GTDMO - DSE - 24th Sep 2019 -(brief)

Linear Algebra.

Vectors and matrices have become an indispensable tool in the last several decades in Economics. Part of this development results from the importance of matrix tools for the statistical component of econometrics; another reason is the increased use of matrix algebra in the economic theory underlying econometric relations. The objective of these lectures is to provide a selective survey of topics in Linear Algebra.

Example 1 (I/O Analysis). In 1973 Wessily Leontiff won the Noble Prize in Economics for his work in input-output analysis. His seminal work allowed for a greater quantification of economic models. Input-output analysis, also called Inter-Industry Analysis, creates an environment where the user can predict the consumption and demand for a system. This system can be as small as a single business or as large as the greater global economy. It is interesting to note that the input-output analysis is robust enough deal with both closed systems, and ones where commodities are flowing into and out the systems (i.e. imports, exports, taxes, etc...). Regardless of the size or composition of the system which is being analyzed the procedure is essentially the same.

Suppose we divide the economy (of a region) into $n = 3$ sectors, we indicate with $\mathbf{x} \in \mathbb{R}^n$ the *production vector*: output of each sector for year. Also let $\mathbf{d} \in \mathbb{R}^n$ be the *demand vector*: value of goods and services demanded from sectors by non-productive part of economy. If the *intermediate demand* are the inputs producers need for production, the Leontief's question: is there a production level such that the total amount produced equals the total demand for production? Or, Is there an $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} =$ intermediate demand $+\mathbf{d}$? We assume that

- hold prices of goods and services constant;
- measure unit of input and output in millions of euro;
- for each sector, there is a unit consumption vector $\mathbf{c}$ listing inputs needed per unit of output of sector.

The intermediate demand values are shown in the Table **??**. For example, if the manufacturing sector produces 100 units, it consumes 50 units from manufacturing, 20 units from agriculture, 10 units from services, and

Purchased from	Manufactoring	Agriculture	Services
Manufactoring	0.50	0.40	0.20
Agriculture	0.20	0.30	0.10
Services	0.10	0.10	0.30

TABLE 1. Inputs Consumed per Unit of Output.

its consumption vector $\mathbf{c}^M$ has components $(0.50,\, 0.20,\, 0.10)^T$: $\mathbf{c}_1^M = 0.50$, $\mathbf{c}_2^M = 0.20$, $\mathbf{c}_3^M = 0.10$. The consumption vectors for the other sectors are, $\mathbf{c}^A = (0.40,\, 0.30,\, 0.10)^T$, $\mathbf{c}^S = (0.20,\, 0.10,\, 0.30)^T$. Suppose sector produces x units of output and has unit consumption vector $\mathbf{c}$, its intermediate demand is $x\mathbf{c}$ (product of a scalar with a vector). Let x_1, x_2, x_3 the units of output produced by the 3 sectors, and suppose the final demand is $d_1 = 50$ units for manufacturing, $d_2 = 30$ units for agriculture, and $d_3 = 20$ units for services. Leontief's question: is there an $\mathbf{x} \in \mathbb{R}^3$ such that

$$\begin{cases} x_1 = 0.5x_1 + 0.4x_2 + 0.2x_3 + 50 \\ x_2 = 0.2x_1 + 0.3x_2 + 0.1x_3 + 30 \\ x_3 = 0.1x_1 + 0.1x_2 + .3x_3 + 20 \end{cases} \Leftrightarrow \begin{cases} 0.5x_1 - 0.4x_2 - 0.2x_3 = 50 \\ -0.2x_1 + 0.7x_2 - 0.1x_3 = 30 \\ -0.1x_1 - 0.1x_2 + 0.7x_3 = 20 \end{cases}$$

We must solve a linear system, but is there a solution? is it unique? how can we calculate it?

We remember that we indicate the vectors in $\mathbb{R}^n$ as a "column of values",

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

and we can also indicate it, when it is convenient, as a row $(x_1, x_2, \ldots, x_n)^T$ using the symbol of transposition $(\ldots)^T$. The heart of linear algebra is in two operations, both with vectors. We add vectors $\mathbf{v}$, $\mathbf{w}$ to get $\mathbf{v} + \mathbf{w}$, we multiply them by numbers c and d to get $c\mathfrak{v}$ and $d\mathbf{w}$. Combining those two operations (adding $c\mathbf{v}$ to $d\mathbf{w}$) gives the **linear combination**, a new vector, $\mathbf{y} = c\mathbf{v} + d\mathbf{w}$. for example, $\mathbf{v} = (1,1)^T$, $\mathbf{w} = (2,3)^T$, $c = 2$, $d = 4$, give the following linear combination,

$$\mathbf{y} = 2\mathbf{v} + 4\mathbf{w} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}.$$

Another example of linear combination with two vectors in three dimensions,

$$\mathbf{u} = 4\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}$$

Remark. Suppose the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are in three-dimensional space, $c, d, e \in \mathbb{R}$,

1) What is the picture of all combinations $c\mathbf{u}$?

