GTDMO – 2019/20

Linear system -
Echelon form and applications (II)

Based on Linear Algebra and Its Applications, David C. Lay, Steven R. Lay, and Judi J. McDonald, PEARSON 5^th ed.

Up to now…

- A matrix is simply a rectangular array of numbers.

- Matrices are used to organize information into categories that correspond to the rows and columns of the matrix.

Example (equilibrium in a Market)

Let us consider **an isolated market for good "i"** . Suppose that both the demand (q_i^d) and supply (q_i^s) of this good is a function of its price (p_i), only, then

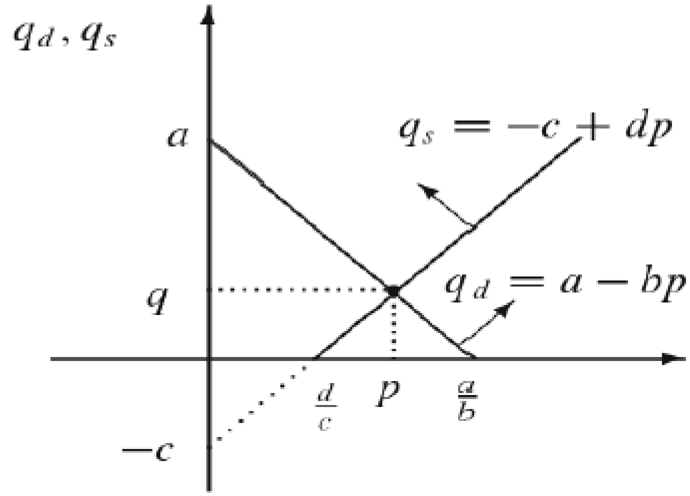
$$q_i^d = \alpha_0 - \alpha_1 p_i$$

$$q_i^s = - \beta_0 + \beta_1 p_i$$

These functions assume that there is a linear relation between the quantity demanded (or supplied) and the price of the good i. **All of the coefficients in and are assumed to be positive.**

The market for good i is said to be in **equilibrium** when the demand for good i is equal to its supply, i.e. $\mathbf{q_i^d = q_i^s}$ Very esay linear system...

$$\hat{p}_i = \frac{\alpha_0 + \beta_0}{\alpha_1 + \beta_1} \qquad \hat{q}_i = \frac{\alpha_0 \beta_1 - \alpha_1 \beta_0}{\alpha_1 + \beta_1}$$



The market equilibrium

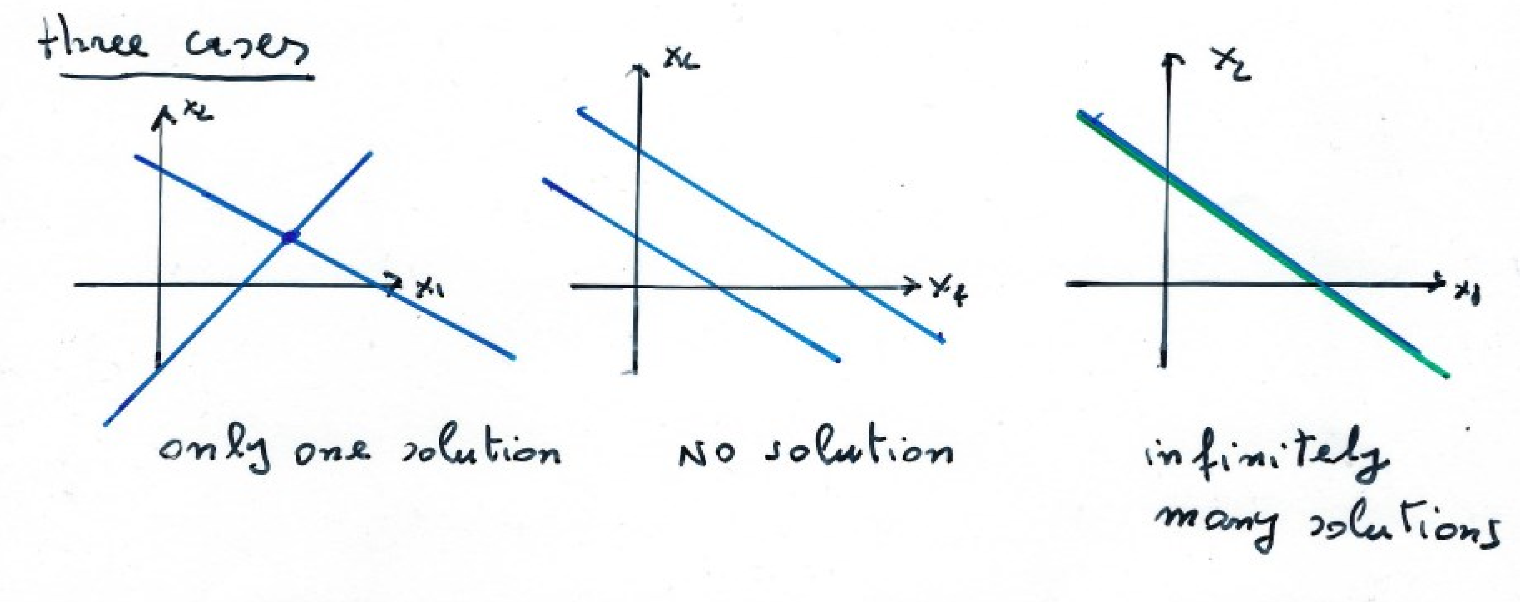
Two equations of the type $\boldsymbol{q} = \boldsymbol{\alpha} + \boldsymbol{\beta p}$ with α and β real constants. Geometrically, the two equations represent the equations of two straight lines in the plane (p, q).

Equilibrium point = solution of the linear system = point in the plane where the lines meet

The row picture

Let examine geometrically a 2x2 linear system

$$\begin{cases} a_{11}\, x_1 + a_{12}\ x_2 = b_1 \\ a_{21}\, x_1 + a_{22}\ x_2 = b_2 \end{cases}$$



more complex (apparently)

Consider an economy with two goods. Suppose that the supply and demand of each good are functions of its own price as well as the price of other good. Such a system can be represented by the following four equations:

$$q_1^d = \alpha_{01} + \alpha_{11} p_1 + \alpha_{12} p_2$$

$$q_1^s = \beta_{01} + \beta_{11} p_1 + \beta_{12} p_2$$

$$q_2^d = \alpha_{02} + \alpha_{12} p_1 + \alpha_{22} p_2$$

$$q_2^s = \beta_{02} + \beta_{12} p_1 + \beta_{22} p_2$$

For such economy, multi-market equilibrium is the quadruple $(\hat{p}_1, \hat{p}_2, \hat{q}_1, \hat{q}_2)$ at which

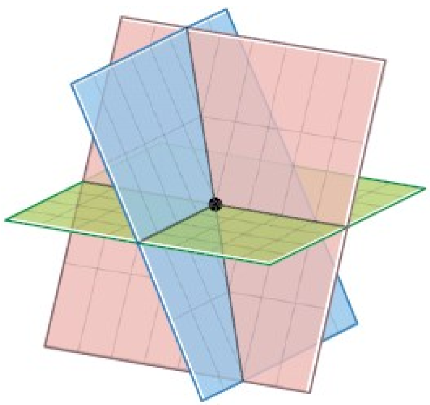
$$q_1^d(\hat{p}_1, \hat{p}_2) - q_1^s(\hat{p}_1, \hat{p}_2) = 0$$

$$q_2^d(\hat{p}_1, \hat{p}_2) - q_2^s(\hat{p}_1, \hat{p}_2) = 0$$

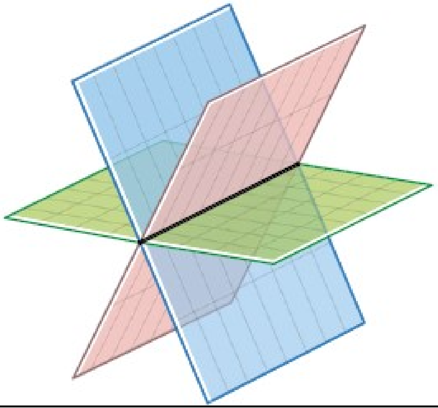
The row picture

$$\begin{cases} a_{11}\,x_1 + a_{12}\;\;x_2 \;\;+\;\; a_{13}\,x_3 \;=\;\; b_1 \\ a_{21}\,x_1 + a_{22}\;\;x_2 \;\;+\;\; a_{23}\,x_3 \;=\;\; b_2 \\ a_{31}\,x_1 + a_{32}\;\;x_2 \;\;+\;\; a_{33}\,x_3 \;=\;\; b_3 \end{cases}$$

