Hypotesis testing

martedì 22 ottobre 2019 12:43

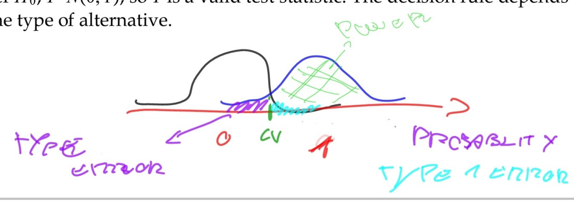
IDE SU ARIEL

Example.
Testing of a mean of a normal distribution (with known variance).

Let $X_1,\ldots,X_n$ be independent, $X_i \sim N(\mu,\sigma^2)$ $(i = 1,\ldots,n)$ for known σ^2.
We are interested in $H_0 : \{\mu = \mu_0\}$ where μ_0 is a known constant. Let

$$T = \sqrt{n}\,\frac{(\bar{X} - \mu_0)}{\sigma}$$

under H_0, $T \sim N(0,1)$, so T is a valid test statistic. The decision rule depends on the type of alternative.



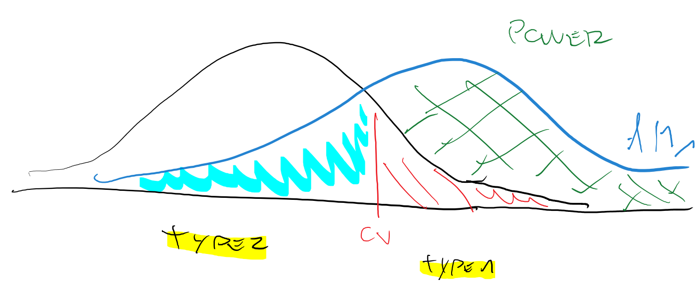
TRADE OFF BETWEEN T1 AND T2
I CANNOT REDUCE TYPE 1 ERROR, BECAUSE TYPE 2 WILL INCREASE

For $H_1 = H_1^+ : \{\mu > \mu_0\}$, the rejection rule is "Reject H_0 if $t > d_1$", where t is the realisation of T, and d_1 is the solution of $P(Z > d_1) = \alpha$, where α is the significance level, and Z is a $N(0,1)$ random variable.

For $H_1 = H_1^- : \{\mu < \mu_0\}$, the rejection rule is "Reject H_0 if $t < -d_1$", where t, d_1 and α are defined as above.

For $H_1 = H_1^\pm : \{\mu \neq \mu_0\}$, the rejection rule is "Reject H_0 if $|t| > d_2$", where t and α are defined as above, d_2 is the solution of $P(Z > d_2) = \alpha/2$, and Z is a $N(0,1)$ random variable.

I HAVE TO FIX T1 ERROR AND GET AS MUCH POWER I CAN



VERY DISTANT ALTERNATIVE

COST T2 WILL BE LESSER

Example.
The Railway Regulation Authority is revising the performance of the Fast Tortoises Railway Company, which is currently running the railway franchise: in the contract it was approved that the length of a regular journey is normally distributed with mean of 2 hours and standard deviation of 0.9. In the last 25 journeys, the average journey time was 2.4 hours. Should the franchise be renewed?

TAKE SAMPLE AVERAGE
CRITICAL VALUE
↓
WHEN SA IS BIG
REJECT

We first check if the problem is correctly specified, and then we address the question about renewing the franchise.
We are told to assume that $X_1,\ldots,X_n$ be independent, $X_i \sim N(\mu,\sigma^2)$ $(i = 1,\ldots,n)$ for $n = 25$, $\sigma^2 = 0.9^2$. We are interested in $H_0 : \{\mu = 2\}$.

WE WILL GET A STANDARD VERSION OF THE SAMPLE AVERAGE

The test statistic is

$$T = \sqrt{25}\,\frac{(\bar{X} - 2)}{0.9} \sim N(0,1),$$

under H_0. We are not told which type of alternative to take, nor the significance value. Assuming that the customers would not mind if the journey is shorter, we take the alternative

$$H_1 : \{\mu > 2\},$$

We also assume that $\alpha = 0.05$, so the critical value is 1.65. So,

1. null hypothesis
$$H_0 : \mu = 2$$
2. the alternative hypothesis
$$H_1 : \mu > 2$$
3. the test statistic
$$T = \sqrt{25}\,\frac{(\bar{X} - 2)}{0.9}$$

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4. the limit distribution of the test statistic under the null
$$T \sim N(0,1)$$
5. the decision rule
$$\text{Reject if } t \geq 1.65$$
6. the size of the test
$$\alpha = 0.05$$
7. the limit distribution of the test statistic under the alternative
$$T = \sqrt{25}\,\frac{(\bar{X} - 2)}{0.9} = \sqrt{25}\,\frac{(\bar{X} - 2 - \mu + \mu)}{0.9}$$
$$= \sqrt{25}\,\frac{(\bar{X} - \mu)}{0.9} + \sqrt{25}\,\frac{(\mu - 2)}{0.9}$$
$$\text{so } T = Z + \sqrt{25}\,\frac{(\mu - 2)}{0.9} \text{ where } Z \sim N(0,1)$$

8. the power of the test
We should compute $P\left(Z + \sqrt{25}\,\frac{(\mu-2)}{0.9} \geq 1.65\right)$ for $\mu > 2$. There are infinite values for this, so we only compute it for a few possible parameters. For $\mu = 2.1$, $1.65 - \sqrt{25}\,\frac{(2.1-2)}{0.9} = 1.0944$, power $1 - 0.86 = 0.14$.

