DSE - GTDMO - (sketch of the solutions) 18^{th} **Oct 2019**

(1) For the matrix A,

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$$

then

$$(2-\lambda)^2 - 1 = 0 \Longleftrightarrow (2-\lambda)^2 = 1 \Longleftrightarrow \lambda = 2 \pm 1.$$

The matrix A has two distinct eigenvalues: $\lambda_1 = 3$, and $\lambda_1 = 1$.
Eigenvector $\mathbf{V}_1 = (x_1, x_2)^T$ corresponding to λ_1:

$$(A - 3I)\,\mathbf{V}_1 = \mathbf{0} \Longrightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = -x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

Eigenvector $\mathbf{V}_2 = (x_1, x_2)^T$ corresponding to λ_2:

$$(A - I)\,\mathbf{V}_1 = \mathbf{0} \Longrightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

Remark. The vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthogonal: this derives from the fact that the matrix A is symmetrical and $\lambda_1 \neq \lambda_2$.
we observe that

$$A\mathbf{V} = \lambda\mathbf{V} \Longrightarrow A^2\mathbf{V} = \lambda A\mathbf{V} \Longrightarrow A^2\mathbf{V} = \lambda^2\mathbf{V}$$

and the eigenvalues of A^2 are $(\lambda_1)^2 = 9$, and $(\lambda_2)^2 = 1$, with eigenvectors as A.
For the inverse of A (the matrix A is invertible, so $\lambda \neq 0$),

$$A\mathbf{V} = \lambda\mathbf{V} \Longrightarrow A^{-1}A\mathbf{V} = \lambda A^{-1}\mathbf{V} \Longrightarrow \frac{1}{\lambda}\mathbf{V} = A^{-1}\mathbf{V},$$

and the eigenvalues of A^{-1} are $(1/\lambda_1) = 1/3$, and $(1/\lambda_2) = 1$, with eigenvectors as A.
Moreover, for tthe matrix $A + 4I$ we have,

$$(A + 4I)\mathbf{V} = A\mathbf{V} + 4I\mathbf{V} \Longrightarrow \lambda\mathbf{V} + 4\mathbf{V} \Longrightarrow (A + 4I)\mathbf{V} = (\lambda + 4)\mathbf{V},$$

and the eigenvalues of $(A+4I)$ are $(\lambda_1+4)=7$, and $(\lambda_2+4)=5$, with eigenvectors as A.
Finally,

$$A = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1=3=\lambda_1\lambda_2 = 3\cdot 1 = 3.$$

(2) For the eigenvalues,

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)-2$$

Then

$$(3-\lambda)(2-\lambda)-2=0 \Longrightarrow \lambda_1=4,\ \lambda_2=1.$$

Eigenvector $\mathbf{V}_1=(x_1,x_2)^T$ corresponding to λ_1:

$$(A-4I)\,\mathbf{V}_1=\mathbf{0} \Longrightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1=x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x_2\in\mathbb{R}.$$

Eigenvector $\mathbf{V}_2=(x_1,x_2)^T$ corresponding to λ_2:

$$(A-I)\,\mathbf{V}_1=\mathbf{0} \Longrightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1=-x_2/2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \quad x_2\in\mathbb{R}.$$

The matrix V with vectors $\mathbf{V}_1$, $\mathbf{V}_2$ as columns

$$V = \begin{pmatrix} 1 & -1/2 \\ 1 & 1 \end{pmatrix}$$

has determinant equal to $3/2\neq 0$, then $rank(V)=2$, and the two eigenvectors are linearly independent.
(3) We have to solve the following homogeneous system,

$$\begin{bmatrix} 3-5 & 1 \\ 2 & 4-5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then $x_1=x_2/2$ and the eigenvector $\mathbf{V}$ is

$$\mathbf{V} = x_2\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad x_2\in\mathbb{R}.$$

For example for $x_2 = 2$, $\mathbf{V} = (1,2)^T$.

Also for the other matrix we have consider a homogeneous system

$$\begin{bmatrix} 3+1 & 3 \\ 4 & 5+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

then $x_1 = -3x_2/2$ and the eigenvector $\mathbf{V}$ is

$$\mathbf{V} = x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

For example for $x_2 = -2$, $\mathbf{V} = (3,-2)^T$.

(4) The eigenvectors $\mathbf{V}$ are obtained by the following homogeneous linear system

$$\begin{bmatrix} 3-2 & 4 & 2 \\ 1 & 6-2 & 2 \\ 1 & 4 & 4-2 \end{bmatrix} \mathbf{V} = \mathbf{0} \iff \begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{bmatrix} \mathbf{V} = \mathbf{0}.$$

Now we have the following equivalent coefficient matrix (after echelon form reduction),

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and two free variables $x_2,\ x_3$ and one basic variable x_1. Then (first row) $x_1 = -4x_2 - 2x_3$ and

$$\mathbf{V} = x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad x_2,\ x_3 \in \mathbb{R}.$$

The vectors $(-4,1,0)^T$, $(-2,0,1)$ are linearly independent and provide a basis, the dimension of the eigenspace is equal to 2.

(5) For the eigenvalues,

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 2 & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) - 4 = 0 \implies \lambda_1 = 7,\ \lambda_2 = 2.$$

Eigenvector $\mathbf{V}_1 = (x_1, x_2)^T$ corresponding to λ_1:

$$(A - I)\,\mathbf{V}_1 = \mathbf{0} \implies \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = x_2/2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_1 = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

Eigenvector $\mathbf{V}_2 = (x_1, x_2)^T$ corresponding to λ_2:

$$(A - I)\,\mathbf{V}_2 = \mathbf{0} \Longrightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $x_1 = -2x_2$ and (we have chosen x_2 as a free variable),

$$\mathbf{V}_2 = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad x_2 \in \mathbb{R}.$$

As a columns of U we choose (for simplicity, any other choice of $x_2 \neq 0$ was possible), $\mathbf{V}_1 = (1, 2)^T$, $\mathbf{V}_2 = (-2, 1)^T$, then

$$U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} U^T = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = 5 \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

The last computation shows that we do not get the matrix M but a scalar multiple. The Theorem requires that U be orthogonal while in the current U matrix the columns are orthogonal but not normalized,

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{5}, \quad \left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = \sqrt{5}.$$

Then we consider the following orthogonal matrix (rescaling of the previous matrix U),

$$Q = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix},$$

now $QQ^T = Q^T Q = I$. Moreover

$$Q \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} Q^T = M$$

and the hypotheses of the Theorem are satisfied with

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$

`If` A `is a real symmetric matrix, then there is an orthogonal matrix` Q `that diagonalizes` A`, that is,` $Q^T A Q = D$`, where` D `is diagonal.`