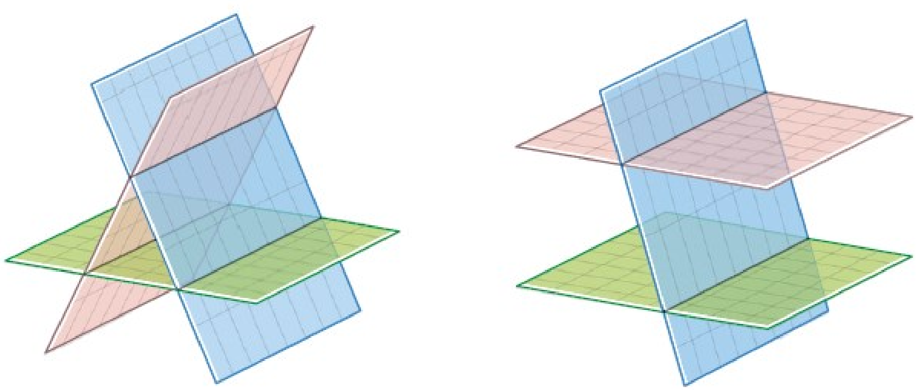
Each row: equation of a plane in $\mathbb{R}^3$



Three planes intersecting in a point: only one solution



Three planes intersecting in a line: infinitely many solutions



Three planes with no intersection: no solution

- From a linear system to a matrix

Linear system		Augmented matrix
$2x - y = 5$	Equation 1	$\begin{bmatrix} 2 & -1 & 5 \end{bmatrix}$
$x + 4y = 7$	Equation 2	$\begin{bmatrix} 1 & 4 & 7 \end{bmatrix}$

$$\begin{cases} 2x - y = 5 \\ x + 4y = 7 \end{cases} \qquad \begin{array}{c} \begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & 7 \end{bmatrix} \\ \begin{array}{ccc} x & y & \end{array} \end{array}$$

  - This matrix is called the augmented matrix of the system.
  - The augmented matrix contains the same information as the system, but in a simpler form.
  - The operations we learned for solving systems of equations can now be performed on the augmented matrix.

Gaussian Elimination

In general, to solve a system of linear equations using its augmented matrix, we use elementary row operations (**that provide an equivalent system**) to arrive at a matrix in a certain form: row-echelon form**.**

Elementary row operations:

1. Add a multiple of one row to another.
2. Multiply a row by a nonzero constant.
3. Interchange two rows.

Elementary Row Operations—Notation

It is possible to use the following notation to describe the elementary row operations:

Symbol	Description
$R_i + kR_j \rightarrow R_i$	Change the *i*th row by adding *k* times row *j* to it. Then, put the result back in row *i*.
kR_i	Multiply the *i*th row by *k*.
$R_i \leftrightarrow R_j$	Interchange the *i*th and *j*th rows.

Example $(x=x_1,\ y=x_2,\ z=x_3)$

System	Augmented Matrix
$\begin{cases} x - y + 3z = 4 \\ x + 2y - 2z = 10 \\ 3x - y + 5z = 14 \end{cases}$	$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 1 & 2 & -2 & 10 \\ 3 & -1 & 5 & 14 \end{bmatrix}$
$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$	$\xrightarrow[R_3-3R_1\to R_3]{R_2-R_1\to R_2} \begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$

Elementary Row Operations and Linear System

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ 2y - 4z = 2 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 2 & -4 & 2 \end{bmatrix}$$

$R_3 - (2/3)\,R_2 \longrightarrow R_3$

$$\begin{cases} x - y + 3z = 4 \\ 3y - 5z = 6 \\ -\frac{2}{3}z = -2 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 3 & 4 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & -\dfrac{2}{3} & -2 \end{bmatrix}$$

Now, we use back-substitution to find that:

(last equation) z=3, so the second equation becomes

3y-15=6 $\Rightarrow$ y=7; finally, from the first equation

x- 7 +9 =4 $\Rightarrow$ x= 2. The solution is (2, 7, 3).

Gaussian Elimination

To solve a system we use:

1. Augmented matrix
2. Row-echelon form
3. Back-substitution

This technique is called Gaussian elimination, in honor of its inventor, the German mathematician C. F. Gauss.

ROW REDUCTION ALGORITHM

- **STEP 1:** Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
- **STEP 2:** Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
- **STEP 3:** Use row replacement operations (add a multiple of one row to another) to create zeros in all positions below the pivot.
- **STEP 4:** Cover the row containing the pivot position, and cover all rows, if any, above it. Apply steps **1–3** to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

SOLUTIONS OF LINEAR SYSTEMS

The system is consistent when the solution set is non empty otherwise the system is inconsistent (the solution set is empty).

The variables x_i corresponding to pivot columns in the echelon form matrix are called **basic variables**. The other variable is called a **free variable**.

Example. Suppose that the augmented matrix of a 3x3 linear system has been changed into the equivalent echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -3 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 free variable

x_1 basic variable

x_2 basic variable

SOLUTIONS OF LINEAR SYSTEMS

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \; \ldots \; 0 \;\; b] \text{ with } b \text{ nonzero.}$$

- Whenever a system is consistent the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables.
- This operation is possible because the coefficient of each basic variable in echelon form is non zero.
- The statement "x_i is free" means that you are free to choose any value for x_i. Each different choice of x_i determines a (different) solution of the system, and every solution of the system is determined by a choice of x_i.
- The *parametric description* of solutions sets is a description of the solution set in which the free variables act as parameters.
- Whenever a system is consistent and has free variables, the solution set has many parametric descriptions.
- If a linear system is consistent the solution is unique when there are no free variables.

Example (cont.) Suppose that the augmented matrix of a 3x3 linear system has been changed into the equivalent echelon form:

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -3 & 7 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, x_3 is free, then

$$\begin{aligned} 2x_1 + x_2 &= 4 - 3x_3 \\ -3x_2 &= 2 - 7x_3 \end{aligned} \quad \Rightarrow \quad x_2 = -2/3 + 7/3\, x_3$$

$$\Rightarrow \quad x_1 = 7/3 - 8/3\, x_3$$

Linear Independence and Dependence of Vectors

Given any set of m (not null) vectors $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$ (with the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \cdots + c_m\mathbf{V}_m$$

where $c_1, c_2, \dots, c_m$ are any scalars. Now consider the (vector) equation

$$c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + \cdots + c_m\mathbf{V}_m = \mathbf{0}$$

Clearly, this vector equation holds if we choose all c_j's zero, because then it becomes **0** = **0**.
If this is the **only m-tuple** of scalars for which this equation holds, then our vectors $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$ are said to form a **linearly independent** set or, more briefly, we call them linearly independent.
Otherwise, if the equation **also holds with scalars not all zero**, we call these vectors **linearly dependent**.
This means that we can express at least one of the vectors as a linear combination of the other vectors. For instance, if c1 ≠ 0, we can solve for $\mathbf{V}_1$

$$\mathbf{V}_1 = k_2\,\mathbf{V}_2 + \dots + k_m\,\mathbf{V}_m \qquad \text{where } k_j = -c_j / c_1$$

EXAMPLE 1

$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ are linearly dependent

$V_2 = 2V_1 \Rightarrow 2V_1 - V_2 = 0 \qquad \alpha_1 = 2,\ \alpha_2 = -1$ (not all zeros)

$v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$; $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent

$$\alpha_1 v_1 + \alpha_2 v_2 = 0 \iff \begin{pmatrix} 3\alpha_1 + \alpha_2 \\ \alpha_1 + 2\alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \alpha_1 = \alpha_2 = 0$$

Definition. Let $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$ be vectors (with the same number of components), . We will refer to span($\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$) as the set of all linear combination of $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$.
span($\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_m$) is a subspace of the space to which the vectors belong.

