DEPARTMENT OF ECONOMICS, MANAGEMENT AND QUANTITATIVE METHODS

Academic Year 2019-2020

B-74-3-B Time Series Econometics

Fabrizio Iacone

Discussion of Exercise Sheet 4

1.

Notice that

$$u_t = \Delta Y_t,$$

so in our example

$$\begin{aligned} u_t &= (0.2 - 0.4) = -0.2 \\ u_{t-1} &= (0.4 - 0.2) = 0.2 \\ u_{t-2} &= (0.2 + 0.1) = 0.3 \end{aligned}$$

As

$$u_t = 0.4 u_{t-1} + 0.2 u_t + \varepsilon_t, \text{ where } \varepsilon_t \ iid\left(0, \sigma^2\right)$$

then u_t is AR(2), and notice that $1 - 0.4z - 0.2z^2 = 0$ has solutions $\frac{0.4+\sqrt{0.96}}{0.4} = 1 \pm \sqrt{6}$ and $1+\sqrt{6} > 1$ whereas $1-\sqrt{6} < -1$ because $-\sqrt{6} < -2$ using the fact that $-2 = -\sqrt{4}$ (numerical solutions are 1.4495, -3.4495) so this is stationary.

so $\widehat{u}_{t+1|t,t-1,t-2} = \widehat{u}_{t+1|t,t-1}$ (i.e., knowledge of u_{t-2} is irrelevant for our forecast)

$$\widehat{u}_{t+1|t,t-1} = 0.4 \times (-0.2) + 0.2 \times (0.2) = -0.04$$

and

$$\widehat{Y}_{t+1|t,t-1,t-2,t-3} = Y_t + \widehat{u}_{t+1|t,t-1,t-2} = 0.2 - 0.04 = 0.16.$$

2.

Assume that

$$\begin{aligned} Y_t &= Y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1} \text{ when } t > 0 \\ Y_0 &= 0 \end{aligned}$$

and that we have a time series $Y_1, ..., Y_T$.

i. We first compute $u_t = \Delta Y_t$ for $t = 2,...,T$; then, we estimate θ in the MA(1) model $u_t = \varepsilon_t + \theta\varepsilon_{t-1}$ by CML.

ii.

$$\widehat{Y}_{T+1|T,\widehat{\varepsilon}_T} = Y_T + \widehat{\theta}\widehat{\varepsilon}_T = 0.2 + 0.8 \times 0.1 = 0.28$$

Note that knowledge of Y_{T-1} is not necessary, once that Y_T, $\widehat{\theta}$ and $\widehat{\varepsilon}_T$ are known.

3.

For $t > 1$,

$$Y_{t-1} = c + Y_{t-2} + \varepsilon_{t-1}$$

so, replacing in Y_t,

$$\begin{aligned} Y_t &= c + Y_{t-1} + \varepsilon_t = c + (c + Y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= c + c + Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t = 2c + Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

If $t - 2 > 0$, i.e. if $t > 2$,

$$Y_{t-2} = c + Y_{t-3} + \varepsilon_{t-2}$$

and

$$\begin{aligned} Y_t &= 2c + (c + Y_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1} + \varepsilon_t \\ &= 2c + c + Y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t = 3c + Y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

Iterating,

$$\begin{aligned} Y_t &= tc + Y_0 + \varepsilon_1 + ... + \varepsilon_{t-1} + \varepsilon_t \\ Y_t &= ct + \sum_{j=1}^{t} \varepsilon_j \end{aligned}$$

so

$$E\left(Y_t\right) = ct.$$

When $c \neq 0$, ct constitutes a deterministic trend (and $\sum_{j=1}^{t} \varepsilon_j$ a random walk).

4.

Introduce a process integrated of order 0 first. Any stationary and invertible ARMA (with mean 0, and of course with finite p and q) is integrated of order 0. (A slight generalisation of the condition would be to allow for any stationary zero mean process with $0 < \sum_{j=-\infty}^{\infty} \gamma_j < \infty$).

i.For any process u_t integrated of order 0, introduce the notation $u_t \in I(0)$. If X_t is such that $X_t = \sum_{j=1}^{t} u_j$, then $X_t \in I(1)$.

ii. In the same way, If Z_t is such that $Z_t = \sum_{j=1}^{t} X_j$, then $Z_t \in I(2)$.

(A non zero mean in u_t, say $E(u_t) = c$, would generate a drift as well: $E(X_t) = ct$).

iii.

The Dickey and Fuller test can be used to establish the order of integration of a process (if the process is integrated of order d, $d = 0, 1, 2, 3, ...$).

Let u_t be a stationary AR process. The ADF test checks if $\phi = 1$ in the model

$$X_t = \alpha + \phi X_t + u_t$$

(where α may or may not be zero, depending on the presence of a linear trend). In this model, a unit root corresponds to $X_t \in I(1)$. In a similar way, two unit roots in Z_t corresponds to $Z_t \in I(2)$.