2) What is the picture of all combinations $c\mathbf{u} + d\mathbf{v}$?

3) What is the picture of all combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$?

The answers depend on the particular vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. If they were zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors the combinations $c\mathbf{u}$ fill a line. The combinations $c\mathbf{u} + d\mathbf{v}$ could fill a plane, but this is not always the case. In fact if $\mathbf{v} = \alpha\mathbf{u}$ for a certain scalar value α ($\mathbf{v}$ belongs to the straight line generated by $\mathbf{u}$), then $c\mathbf{u} + d\mathbf{v} = c\mathbf{u} + d\alpha\mathbf{u} = (c + d\alpha)\mathbf{u}$ and the combinations of $\mathbf{u}$ and $\mathbf{v}$ fill a straight line. We observe that, in this case, $\mathbf{v} - \alpha\mathbf{u} = \mathbf{0}$ ($\mathbf{0}$ is the zero vector, or null vector), i.e. there is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ that provides the null vector.

The the linear combinations of the three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ typically fill three-dimensional space. If one vector, for example $\mathbf{v}$, belongs to the line generated by a different vector, for example $\mathbf{u}$, but the third vector does not belong to it, the combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill a plane. Finally if two vectors, for example $\mathbf{v}$ and $\mathbf{w}$, are scalar multiple of the remaining vector $\mathbf{u}$ the combinations fill a line. We note that in these last two cases there are scalar values α, β, γ (not all zero), such that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{0}$.

Figure **??** shows two set of vectors, the first two vectors $\mathbf{u}$ and $\mathbf{v}$ are the same in both pictures. If we only look at the combinations of those two vectors, we will get a two-dimensional plane. The key question is whether the third vector is in that plane:

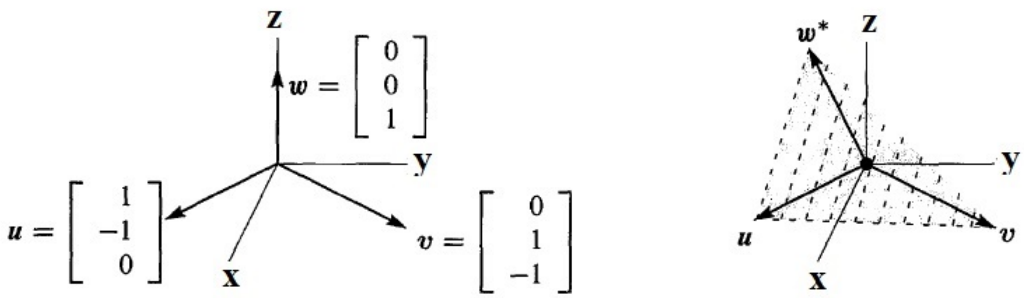

FIGURE 1. Independent vectors **u**, **v**, **w**, dependent vectors **u**, **v**, $\mathbf{w}^*$ in a plane.

Independence **w** is not in the plane of **u** and **v**.
Dependence $\mathbf{w}^*$ is in the plane of **u** and **v**.
In the last case, dependence, $\mathbf{w}^* = c\mathbf{u} + d\mathbf{v}$ for some scalar c and d.
Definition. The not zero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, $m \geq 1$, are said to be **linearly independent** if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

implies that all the scalars $c_1, c_2, \ldots, c_m$ must equal 0.
The vectors **u**, **v**, **w** in Figure **??** are linearly independent.
Definition. The not zero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$, $m \geq 1$, are said to be **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_m$ not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

The vectors **u**, **v**, $\mathbf{w}^*$ in Figure **??** are linearly dependent.
Remark. A set of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others.
Example 2. The vectors $\mathbf{v}_1 = (1,0,0)^T$, $\mathbf{v}_2 = (0,1,0)^T$, $\mathbf{v}_3 = (0,0,1)^T$ are linearly independent, in fact

