$P^{(1)}, P^{(2)}, \ldots$. *For any n, we then have that*

$$\mu^{(n)} = \mu^{(0)} P^{(1)} P^{(2)} \cdots P^{(n)} .$$

Proof Follows by a similar calculation as in the proof of Theorem 2.1. □

Problems

2.1 (5) Consider the Markov chain corresponding to the random walker in Figure 1, with transition matrix P and initial distribution $\mu^{(0)}$ given by (11) and (14).

(a) Compute the square P^2 of the transition matrix P. How can we interpret P^2? (See Theorem 2.1, or glance ahead at Problem 2.5.)

(b) Prove by induction that

$$\mu^{(n)} = \begin{cases} (0, \frac{1}{2}, 0, \frac{1}{2}) & \text{for } n = 1, 3, 5, \ldots \\ (\frac{1}{2}, 0, \frac{1}{2}, 0) & \text{for } n = 2, 4, 6, \ldots . \end{cases}$$

2.2 (2) Suppose that we modify the random walk example in Figure 1 as follows. At each integer time, the random walker tosses *two* coins. The first coin is to decide whether to stay or go. If it comes up heads, he stays where he is, whereas if it comes up tails, he lets the second coin decide whether he should move one step clockwise, or one step counterclockwise. Write down the transition matrix, and draw the transition graph, for this new Markov chain.

2.3 (5) Consider Example 2.1 (the Gothenburg weather), and suppose that the Markov chain starts on a rainy day, so that $\mu^{(0)} = (1, 0)$.

(a) Prove by induction that

$$\mu^{(n)} = (\tfrac{1}{2}(1 + 2^{-n}), \tfrac{1}{2}(1 - 2^{-n}))$$

for every n.

(b) What happens to $\mu^{(n)}$ in the limit as n tends to infinity?

2.4 (6)

(a) Consider Example 2.2 (the Los Angeles weather), and suppose that the Markov chain starts with initial distribution $(\frac{1}{6}, \frac{5}{6})$. Show that $\mu^{(n)} = \mu^{(0)}$ for any n, so that in other words the distribution remains the same at all times.[7]

(b) Can you find an initial distribution for the Markov chain in Example 2.1 for which we get similar behavior as in (a)? Compare this result to the one in Problem 2.3 (b).

2.5 (6) Let $(X_0, X_1, \ldots)$ be a Markov chain with state space $\{s_1, \ldots, s_k\}$ and transition matrix P. Show, by arguing as in the proof of Theorem 2.1, that for any $m, n \geq 0$ we have

$$\mathbf{P}(X_{m+n} = s_j \mid X_m = s_i) = (P^n)_{i,j} .$$

[7] Such a Markov chain is said to be in **equilibrium**, and its distribution is said to be **stationary**. This is a very important topic, which will be treated carefully in Chapter 5.

2.6 (8) **Functions of Markov chains are not always Markov chains.** Let $(X_0, X_1, \ldots)$ be a Markov chain with state space $\{s_1, s_2, s_3\}$, transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and initial distribution $\mu^{(0)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. For each n, define

$$Y_n = \begin{cases} 0 & \text{if } X_n = s_1 \\ 1 & \text{otherwise.} \end{cases}$$

Show that $(Y_0, Y_1, \ldots)$ is *not* a Markov chain.

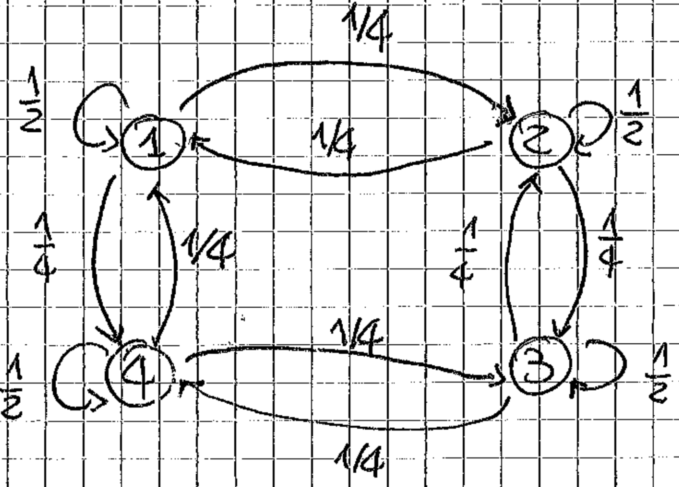
2.7 (9) **Markov chains sampled at regular intervals are Markov chains.** Let $(X_0, X_1, \ldots)$ be a Markov chain with transition matrix P.

(a) Define $(Y_0, Y_1, \ldots)$ by setting $Y_n = X_{2n}$ for each n. Show that $(Y_0, Y_1, \ldots)$ is a Markov chain with transition matrix P^2.

(b) Find an appropriate generalization of the result in (a) to the situation where we sample every k^{th} (rather than every second) value of $(X_0, X_1, \ldots)$.

