Computer simulations of Markov Chains

All major programming languages have a random numbers generator, producing sequences of i.i.d. numbers

$$U_0, U_1, U_2, \dots \sim U(0,1)$$

(Actually they are pseudo-random numbers generators $\Rightarrow$ possible problem, but we disregard it)

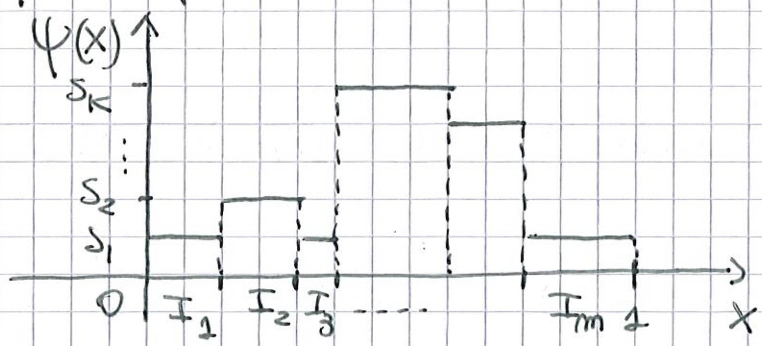
Starting from this, we want to simulate a Markov Chain $\{X_0, X_1, X_2, \dots\}$ with given state space $S = \{s_1, s_2, \dots, s_k\}$, initial distrib. $\underline{\mu}^{(0)}$ and transition matrix P (constant $\Rightarrow$ homog. MC).

Main ingredients: initiation function and update function.

The initiation function $\psi : [0,1] \to S$ is a function that we use to generate the starting value X_0.

Assumptions:

(i) ψ is piecewise constant:



$[0,1]$ partitioned into intervals $\{I_1, I_2, \dots, I_m\}$ where ψ is constant

(ii) for each $s \in S$, the total length of the intervals on which $\varphi(x) = s$ equals $\mu^{(0)}(s) = \mathbb{P}(X_0 = s)$

$$\sum_{\{j : \varphi(x) = s \text{ on } I_j\}} \ell(I_j) = \mu^{(0)}(s).$$

$\ell(I_j) =$ length I_j

A method to build an initiation function which satisfies (i) and (ii) is the following (<u>inverse transform method</u>):

Let $S = \{s_1, \dots, s_K\}$ $\mu^{(0)} = (\mu^{(0)}(s_1), \dots, \mu^{(0)}(s_K))$

we can set:

$$\varphi(x) = \begin{cases} s_1 & \text{for } x \in [0, \mu^{(0)}(s_1)) =: I_1 \\ s_2 & \text{for } x \in [\mu^{(0)}(s_1), \mu^{(0)}(s_1) + \mu^{(0)}(s_2)) =: I_2 \\ \vdots & \\ s_i & \text{for } x \in \left[\sum_{j=1}^{i-1} \mu^{(0)}(s_j), \sum_{j=1}^{i} \mu^{(0)}(s_j)\right) =: I_i \\ \vdots & \\ s_K & \text{for } x \in \left[\sum_{j=1}^{K-1} \mu^{(0)}(s_j), 1\right] =: I_K \end{cases}$$