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Leftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \; c_1 = c_2 = c_3 = 0.$$

Example 3. The vectors $\mathbf{v}_1 = (1,2)^T$, $\mathbf{v}_2 = (2,4)^T$ are linearly dependent, in fact

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \begin{pmatrix} c_1 + 2c_2 \\ 2c_1 + 4c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} c_1 + 2c_2 = 0 \\ 2c_1 + 4c_2 = 0 \end{pmatrix}$$

If we divide the second equation by two we obtain the first equation, then the linear combination is equal to the null vector when $c_1+2c_2 = 0 \Leftrightarrow c_1 = -2c_2$. Choosing, for example, $c_2 = 1$ (then $c_1 = -2$), we have $-2\mathbf{v}_1+\mathbf{v}_2 = \mathbf{0}$. From another point of view it follows that $-2\mathbf{v}_1 = -\mathbf{v}_2 \Rightarrow \mathbf{v}_2 = 2\mathbf{v}_1$.

Linear system.

A linear equation in the variables x_1, x_2, $\ldots$, x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b_1$$

where b and the coefficients $a_1, a_2, \ldots, a_n$ are real numbers, usually known in advance, while the subscript n may be any positive integer. The following equations

$$4x_1 + 5x_2 - x_3 = 0, \;\; 3\sqrt{2}x_1 + 3x_2 = 5$$

are both linear, while the equations

$$4x_1 + 5x_2 = x_1x_2, \;\; 3\sqrt{x_1} - x_2 + x_3 = 7,$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables, say, $x_1, x_2, \ldots, x_n$. An example is

$$\begin{array}{lll} x_1 - 2x_2 & +5x_3 & = 8 \\ x_1 & -4x_3 & = 7 \end{array}$$

A **solution** of the system is a list $s_1, s_2, \ldots, s_n$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively. For instance, $(11, 4, 1)$ is a solution of system of the last example because, when these values are substituted for x_1, x_2, x_3, respectively, the equations simplify to $8 = 8$ and $7 = 7$. The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Example 4. Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is (now $x = x_1$, $y = x_2$)

$$\begin{aligned} x \quad & -2y = 1 \\ 3x \quad & +2y = 11 \end{aligned}$$

We begin a row at a time. The first equation $x - 2y = 1$ produces a straight line in the xy plane. The second line in this "row picture" comes from the second equation $3x + 2y = 11$ (see Figure **??**). The intersection point where

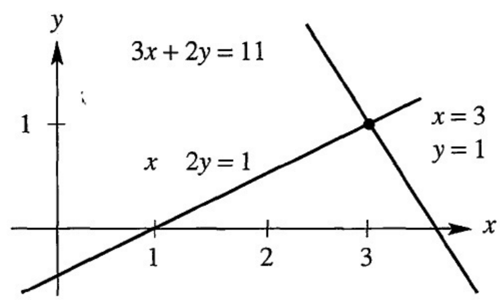

FIGURE 2. Row picture: The point $(3, 1)$ where the lines meet is the solution.

the two lines meet, the point $(3, 1)$ lies on both lines. That point solves both equations at once. This is the solution to our system of linear equations.

Turn now to the column picture. I want to recognize the same linear system as a "vector equation". Instead of numbers we need to see vectors. If you separate the original system into its columns instead of its rows, you get a vector equation:

$$x \begin{pmatrix} 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix} = \mathbf{b}.$$

The problem is to find the combination of those vectors that equals the vector on the right. We are multiplying the first column by x and the second column by y, and adding. With the right choices $x = 3$ and $y = 1$ (the same numbers as before), this produces 3(**column 1**) + 1 (**column 2**) = **b** (see Figure **??**).

The essential information of a linear system can be recorded compactly in

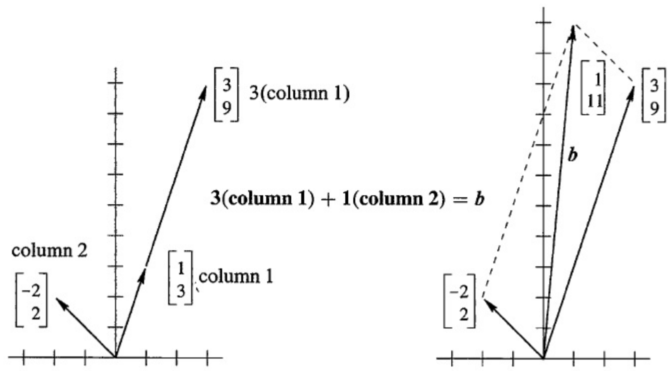

FIGURE 3. Column picture: A combination of columns produces the right side.

a rectangular array called a **matrix**. Given the system of the Example 4. above, the matrix

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

is called the **coefficient matrix** (or matrix of coefficients) of the system, and

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$$

is called the **augmented matrix** of the system. An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the right sides of the equations. The size of a matrix tells how many rows and columns it has. The augmented matrix above has 2 rows and 3 columns and is called a 2×3 (read 3 by 4) matrix (the number of rows always comes first).