Definition. A set of n (not null) linearly independent vectors $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n$ in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$,

$$span(\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_n) = \mathbb{R}^n$$

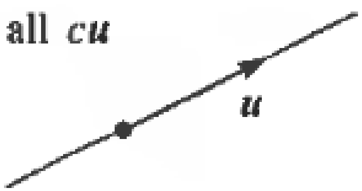
Example. In $\mathbb{R}^3$ with linear independent vectors:

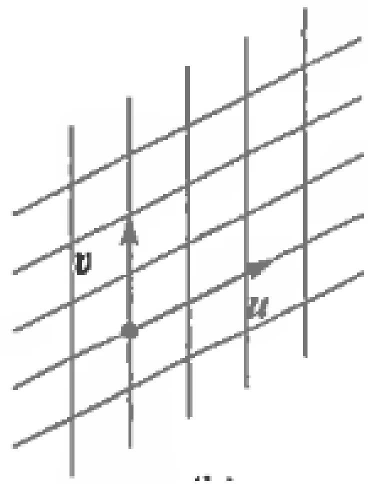
The combinations cu fill a ***line***.

The combinations $cu + dv$ fill a ***plane***.

The combinations $cu + dv + ew$ fill ***three-dimensional space***.

Line from all cu





Plane from all $cu + dv$

How to proceed using the echelon form? Let, $\mathbf{V}_i \in \mathbb{R}^n \Rightarrow \mathbf{V}_i = \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{in} \end{pmatrix}$

Linear combination equal to the null vector:

$$c_1 \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{pmatrix} + c_2 \begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{pmatrix} + \ldots c_m \begin{pmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

corresponds to the following homogeneous linear system

$$\begin{aligned} v_{11}c_1 + v_{21}c_2 + \ldots + v_{m1}c_m &= 0 \\ v_{12}c_1 + v_{22}c_2 + \ldots + v_{m2}c_m &= 0 \\ \ldots \quad \ldots \quad \ldots & \\ v_{1n}c_1 + v_{2n}c_2 + \ldots + v_{mn}c_m &= 0 \end{aligned}$$

Then we consider **only** the **coefficients matrix** because the **known term** is equal to the **null.**
The i-th column of the coefficient matrix is equal to the vector V_i.
Now we Compute its echelon form.

Example.

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{V}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{V}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{V}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Compute the echelon form:

if if there is at least one zero row (all entries are equal to zero) **LINEAR DEPENDENCE**

if all the lrows are non-zero rows (at least one non-zero entry) **LINEAR INDEPENDENCE**

Example (cont.)

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The vectors are linearly independent

Example

$$\mathbf{V}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{V}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{V}_4 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -4 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

Zero row

The vectors
are linearly dependent

Definition. The **rank** of a matrix A, denoted by rank(A), is the number of linearly independent columns.

Remark. it is possible to prove that the number of linearly independent columns is equal to the number of linearly independent rows

Remark. It is possible to prove that the number of non-zero rows in the echelon form of a matrix is equal to the rank of the matrix.

Example Determine the rank of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. B= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

three rows not zero, then rank(A) = 3

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Two rows not zero, one zero row then rank(B) = 2

For a matrix A: $\begin{Bmatrix} \text{number of} \\ \text{pivot columns} \end{Bmatrix} + \begin{Bmatrix} \text{number of} \\ \text{nonpivot columns} \end{Bmatrix} = \begin{Bmatrix} \text{number of} \\ \text{columns} \end{Bmatrix}$

and the number of the pivot columns is equal to the rank(A).

Fundamental Theorem for Linear Systems

(a) Existence.
A linear system of m equations in n unknowns $x_1, \ldots, x_n$ *is* **consistent**, *that is, has solutions, if and only if the coefficient matrix* $\mathbf{A}$ *and the augmented matrix* $\tilde{\mathbf{A}}$ *have the same rank.*

(b) Uniqueness.
The system has precisely one solution if and only if this common rank r of $\mathbf{A}$ *and* $\tilde{\mathbf{A}}$ *equals n.*

(c) Infinitely many solutions.
If this common rank r is less than n, the system has infinitely many solutions. All of these solutions are obtained by determining r suitable unknowns (whose submatrix of coefficients must have rank r) in terms of the remaining $n - r$ *unknowns, to which arbitrary values can be assigned (free variables)*

The Column picture

Let us consider the linear system

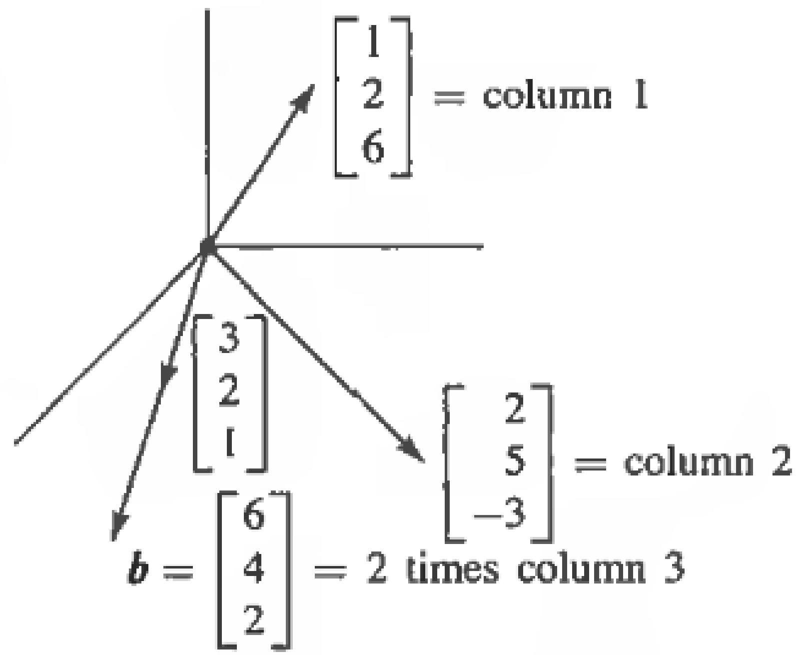
$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{array}$$

The column picture combines three columns to produce the vector (6, 4, 2). Then, the column picture start with the vector form of the equations:

$$x\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The Column picture

The unknown numbers x, y, z are the coefficients in this linear combination. We want to multiply the three column vectors by the correct numbers x. y, z to produce **b** = (6, 4, 2).



Correct combination $0\begin{bmatrix}1\\2\\6\end{bmatrix}+0\begin{bmatrix}2\\5\\-3\end{bmatrix}+2\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}6\\4\\2\end{bmatrix}.$

The Column picture. *Example (Diet)*

The formula for the Cambridge Diet (a popular diet in the 1980s) was based on years of research. A team of scientists headed by Dr. Alan H. Howard developed this diet after more than eight years of clinical work with obese patients. **The very low-calorie powdered formula diet combines a precise balance of carbohydrate, high-quality protein, and fat, together with vitamins, minerals, trace elements, and electrolytes.**
To achieve the desired amounts and proportions of nutrients, Dr. Howard had to incorporate a large variety of foodstuffs in the diet. Each foodstuff supplied several of the required ingredients, but not in the correct proportions. For instance, nonfat milk was a major source of protein but contained too much calcium. So soy flour was used for part of the protein because soy flour contains little calcium. However, soy flour contains proportionally too much fat, so whey was added since it supplies less fat in relation to calcium. Unfortunately, whey contains too much carbohydrate …
The following example illustrates the problem on a small scale. Listed in Table are three of the ingredients in the diet, together with the amounts of certain nutrients supplied by 100 grams (g) of each ingredient.

