorrelation goes to 0 because there is no more correlation on epsilon t and Yt.

Example.

For example, AR(2),

$$\begin{aligned}\gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 + \sigma^2\\ \gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_{-1}\\ \gamma_2 &= \phi_1\gamma_1 + \phi_2\gamma_0\end{aligned}$$

and notice that $\gamma_1 = \gamma_{-1}$, so replacing γ_1 and γ_2,

$$\gamma_1 = \frac{\phi_1}{1-\phi_2}\gamma_0, \quad \gamma_2 = \left(\frac{\phi_1^2}{1-\phi_2} + \phi_2\right)\gamma_0$$

$$\gamma_0 = \frac{(1-\phi_2)}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right]}\sigma^2$$

and

$$\begin{aligned}\rho_1 &= \phi_1 + \phi_2\rho_1\\ \rho_2 &= \phi_1\rho_1 + \phi_2\end{aligned}$$

so

$$\begin{aligned}\rho_1 &= \frac{\phi_1}{1-\phi_2}\\ \rho_2 &= \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2}.\end{aligned}$$

AR(2). Now it's a system not only for gamma0 and gamma1 but also for gamma2. I need to know phi1, phi2 and sigma^2.

Gamma -1 is the autocovariance one lag behind, but the moments depend only on the distant in time so doesn't matter if we go backward or straight ward (davanti) .

Numerical exercise

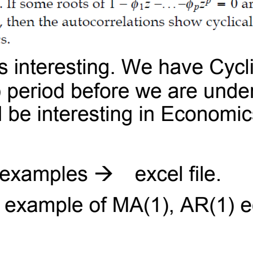
★ AR(2) If the roots of $1 - \phi_1 z - \phi_2 z^2 = 0$ are complex, then the autocorrelations show a cyclical dynamics. This is very important because both economics and natural phenomena often display cyclical dynamics.

Example: $\phi_1 = 0.75$, $\phi_2 = -0.45$.

Solutions of $1 - 0.75z + 0.45z^2 = 0$ are $z_{1,2} = 0.833\,33 \pm 1.236i$.

Note that $|z_i| = \sqrt{0.833\,33^2 + 1.236^2} = 1.4907$ so this is process is stationary.

Autocorrelation function:



$\phi_1 = 0.75, \phi_2 = -0.45$

★ AR(p). If some roots of $1 - \phi_1 z - \ldots - \phi_p z^p = 0$ are complex, then the autocorrelations show cyclical dynamics.

→ $Y_t = c + \phi_1 Y_{t-1} + \ldots \phi_p Y_{t-p} + \varepsilon_t + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q}$

→ $(1 + \phi_1 L - \phi_2 L^2 \ldots \phi_p L^p) Y_t = c + (1 + \theta_1 L + \ldots + \theta_q L^q)\varepsilon_t$

→ $\Phi(L) Y_t = \Theta(L)\varepsilon_t$

C=0 SO WE IGNORE IT

TO HAVE INVERTIBILITY I WANT: $\boxed{Y_t = \sum -\pi_s Y_{t-s} + \varepsilon_t}$

THIS IS $\boxed{\Pi(L) Y_t = \varepsilon_t}$

SO I NEED THAT $\Theta(L)^{-1}$ EXIST

$\Pi(L) = \Phi(L)\Theta(L)^{-1}$

$\Theta(L)^{-1}\Phi(L) Y_t = \varepsilon_t$

$\gamma_2 = E[(Y_t - \mu)(Y_{t-2} - \mu)]$

$= E([\phi(Y_{t-1} - \mu) + \varepsilon_t + \theta\varepsilon_{t-1}](Y_{t-2} - \mu))$

$= E(\phi(Y_{t-1} - \mu)(Y_{t-2} - \mu)) + E(\varepsilon_t(Y_{t-2} - \mu)) + E(\theta\varepsilon_{t-1}(Y_{t-2} - \mu))$

ARMA(1,1) $Y_t = -0.5 Y_t + \varepsilon_t + 0.5\varepsilon_{t-1}$

$Y_t = \varepsilon_t$

AR(1) $Y_{t+1} = \phi Y_t + \varepsilon_{t+1}$

$Y_{t+1|t} = \phi Y_t$

AR(2) is interesting. We have Cyclical dynamics of autocorrelation if square root of z. We have to period before we are under average and higher of average. This is a cycle and that will be interesting in Economics!

ARMA examples → excel file.

We got example of MA(1), AR(1) ecc.

Put -0.5 in the parameters and we get autocorrelation function. If we do - 2 the autocorrelation function doesn't change because as we know invertibility.

There are things that are negatively correlated and then I can put in AR1 example a negative value. If we want more smother dynamic we can have AR2. If I want more dependence I can put 0.4 and 0.3 in the PSI value. As long we change parameter it adapt to various different situation.

Derive impulse refer function IRF.

I want the invert of the polynomial Phi of L. Phi of L ^-1 must be equal to PSI of L. Moltiplicate on the other side and then we will obtain (slide). I we multiply them together we get 1 + Psi1 L + Psi2 L^2 and a regression of these guy.

I only have power of L so I can derive values. I can derive these to this polynomial and using these algorithm. The model is generating a cycle.

ARMA will combine AR and MA models.