So, provided that $Y_t \in I(d)$ with $d = 0, 1, 2, 3, ...$, the procedure we can use is the following:

First test (with the ADF test) for a unit root in Y_t. If the null hypothesis is rejected, then conclude that Y_t has no unit roots. If it is not rejected, then conclude that Y_t has at least one unit root. Therefore, test for a unit root in ΔY_t: if the null hypotesis is rejected, conclude that $\Delta Y_t \in I(0)$, so $Y_t \in I(1)$. Otherwise, conclude that ΔY_t has at least one unit root, and then test for a unit root in $\Delta^2 Y_t$: if the null hypotesis is rejected, conclude that $\Delta^2 Y_t \in I(0)$, so $Y_t \in I(2)$. Continue with this procedure until, after d differences, then $\Delta^d Y_t \in I(0)$, so $Y_t \in I(d)$.

[Note: in the response to part (i) many students said that X_t is $I(1)$ if ΔX_t is $I(0)$: this is correct but, unless the definition of $I(0)$ is also given, this does not constitute an explanation of what $X_t \in I(1)$ means. Other students said that X_t is $I(1)$ if ΔX_t is stationary: this is not correct as, for example, consider ε_t iid$(0, \sigma^2)$ and $X_t = \varepsilon_t$ (so X_t is $I(0)$): then, $\Delta X_t = \varepsilon_t - \varepsilon_{t-1}$ is MA(1) and stationary, but as we have seen X_t is not $I(1)$. It is worth noticing that in this case ΔX_t is not $I(0)$, as we can easily verify by computing $\sum_{j=-\infty}^{\infty} E(\Delta X_t \Delta x_{t+j}) = 0$ (which is not possible for $I(0)$ processes (indeed, $\Delta X_t \in I(-1)$ in this example). Other students still said X_t is $I(1)$ if X_t is not stationary but ΔX_t is stationary: again, this is not correct: the point here is still that the class of stationary processes is much wider than the class of $I(0)$ processes. Consider for example $u_t = \sum_{j=1}^{\infty} j^{\delta-1}\varepsilon_{t-j}$, for $\delta \in [0, 1/2)$: then this is an infinite moving average and, it is possible to show that $\sum_{j=1}^{\infty} j^{2(\delta-1)} \leq C$ so that u_t is stationary, and yet $\sum_{j=-\infty}^{\infty} E(u_t u_{t+j}) = \infty$ so u_t is not $I(0)$.

$\sum_{j=1}^{\infty} j^{-1.2}$

$\int_1^{\infty} j^{2(\delta-1)} dj = \int j^{2\delta-2}\, dj = \frac{j^{2\delta-1}}{2\delta-1}$

5.

(i)

The t statistic takes value $\frac{0.75-1}{0.07} = -3.5714$. As $-3.5714 < -2.86$, the realization of the t statistic is below the critical value, so null hypothesis is rejected.

(ii) This test assumes the Case 1 model. With size 5%, both Case 1 and Case 2 versions have the same probability of not rejecting $H_0 : \{\phi = 1\}$ when it is true (i.e., to conclude that there is a unit root when there is one). So, we compare them on the basis of the power, i.e. the probability of Rejecting H_0 when it is false (i.e., to conclude that there is no unit root, when indeed there is not). So, we focus on $|\phi| < 1$. If $\alpha = 0$, then the estimates of ϕ under both Case 1 and Case 2 have the same limit distributions: however, because the critical value for Case 2 is smaller, it is less easy to reject H_0 (thus correctly concluding that there is no unit root). If $\alpha \neq 0$, the estimate of ϕ in Case 2 is unaffected, but the estimate of ϕ in Case 1 is now inconsistent, and it moves closer to 1, thus the power is reduced (more heavily the larger is α).

As we do not know anything about α, Case 2 seems the safe option.

Some students may have run the test for Case 1 too: $\frac{0.89-1}{0.05} = -2.2$: again, this is below the critical value (which is -1.95), thus the null hypothesis is rejected. However, running this second test is not a good idea. This is because the second test does not provide extra information: if (as in this example) we reach the same conclusion (we do not reject H_0), then this second test did not add any information; if on the other hand the second test would lead to the opposite conclusion, what should we do? In view of the fact that we do not know anything about α, it is safer to choose Case 2 as it has more power: the information from the Case 1 test is then ignored, so the second test was useless. Worse than that, it may be misleading, as running the second test and looking at the two tests jointly would alter the size of the test.

(iii) . This is a Case 2 test: As $-\frac{0.08}{0.03} = -2.6667 > -2.86$, the null hypothesis is not rejected (unit root).