Let A the coefficient matrix, we can write the linear system of the Example 4 as

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$$

What does it mean? We observe that the term on the left of the first equation, $(x - 2y)$, is equal to the dot product between the first row of A, $(1, -2)$, and the vector of the unknowns (x, y). Again, the term on the left of the second equation, $(3x + 2y)$, is equal to the dot product between the second row of A, $(3, 2)$, and the vector of the unknowns (x, y). For general linear

system, for each equation we have to compute the dot product between the corresponding row of the coefficient matrix and the vector of the unknowns (the dot product can be calculated because the number of columns of the coefficient matrix is equal to the number of unknowns).
If A is an $n \times m$ nmatrix, that is a matrix with n rows and m columns, then the scalar entry in the $i-$th row and $j-$th column of A is denoted by a_{ij} and is called the (i, j) entry of A. For instance, the $(3, 2)$ entry is the number a_{32} in the third row, second column. Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^n$. Often, these columns are denoted by $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m$ and the matrix A is written as $A = [\mathbf{c}_1\, \mathbf{c}_2\, \ldots\, \mathbf{c}_m]$.
The following general fact is fundamental for linear systems.

A system of linear equations has
1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.
Example 5. We consider the following 2×2 linear systems (the first equa-

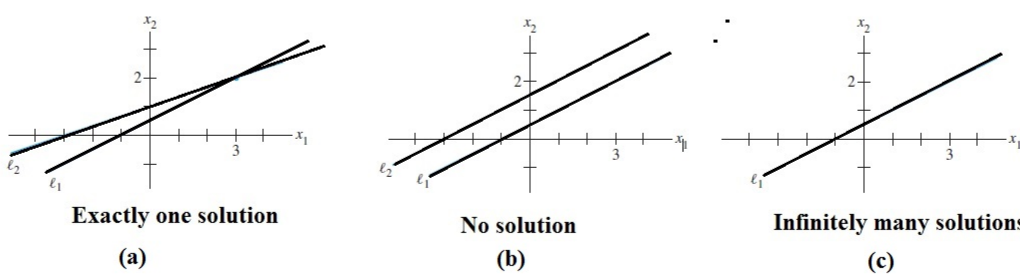

FIGURE 4. Possible cases for a linear system.

tion is the same for all three systems, the second one changes),

$$(a) \begin{cases} x_1 & -2x_2 = -1 \\ -x_1 & +3x_2 = 3 \end{cases} \qquad (b) \begin{cases} x_1 & -2x_2 = -1 \\ -x_1 & +2x_2 = 3 \end{cases} \qquad (c) \begin{cases} x_1 & -2x_2 = -1 \\ -x_1 & +2x_2 = 1 \end{cases}$$

For the system (a), adding the two equations we obtained $x_2 = 2$, then $x_1 = 3$ and we have exactly one solution. Proceeding in the same way for the system (b) we have $0 = 2$ which is not possible: the system has no solution. Finally, for the system (c) we observe that the first equation can be obtained from the second one by multiplying it by -1, so we have only one linear constraint: the system has infinitely many solutions. In Figure **??** we show the geometric interpretation for the set of solutions of the three systems. Systems (a) and (c) are consistent while system (b) is inconsistent.

Solving a Linear System I. We intend to describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is to replace one system with an equivalent system (i.e., one with the same solution set) that is easier to solve. Roughly speaking, use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations. Three basic operations are used to simplify a linear system: replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. These three operations do not change the solution set of the system.