Amounts (g) Supplied per 100 g of Ingredient				Amounts (g) Supplied by Cambridge Diet in One Day
Nutrient	**Nonfat milk**	**Soy flour**	**Whey**	
Protein	36	51	13	33
Carbohydrate	52	34	74	45
Fat	0	7	1.1	3

If possible, find some combination of nonfat milk, soy flour, and whey to provide the exact amounts of protein, carbohydrate, and fat supplied by the diet in one day.

Let x_1, x_2, and x_3, respectively, denote the number of units (100 g) of these foodstuffs. The product

$$(x_1 \text{ units of nonfat milk}) \times (\text{protein per unit of nonfat milk})$$

gives the amount of protein supplied by x_1 units of nonfat milk. To this amount, we
would then add similar products for soy flour and whey and set the resulting sum equal to the amount of protein we need. Analogous calculations would have to be made for each nutrient.

$$
\begin{aligned}
36\,x_1 + 51\,x_2 + 13\,x_3 &= 33 \\
52\,x_1 + 34\,x_2 + 74\,x_3 &= 45 \\
7\,x_2 + 1.1\,x_3 &= 3
\end{aligned}
$$

A different method, and one that is conceptually simpler, is to consider a "nutrient vector" for each foodstuff and build just one vector equation. The amount
of nutrients supplied by x_1 units of nonfat milk is the

(Scalar x_1 units of nonfat milk) (Vector nutrients per unit of nonfat milk)=

$$x_1 \qquad\qquad\qquad\qquad \mathbf{V}_1$$

Let $\mathbf{V}_2$ and $\mathbf{V}_3$ be the corresponding vectors for soy flour and whey, respectively, and let **b** be the vector that lists the total nutrients required (the last column of the table). Then $x_2 \mathbf{V}_2$ and $x_3 \mathbf{V}_3$ give the nutrients supplied by x_2 units of soy flour and x_2 units of whey, respectively. So the equation is

$$x_1 \mathbf{V}_1 + x_2 \mathbf{V}_2 + x_3 \mathbf{V}_3 = \mathbf{b}$$

INNER PRODUCT

Geometric concepts of length, distance, and perpendicularity, which are well known for $\mathbf{R}^2$ and $\mathbf{R}^3$, are defined here for $\mathbf{R}^n$. These concepts provide powerful geometric tools for solving many applied problems.
All three notions are defined in terms of the inner product of two vectors.

If **u** and **v** are vectors in $\mathbf{R}^n$, the number <**u, v**> **= u • v**

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

is called the **inner product** of **u** and **v**. This inner product, is also referred to as a **dot product**.

Example. Compute $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$.

$\mathbf{u} \bullet \mathbf{v} =$ 2x3 + (-5)x2 + (-1) x (-3) = -1

$\mathbf{v} \bullet \mathbf{u} =$ 3x2 + 2 x(-5) + (-3) x (-1) = -1

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

b. $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

c. $(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$

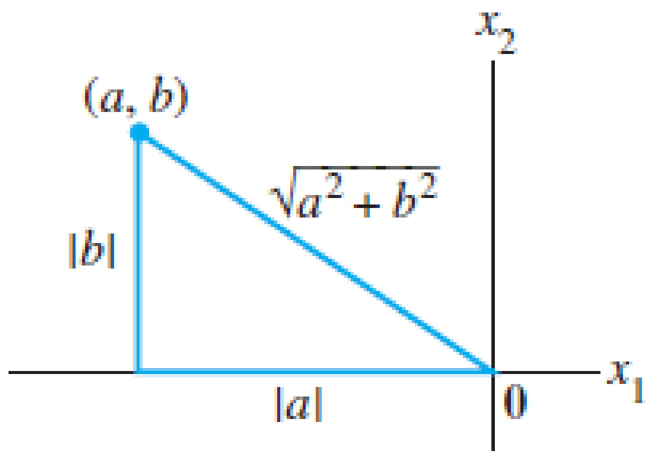
d. $\mathbf{u}\cdot\mathbf{u} \geq 0$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

The Length of a Vector

The **length** (or **norm**) of **v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$$

Example. In the space $\mathbf{R}^2$ If we identify **v** with a geometric point in the plane, as usual, then || **v** || coincides with the standard notion of the length of the line segment from the origin to **v** =(a,b).
This follows from the Pythagorean Theorem applied to a triangle such as the one shown in the following figure.



For any scalar c, the length of $c\mathbf{v}$ is |c| times the length of v. That is,
|| c **v** || = |c| || **v** ||

Distance

For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, written as dist$(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Example. Compute the distance between the vectors **u** = (7,1) and **v**=(3,2).

$$\mathbf{u} - \mathbf{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

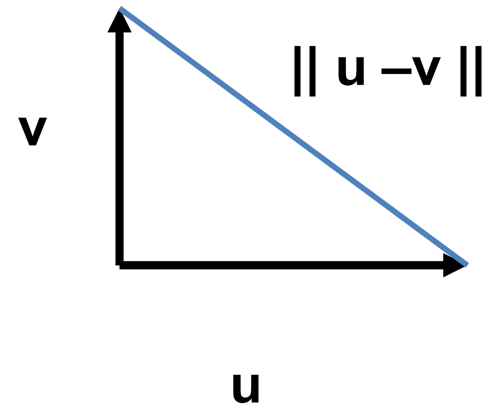
$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

Example. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned} \text{dist}(\mathbf{u}, \mathbf{v}) &= \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} \\ &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2} \end{aligned}$$

Orthogonal Vectors

Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u}\cdot\mathbf{v} = 0$.



Remark.

$$
\begin{aligned}
[\,\mathrm{dist}\,(\mathbf{u}, -\mathbf{v})\,]^2 &= \|\mathbf{u} - (-\mathbf{v})\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 \\
&= (\mathbf{u} + \mathbf{v})\cdot(\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u}\cdot(\mathbf{u} + \mathbf{v}) + \mathbf{v}\cdot(\mathbf{u} + \mathbf{v}) \\
&= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\
&= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}
\end{aligned}
$$

Then, if **u** and **v** are orthogonal $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$

THE MATRIX FORM OF THE EQUATIONS

$\boxed{Ax} = b$ $\quad\quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}$

→ Ax comes from dot (inner) product, each row times the column x

$$Ax = \begin{bmatrix} (\text{row } 1) \cdot x \\ (\text{row } 2) \cdot x \\ (\text{row } 3) \cdot x \end{bmatrix}$$

$$Ax = x_1 (\text{column } 1) + x_2 (\text{column } 2) + x_3 (\text{column } 3)$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned} \Rightarrow \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow A\mathbf{x} = \mathbf{b}$$

And A**x** ≤ **b** means:

$$\begin{array}{lllll} a_{11}x_1 & +a_{12}x_2+ & \ldots & +a_{1n}x_n & \leq b_1 \\ a_{21}x_1 & +a_{22}x_2+ & \ldots & +a_{2n}x_n & \leq b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2+ & \ldots & +a_{mn}x_n & \leq b_m \end{array}$$

An example of linear programming model

Let: $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ n decision variables

Z = linear objective function

Requirement: Maximization of the linear function Z

$$Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots\ldots + c_nx_n = \mathbf{c} \bullet \mathbf{x} \quad \text{(dot product)}$$

subject to the following constraints ($A\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$)

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_n \\
\text{all } x_j &\ge 0
\end{aligned}$$

Models Without Unique Optimal (max) Solutions:

Infeasibility: Occurs when a model has no feasible point.