μ	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7
size	0.05							
Power		0.14	0.30	0.51	0.72	0.87	0.95	0.99

I DON'T KNOW THE TRUE VALUE

9. the realisation of the test statistic
$$t = \sqrt{25}\,\frac{(\bar{x} - 2)}{0.9} = \sqrt{25}\,\frac{(2.4 - 2)}{0.9} = 2.22$$
10. wether the null hypothesis is rejected or not.
Since $t > 1.65$, the realisation of the test is in the rejection area so H_0 is rejected. The Fast Tortoise will lose the franchise.

Example.
Testing of a mean of a normal distribution with unknown variance.

Let $X_1,\ldots,X_n$ be independent, $X_i \sim N(\mu,\sigma^2)$ $(i = 1,\ldots,n)$.
We are interested in $H_0 : \{\mu = \mu_0\}$ where μ_0 is a known constant. We do not know σ^2, but we estimated $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Let

$$T = \sqrt{n}\,\frac{(\bar{X} - \mu_0)}{S}$$

under H_0, $T \sim t_{n-1}$, so T is a valid test statistic. The decision rule depends on the type of alternative.

WHAT IS THE PRINCIPLE OF TESTING?

WHY IS THIS GOOD?

- WE CAN PROVE THIS IS THE BEST TEST

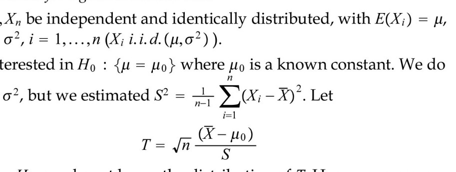
- WHIT NEGATIVE ALTERNATIVES WE TAKE CRITICAL VALUE AT THE LEFT

For $H_1 = H_1^+ : \{\mu > \mu_0\}$, the rejection rule is "Reject H_0 if $t > d_1$", where t is the realisation of T, and d_1 is the solution of $P(T_{n-1} > d_1) = \alpha$, where α is the significance level, and T_{n-1} is a t_{n-1} distributed random variable.

For $H_1 = H_1^- : \{\mu < \mu_0\}$, the rejection rule is "Reject H_0 if $t < -d_1$", where t, d_1 and α are defined as above.

For $H_1 = H_1^\pm : \{\mu \neq \mu_0\}$, the rejection rule is "Reject H_0 if $|t| > d_2$", where t and α are defined as above, d_2 is the solution of $P(T_{n-1} > d_2) = \alpha/2$, and T_{n-1} is a t_{n-1} random variable.

CRITICAL VALUE WILL BE 1.96



Testing the mean for a general distribution.
Let $X_1,\ldots,X_n$ be independent and identically distributed, with $E(X_i) = \mu$, $Var(X_i) = \sigma^2$, $i = 1,\ldots,n$ (X_i $i.i.d.(\mu,\sigma^2)$).
We are interested in $H_0 : \{\mu = \mu_0\}$ where μ_0 is a known constant. We do not know σ^2, but we estimated $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Let

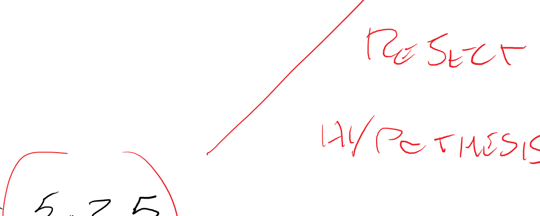
$$T = \sqrt{n}\,\frac{(\bar{X} - \mu_0)}{S}$$

even under H_0, we do not know the distribution of T. However, as $n \to \infty$, $T \to_d N(0,1)$, so T is a valid test statistic in large samples. The decision rule depends on the type of alternative. The decision rule given for the normal distribution of X_i (with known σ^2) are applied.

$Y_1 \ldots Y_t \quad E(Y_t) = \mu \quad H_1 : \{\mu \neq 3\}$

$\bar{Y} = 4 \quad H_0 : \{\mu = 3\}$

$$\sqrt{T}\,(\bar{Y} - \mu) \xrightarrow{d} N\left(0, \sum_{-\infty}^{\infty} \gamma_j\right)$$



$$\frac{\sqrt{T}\,(\bar{Y} - \mu)}{\sqrt{\sum_{-\infty}^{\infty} \gamma_j}} \xrightarrow{d} N(0,1)$$

$$\sqrt{100}\,\frac{4 - 9}{\sqrt{3625}} = 5.25$$

REJECT HYPOTHESIS

COMPUTER CALCULATE PVALUE

IN THIS CASE PV = 0,000 001

IF I CHOOSE 0.5 THE PV < SIZE SO I REJECT OTHERWISE I ACCEPT.

PV < SIZE => REJECT H_0