<u>why it works?</u>

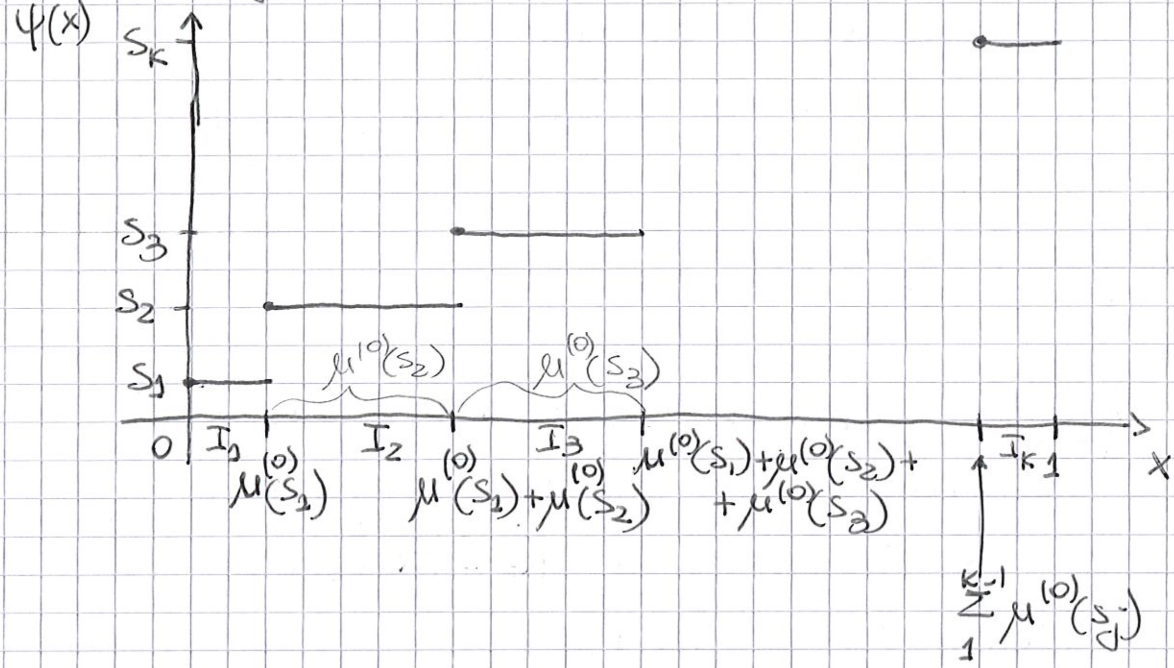
(i) is clearly satisfied ($\varphi(x)$ is piecewise constant)

Let's verify that (ii) is also satisfied:

For each s_i $\psi(x) = s_i$ only on the interval $\left[\sum_{j=1}^{i-1} \mu^{(0)}(s_j), \sum_{j=1}^{i} \mu^{(0)}(s_j) \right)$, whose length is

$$\sum_{j=1}^{i} \mu^{(0)}(s_j) - \sum_{j=1}^{i-1} \mu^{(0)}(s_j) = \mu^{(0)}(s_i) \quad \text{ok!}$$

graphically:



remember that since $\mu^{(0)}$ is a distribution, $\sum_{i=1}^{K} \mu^{(0)}(s_i) = 1 \Rightarrow$ we are partitioning correctly the interval $[0,1]$

Thus we can simulate X_0 using the initiation function $\psi(x)$ as follows:

<u>Algorithm to simulate X_0</u>

a) Generate a random number $U \sim U(0,1)$
b) Identify the interval I_ℓ in which U is fallen and set $X_0 = s_\ell$

Usually b) can be performed with a "while cicle":

b) counter = 0
   scan = 0
   while U > scan {
       counter = counter + 1
       scan = scan + $\mu^{(0)}(s_{counter})$
       }
   output : $X_0 = s_{counter}$

— ∘ —

The Update function $\phi : S \times [0,1] \to S$ is the distribution by which we generate X_{n+1} from X_n ⇒ iterating it we generate the entire chain $(X_0, X_1, \dots)$.

$\phi(s, x)$ takes as input a state $s \in S$ and a number $x \in [0,1]$ (a probability) and produces another state $s' \in S$ as output.