Unboundness: Occurs when the objective can become infinitely large

No Unique solution:

Alternate solution: Occurs when more than one point optimizes the objective function

If the feasibility region is not empty we can apply the KKT Theorem (necessary and sufficient condition because Z is convex)

Extreme point (or Simplex filter) Theorem:

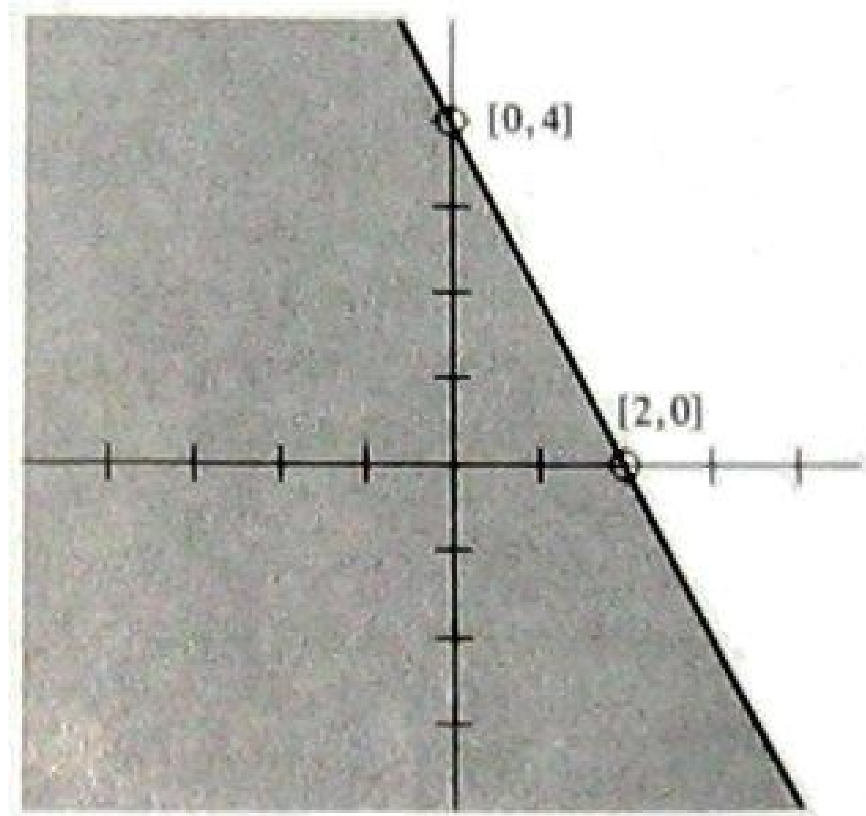
If the maximum or minimum value of a linear function defined over a polygonal convex region exists, then it is to be found at the boundary of the region.

Region of Feasibility

- Graphical region describing all feasible solutions to a linear programming problem
- In 2-space: polygon, each edge a constraint
- In 3-space: polyhedron, each face a constraint

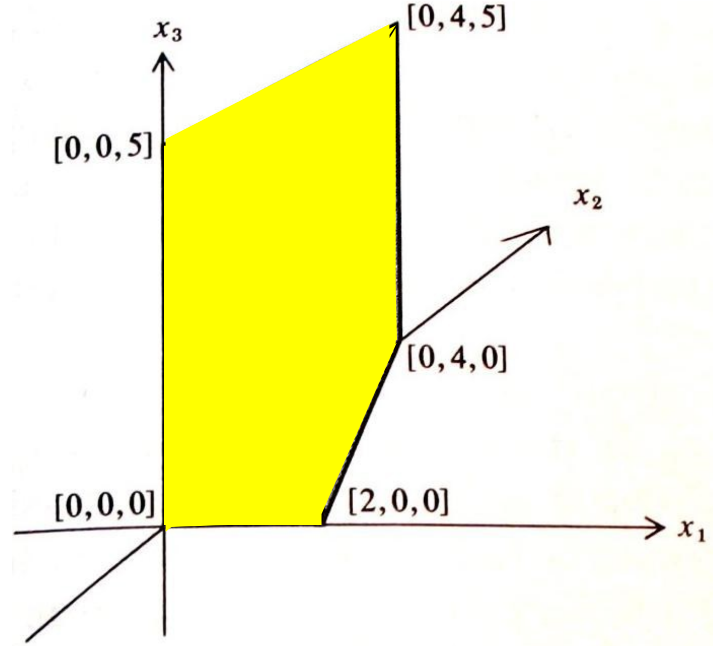
Feasibility in 2-Space

- $2x_1 + x_2 \leq 4$
- In an LP environment, restrict to Quadrant I since $x_1, x_2 \geq 0$



Feasibility in 3-Space

- maximize $3x_1 + 2x_2 + 5x_3$
  subject to
  ≥ 0



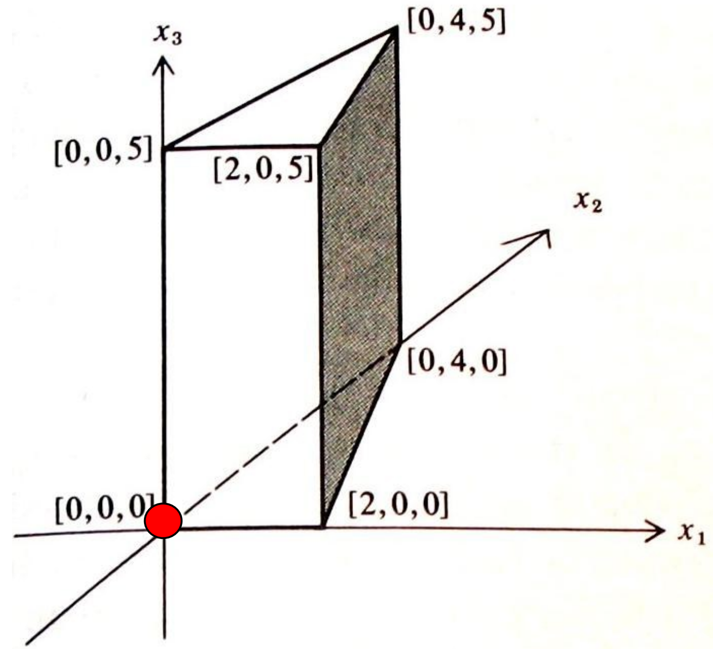
- Five total constraints; therefore 5 faces to the polyhedron

Simplex Method

- Every time a new dictionary is generated:
  - Simplex moves from one vertex to another vertex along an edge of polyhedron
  - Analogous to increasing value of a non-basic variable until bounded by basic constraint
  - Each such point is a feasible solution

Simplex Illustrated: Initial Dictionary

$$
\begin{aligned}
x_4 &= 4 - 2x_1 - \; x_2 \\
&= 5 - \qquad\qquad\quad x_3 \\
\hline
z &= \qquad 3x_1 + 2x_2 + 5
\end{aligned}
$$



Current solution:

$x_1 = 0$
$x_2 = 0$
$x_3 = 0$

Simplex Illustrated: First Pivot

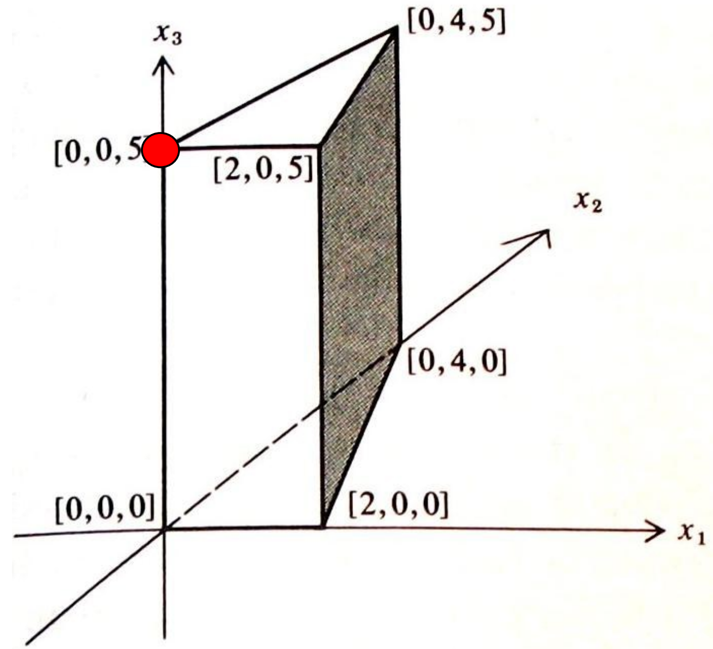
$$\quad = 4 - 2x_1 - x_2$$

$$x_3 = 5 - x_5$$

$$z = 25 + 3\quad + 2x_2 - 5x_5$$

Current solution:

$x_1 = 0$
$x_2 = 0$
$x_3 = 5$

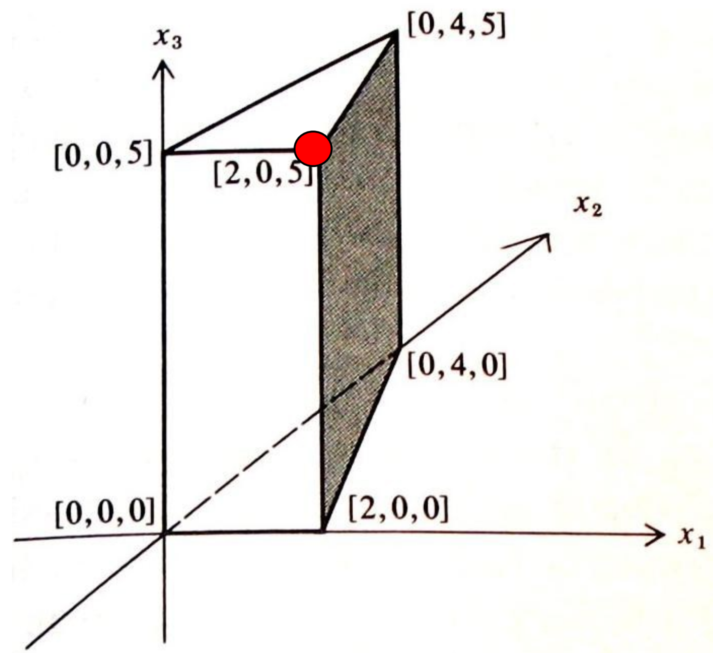


Simplex Illustrated: Second Pivot

$$
\begin{aligned}
\quad\;\; &= 2 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_4 \\
x_3 &= 5 - x_5 \\
\hline
z &= 31 + \tfrac{1}{2}\quad\;\; - \tfrac{3}{2}x_4 - 5x_5
\end{aligned}
$$

Current solution:

$x_1 = 2$
$x_2 = 0$
$x_3 = 5$

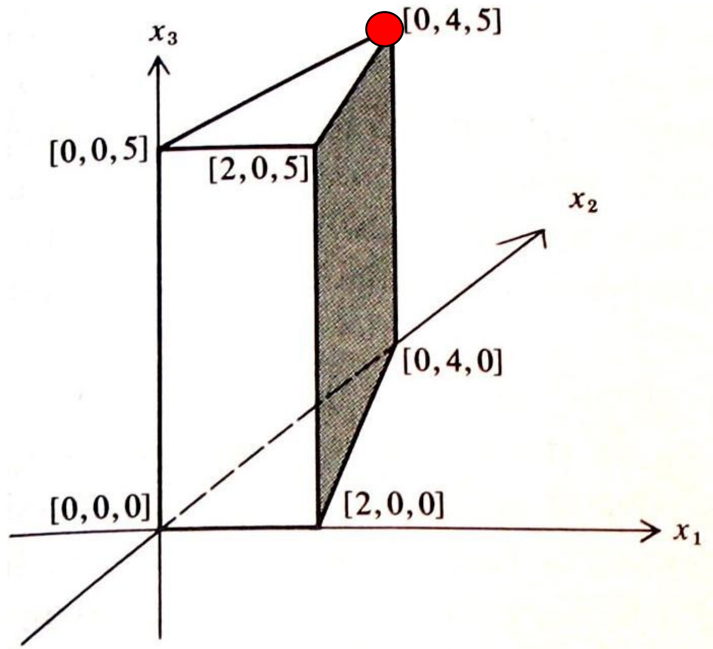


Simplex Illustrated: Final Pivot

$$
\begin{aligned}
x_2 &= 4 - 2x_1 - x_4 \\
x_3 &= 5 - x_5 \\
\hline
z &= 33 - 7x_1 - 2x_4 - 5x_5
\end{aligned}
$$

Final solution (optimal):

$x_1 = 0$
$x_2 = 4$
$x_3 = 5$



Simplex Review and Analysis

- Simplex pivoting represents traveling along polyhedron edges
- Each vertex reached tightens one constraint (and if needed, loosens another)
- May take a longer path to reach final vertex than needed

Simplex Weaknesses: Exponential Iterations: Klee-Minty Reviewed

$$
\begin{array}{rcrcrcr}
100x_1 & + & 10x_2 & + & x_3 & = & z \\
x_1 & & & & & \leq & 1 \\
20x_1 & + & x_2 & & & \leq & 100 \\
200x_1 & + & 20x_2 & + & x_3 & \leq & 10,000
\end{array}
$$

$$x_1, x_2, x_3 \geq 0$$

- Cases with high complexity (2^n-1 iterations)
- Normal complexity is O(m^3)
- How was this problem solved?

The Graphic Method

- Use geometry to quickly solve LP problems in 2 variables
- Plot all restrictions in 2D plane (x_1, x_2)
- Result plus axes forms polyhedron
  - Region of feasible solutions
- Draw any line with same slope as objective function through polyhedron
- "Move" line until leaving feasible region
  - i.e., Find parallel tangent

Graphic Method Example:
Step 1: Plot Boundary Conditions

max $5x_1 + 4x_2$

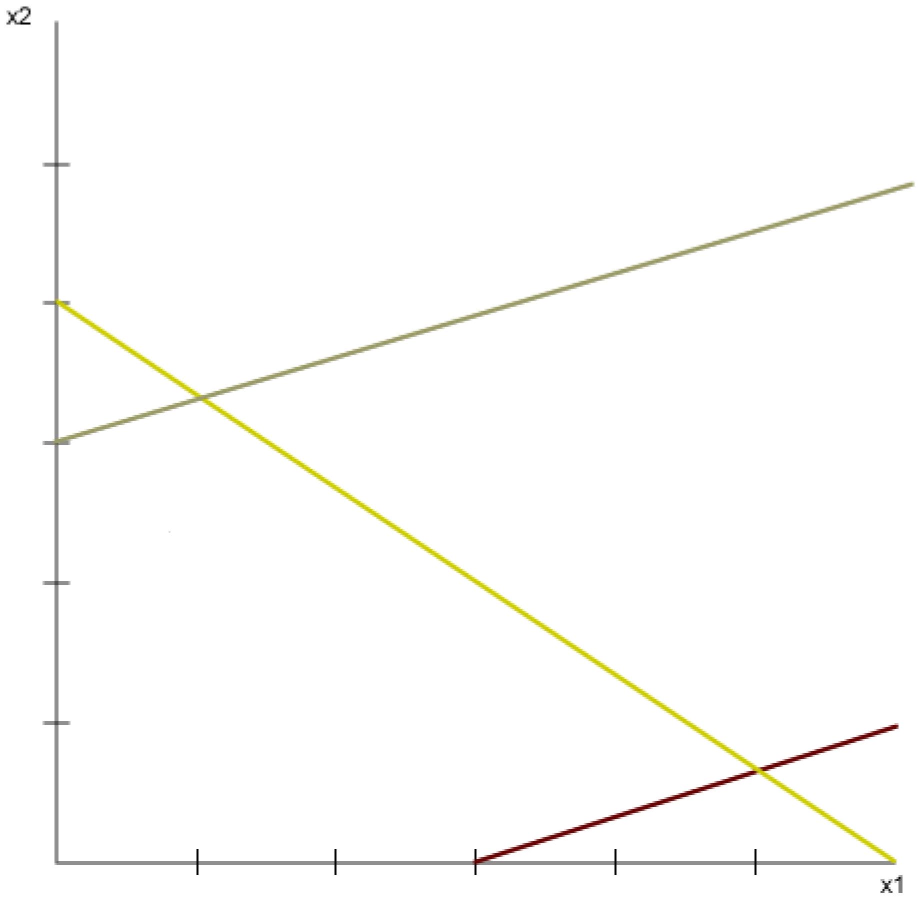
subject to:

$x_1 - 3x_2 \leq 3$

$2x_1 + 3x_2 \leq 12$

$-2x_1 + 7x_2 \leq 21$

$x_1, x_2 \geq 0$



Graphic Method Example: Step 2: Determine Feasibility

max $5x_1 + 4x_2$

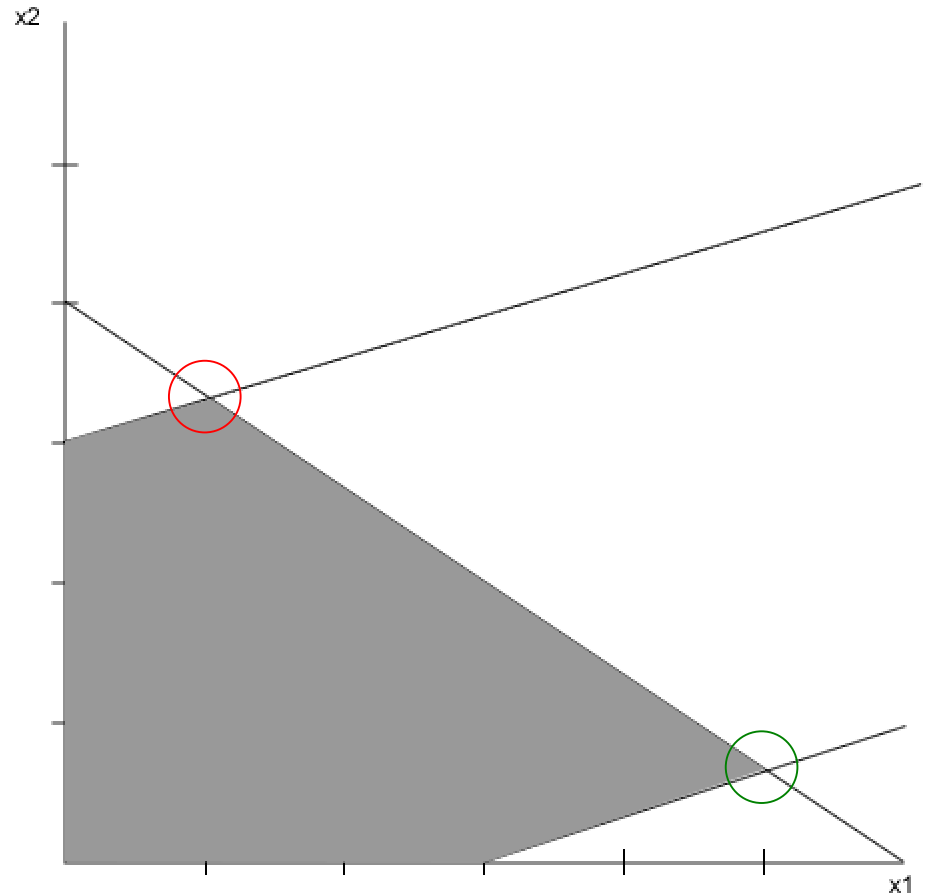
subject to:

$x_1 - 3x_2 \leq 3$

$2x_1 + 3x_2 \leq 12$

$-2x_1 + 7x_2 \leq 21$

$x_1, x_2 \geq 0$



Based only on this, where might the optimal solution be?

Graphic Method Example: Step 3: Plot Objective = *c*

max $5x_1 + 4x_2$

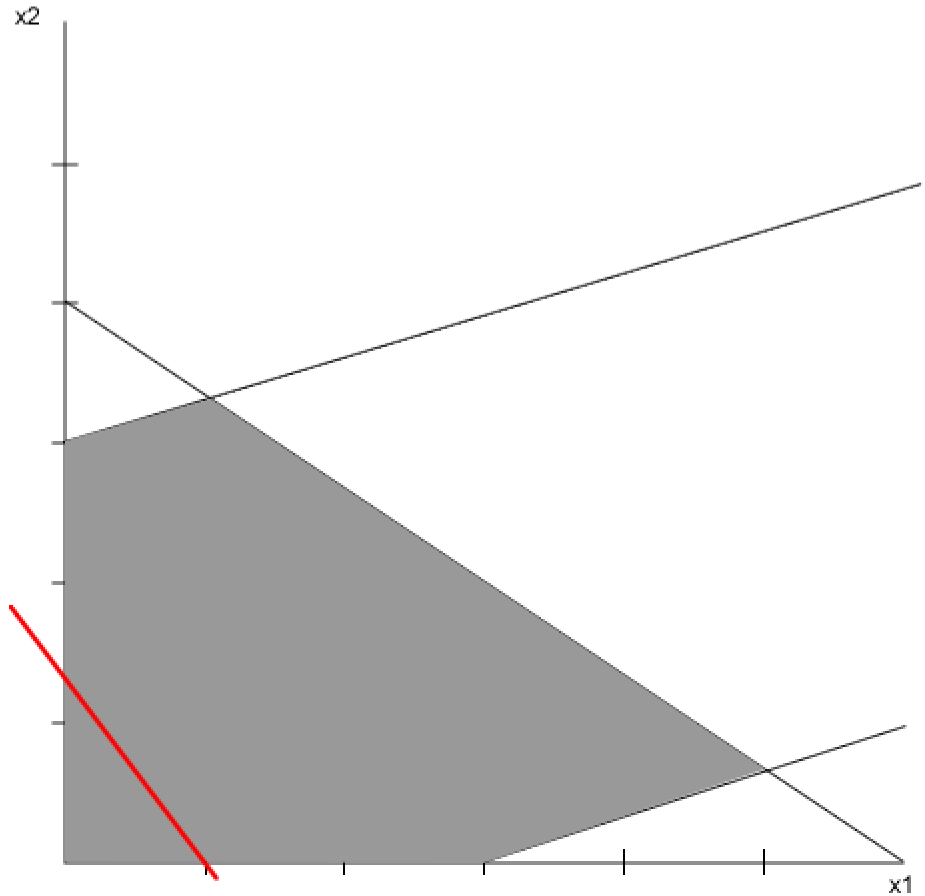
subject to:

$x_1 - 3x_2 \leq 3$

$2x_1 + 3x_2 \leq 12$

$-2x_1 + 7x_2 \leq 21$

$x_1, x_2 \geq 0$



Graphic Method Example: Step 4: Find Parallel Tangent

max 5x1 + 4x2

subject to:

$x_1 - 3x_2 \leq 3$

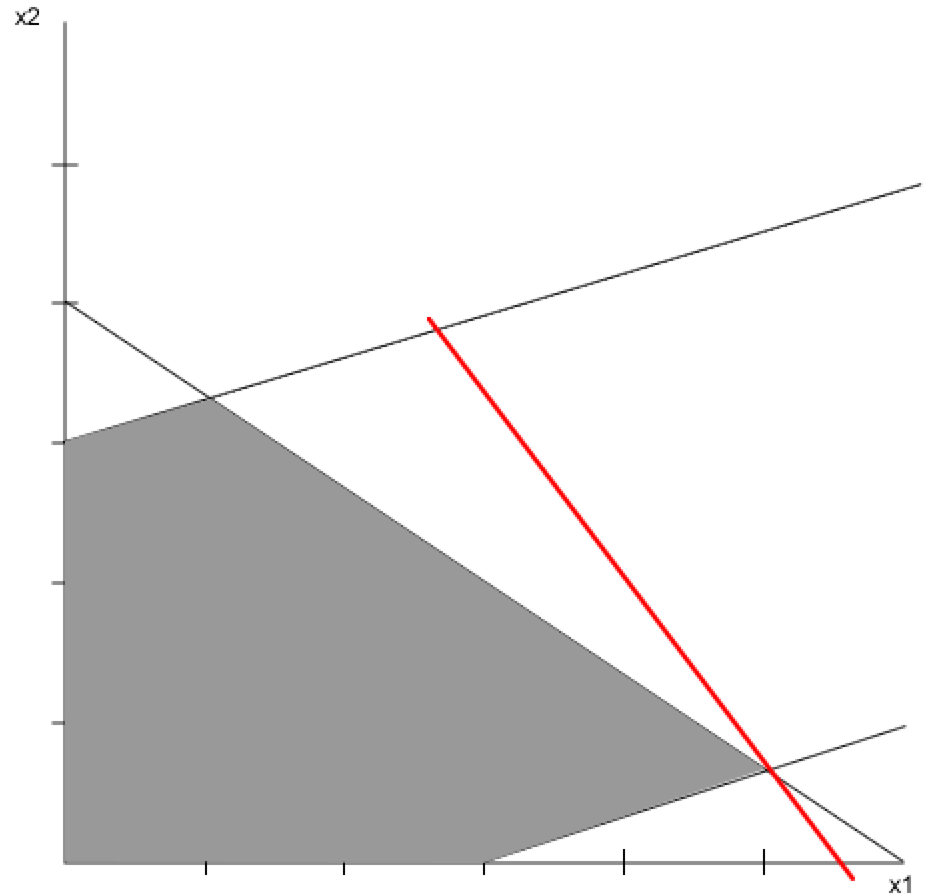
$2x_1 + 3x_2 \leq 12$

$-2x_1 + 7x_2 \leq 21$

$x_1, x_2 \geq 0$

Optimal solution:

X1=5, x2=2/3, z=83/3



Graphic Method Discussion

- Pro:
  - Works for any number of constraints
  - Fast, especially with graphing tool
  - Gives visual representation of tradeoff between variables
- Con:
  - Only works well in 2D (feasible but difficult in 3D)
  - For very large number of constraints, could be annoying to plot
  - For large range / ratio of coefficients, plot size limits precision and ability to quickly find tangent

Second Graphic Method Example

max $4x_1 + 6x_2$

subject to:

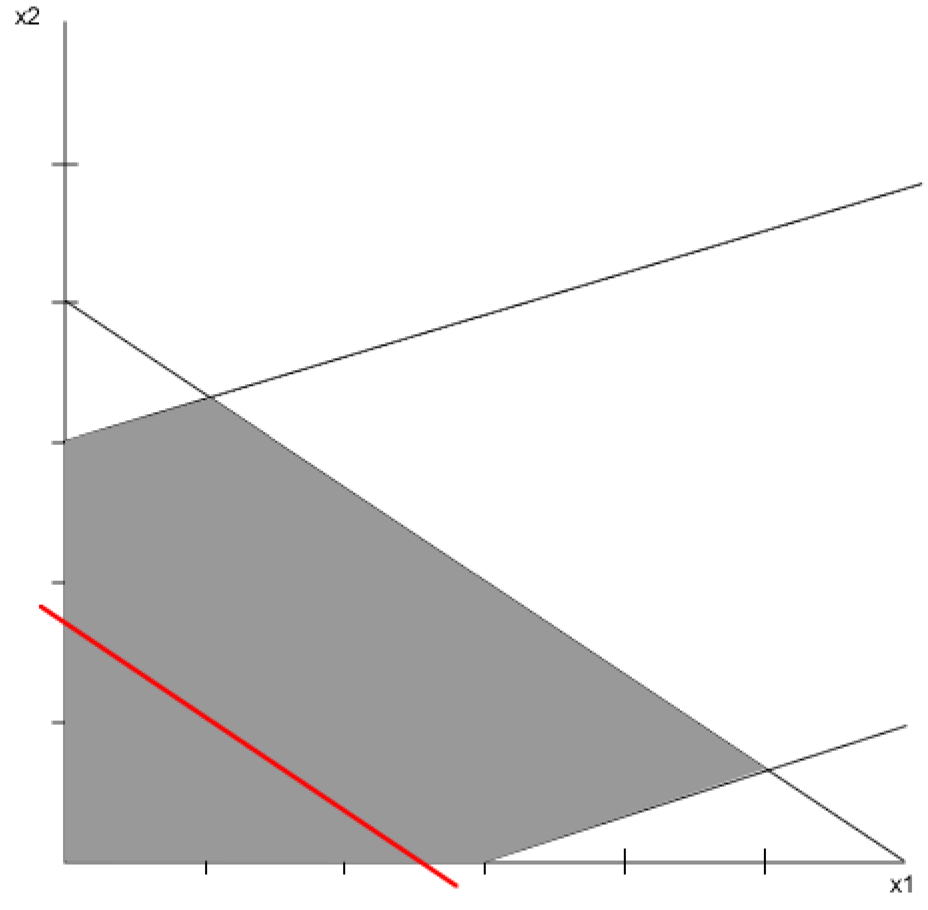
$x_1 - 3x_2 \leq 3$

$2x_1 + 3x_2 \leq 12$

$-2x_1 + 7x_2 \leq 21$

$x_1, x_2 \geq 0$

Same constraints; new objective. What changes?



Second Graphic Method Example: No Tangent Exists

max $4x_1 + 6x_2$

subject to:

$x_1 - 3x_2 \leq 3$

$2x_1 + 3x_2 \leq 12$

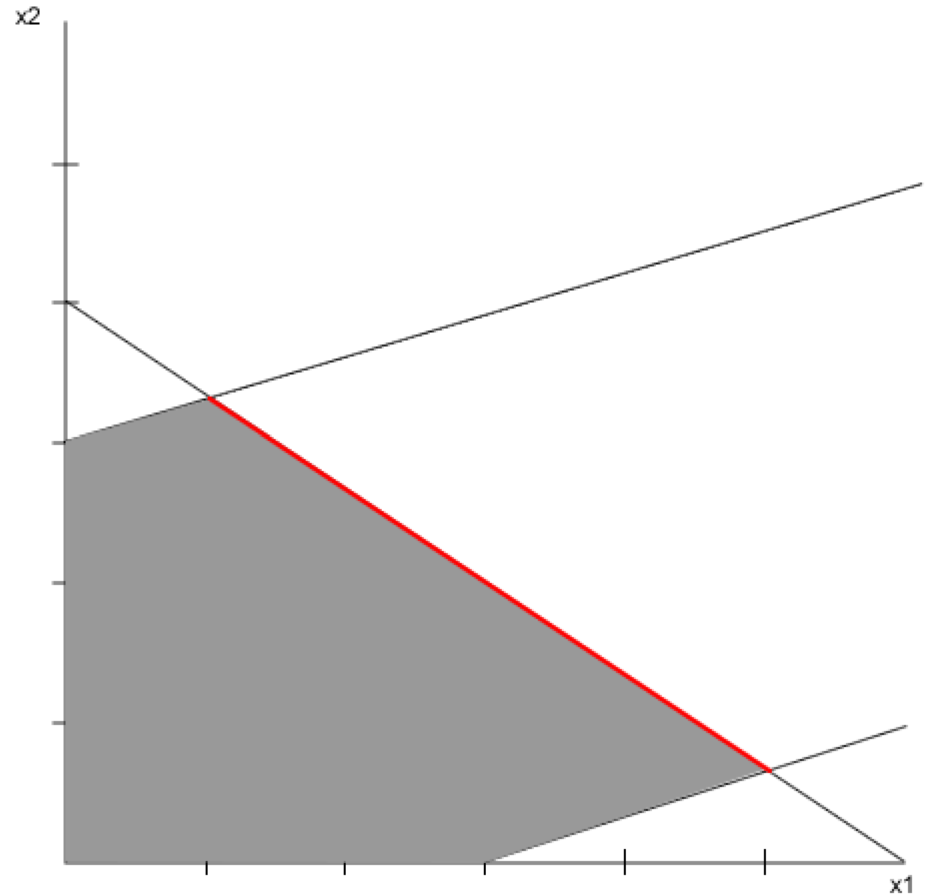
$-2x_1 + 7x_2 \leq 21$

$x_1, x_2 \geq 0$

Optimal solution:

$1.05 \leq x_1 \leq 5$,

$2x_1 + 3x_2 = 12$, $z=24$



Sensitivity analysis (how the solution changes when we change the coefficients c_i and/or the constraints