Example 6. The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison,

$$\left\{\begin{array}{llll} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}\right. \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

Keep x_1 in the first equation, $x_1 = 2x_2 - x_3$, and eliminate it from the other equations,

$$\left\{\begin{array}{llll} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ & 10x_2 & -10x_3 & = 10 \end{array}\right. \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now, use the x_2 in equation 2 to eliminate the $10x_2$ in equation 3. The result of this calculation is written in place of the previous third equation (row):

$$\left\{\begin{array}{llll} x_1 & -2x_2 & +x_3 & =0 \\ & 2x_2 & -8x_3 & =8 \\ & & 30x_3 & =-30 \end{array}\right. \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

The final system has a “triangular form”. Now, from the last equation, $x_3 = -1$, move back to the x_2 in equation 2: $x_2 = 4 + 4x_3 \Rightarrow x_2 = 0$. Finally, using the computed values of x_2 and x_3 and the first equation we obtain: $x_1 = 2x_2 - x_3 \Rightarrow x_1 = 1$. The work is essentially done and it shows that the only solution of the original system is $(1, 0, -1)$. However it is a good practice to check the work and that $(1, 0, -1)$ is a solution: substitute these values into the left side of the original system, and verify that the results agree with the right side of the original system.

Solving a Linear System II. Example 6 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations correspond to the following operations on the augmented matrix,

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Two matrices are called row equivalent if there is a sequence of elementary row operations that transforms one matrix into the other. It is important to note that row operations are reversible. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement:

`If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.`

Example 6 rev. The augmented matrix $\bar{A}$ of the system of the Example

6 is the following

$$\bar{A}\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

In the second equation the unknown x_1 does not appear, $\bar{a}_{21} = 0$ (as if it had already been eliminated) while in the third equation we have the term $5x_1$. The elimination step is equivalent to the search for a suitable linear combination between [*equation* 1] and [*equation* 3] to replace instead of [*equation* 3] in such a way that the coefficient of x_1 becomes equal to zero:

$$[equation\,3]+K[equation\,1] \Rightarrow (5+K)x_1+(0-2K)x_2+(-5+K)x_3 = (10+0K)$$

where K is a suitable constant. For $K = -5$ we have the new [*equation* 3] is $0x_1 + 10x_2 - 10x_3 = 10$:

$$\begin{array}{ll} -5\,[equation\ 1] & -5x_1 + 10x_2 - 5x_3 = 0 \\ +\,[equation\ 3] & 5x_1 \qquad\qquad - 5x_3 = 10 \\ ------------ & -------------- \\ new\,[equation\,3] & 0x_1 + 10x_2 - 10x_3 = 10 \end{array}$$

The equivalent augmented matrix becomes

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

Now we want to obtain a new augmented matrix with null $(3, 2)$-entry (i.e. we eliminate the variable x_2) using a suitable linear combination between the third and the second equation (second and third row of the new augmented matrix). Then, we consider the following combination

$$[equation\,3] + K[equation\,2] \Rightarrow (10 + 2K)x_2 + (-10 - 8K)x_3 = (10 + 8K)$$

where we have indicated again with K the constant to be determined. For $K = -5$ the new third equation becomes $0x_2 + 30x_3 = -30$,

$$\begin{array}{ll} -5\,[equation\ 2] & -10x_2 + 40x_3 = -40 \\ +\,[equation\ 3] & 10x_2 - 10x_3 = 10 \\ ------------ & ---------- \\ new\,[equation\,3] & 30x_3 = -30 \end{array}$$

The final augmented matrix is the following,

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

Remark. Until now we have not used the interchange of two rows because it was not necessary. Now consider the linear system,

$$\begin{array}{llll} & x_2 & -2x_3 & = -3 \\ x_1 & +x_2 & +x_3 & = 2 \\ 2x_1 & -x_2 & -2x_3 & = 3 \end{array}$$

To obtain an x_1 in the first equation, interchange rows 1 and 2 (or rows 1 and 3),

$$\begin{array}{llll} x_1 & +x_2 & +x_3 & = 2 \\ & x_2 & -2x_3 & = -3 \\ 2x_1 & -x_2 & -2x_3 & = 3 \end{array}$$

and proceed with the elimination algorithm. Writing only the augmented matrices and schematizing the linear combinations between rows (equations) as $R_i + KR_j \to R_i$ (the i-th row is updated by adding to itself the j-th row multiplied by the constant K), we obtain

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 2 & -1 & -2 & 3 \end{bmatrix} \xrightarrow{R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -4 & -1 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -4 & -1 \end{bmatrix} \xrightarrow{R_3 + 32R_2 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -10 & -10 \end{bmatrix}$$

From the third equation we obtain $x_3 = 1$ and proceeding with the back-Substitution: $x_2 = -1$, $x_1 = 2$.