Solutions Problems (Jonn O. Häggström, pages 15-16)

$\boxed{2.2}$



We assume that coins are fair, thus at each toss we have prob $\frac{1}{2}$ of the two outcomes

1^{st} coin tells us that : $P_{ii} = \frac{1}{2}$

The remaining probabilities on the rows must sum to one

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

The second coin chooses equally clockw. and counterclockw. $\Rightarrow$ prob $= \frac{1}{4}$ clockwise

$\frac{1}{4}$ counterclockwise

$\boxed{2.3}$ Gothenburg weather : $S = \{s_1, s_2\}$

s_1 = rain s_2 = sunshine

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

$$\mu^{(0)} = (1, 0)$$

(a) $\mu^{(1)} = \mu^{(0)} P = [1, 0] \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} =$

$$= \left[\frac{3}{4}, \frac{1}{4}\right] = \left[\frac{1}{2}\left(1 + \frac{1}{2}\right), \frac{1}{2}\left(1 - \frac{1}{2}\right)\right]$$

OK

assume that

$$\mu^{(n-1)} = \left[\frac{1}{2}\left(1 + 2^{-(n-1)}\right), \frac{1}{2}\left(1 - 2^{-(n-1)}\right)\right]$$

$$\mu^{(n)} = \mu^{(n-1)} P =$$

$$= \left[\frac{1}{2}\left(1 + 2^{-(n-1)}\right), \frac{1}{2}\left(1 - 2^{-(n-1)}\right)\right] \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} =$$

$$= \Big[\underbrace{\frac{3}{8}\left(1 + 2^{-(n-1)}\right) + \frac{1}{8}\left(1 - 2^{-(n-1)}\right)}_{(A)},$$

$$\underbrace{\frac{1}{8}\left(1 + 2^{-(n-1)}\right) + \frac{3}{8}\left(1 - 2^{-(n-1)}\right)}_{(B)}\Big] =$$

$$(A) = \frac{3}{8} + 3 \cdot 2^{-(n-1)-3} + \frac{1}{8} - 2^{-(n-1)-3} =$$

$$= \frac{1}{2} + 2 \cdot 2^{-(n-1)-3} = \frac{1}{2} + 2^{-(n-1)-2} =$$

$$= \frac{1}{2} + 2^{-n-1} = \frac{1}{2}\left(1+2^{-n}\right)$$

$$(B) = \frac{1}{8} + 2^{-(n-1)-3} + \frac{3}{8} - 3 \cdot 2^{-(n-1)-3} =$$

$$= \frac{1}{2} - 2 \cdot 2^{-(n-1)-3} = \frac{1}{2} - 2^{-n+1-3+1} =$$

$$= \frac{1}{2} - 2^{-n-1} = \frac{1}{2}\left(1-2^{-n}\right)$$

$$\Rightarrow \mu^{(n)} = \left[\frac{1}{2}\left(1+2^{-n}\right), \frac{1}{2}\left(1-2^{-n}\right)\right]$$

□

(b) When $n \to \infty$, $2^{-n} \to 0$ thus

$$\mu^{(n)} \longrightarrow \left[\frac{1}{2}, \frac{1}{2}\right]$$

which is equivalent to predict the weather of tomorrow tossing a fair coin to choose between S_1 and S_2, i.e. rain and sunshine are equally likely

2.4 Los Angeles weather

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.1 & 0.9 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}$$

$$\mu^{(0)} = \left[\frac{1}{6} , \frac{5}{6} \right]$$

a)

Let's proceed by induction: Compute $\mu^{(1)}$

$$\mu^{(1)} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix} =$$

$$= \left[\frac{1}{12} + \frac{5}{60} , \frac{1}{12} + \frac{45}{60} \right] =$$

$$= \left[\frac{10}{60} , \frac{50}{60} \right] = \left[\frac{1}{6} , \frac{5}{6} \right] \quad \text{Ok}$$

If we now assume that $\mu^{(n-1)} = \left[\frac{1}{6}, \frac{5}{6}\right]$ we have, as before,

$$\mu^{(n)} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1/2 \\ 1/10 & 9/10 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

$\square$

b) Gothenburg weather: $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

$\mu^{(0)} = \begin{bmatrix} x & y \end{bmatrix}$ with $x + y = 1$ (*)

$\mu^{(0)}$ does not change if

$$\begin{bmatrix} x & y \end{bmatrix} P = \begin{bmatrix} x & y \end{bmatrix}$$

that is

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{cases} \frac{3}{4}x + \frac{1}{4}y = x \\ \frac{1}{4}x + \frac{3}{4}y = y \end{cases} \Rightarrow \begin{cases} -x + y = 0 \\ x - y = 0 \end{cases} \Rightarrow x = y$$

now consider (*) $\Rightarrow$

$$\begin{cases} x = y \\ x + y = 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \end{cases} \qquad \mu^{(0)} = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Thus we have found the stationary distribution.