Assumptions :

(i) for any fixed s_i, $\phi(s_i, x)$, as function of x, is piecewise constant

(ii) for each fixed $s_i, s_j \in S$, the total length of the intervals on which $\phi(s_i, x) = s_j$ equals $p_{ij} = \mathbb{P}(X_{n+1} = s_j \mid X_n = s_i)$ (element ij of P)

Remember that "rows of P sum to one"

$$P = \begin{bmatrix} \cdots & & \\ p_{i1} & \cdots & p_{ik} \\ \cdots & & \end{bmatrix}$$ i-th row: can be seen as a distribution

Thus we may define an update function ϕ satisfying (i) and (ii) similarly as before, replacing $\mu^{(0)}$ with the i-th row of P:

$$\phi(s_i, x) = \begin{cases} s_1 & \text{for } x \in [0, p_{i1}] = I_1 \\ s_2 & \text{for } x \in [p_{i1}, p_{i1}+p_{i2}) = I_2 \\ \vdots & \\ s_j & \text{for } x \in \left[\sum_{\ell=1}^{j-1} p_{i\ell}, \sum_{\ell=1}^{j} p_{i\ell}\right) = I_j \\ \vdots & \\ s_k & \text{for } x \in \left[\sum_{\ell=1}^{k-1} p_{i\ell}, 1\right] = I_k \end{cases}$$

It can be proven as before that this definition is correct and satisfies the assumptions.

Algorithm to simulate X_{n+1} from X_n

a) Generate a random number $U \sim U(0,1)$

b) consider $\phi(X_n, x)$: check in which interval I_ℓ U is fallen and set $X_{n+1} = s_\ell$

General algorithm to simulate the MC

$X_0 = \psi(U_0)$ $\qquad$ $U_i \sim U(0,1)$ independent

$X_1 = \phi(X_0, U_1)$

$X_2 = \phi(X_1, U_2)$

$\vdots$

Example - the Gothenburg weather

Remember that in our example we had

$S = \{s_1, s_2\}$ $\quad s_1 =$ "rain" $\quad s_2 =$ "sunshine"

and transition matrix

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

Suppose that we start the MC in a rainy day $\Rightarrow X_0 = s_1$ and $\mu^{(0)} = (1, 0)$

$\Rightarrow$ Initiation function : $\psi(x) = s_1 \quad \forall x$

Update function :

$$\phi(s_1, x) = \begin{cases} s_1 & \text{for } x \in [0, 0.75) \\ s_2 & \text{for } x \in [0.75, 1] \end{cases}$$

$$\phi(s_2, x) = \begin{cases} s_1 & \text{for } x \in [0, 0.25) \\ s_2 & \text{for } x \in [0.25, 1] \end{cases} \qquad (*)$$

Simulation of an inhomogeneous Markov Chain

Extension (quite simple) of previous method:

Let $(X_0, X_1, \dots)$ be an inhomogeneous MC with state space $S = \{s_1, s_2, \dots, s_k\}$

initial distribution $\mu^{(0)}$

transition matrices $P^{(0)}, P^{(1)}, P^{(2)}, \dots$

We want to simulate this MC.

We can obtain the initiation function ψ and the starting value X_0 exactly like in the homogeneous case.

For the updating we need a sequence of updating functions $\phi^{(1)}, \phi^{(2)}, \dots$ which are computed as before, but using at each time step the corresponding transition matrix:

$$\phi^{(n)}(s_i, x) = \begin{cases} s_1 & \text{for } x \in [0, p_{i1}^{(n)}) \\ s_2 & \text{for } x \in [p_{i1}^{(n)}, p_{i1}^{(n)} + p_{i2}^{(n)}) \\ \vdots & \\ s_j & \text{for } x \in \left[\sum_{l=1}^{j-1} p_{il}^{(n)}, \sum_{l=1}^{j} p_{il}^{(n)}\right) \\ \vdots & \\ s_k & \text{for } x \in \left[\sum_{l=1}^{k-1} p_{il}^{(n)}, 1\right] \end{cases}$$

The inhomogeneous MC is simulated by setting

$$\begin{aligned} X_0 &= \psi(U_0) \\ X_1 &= \phi^{(1)}(X_0, U_1) \\ X_2 &= \phi^{(2)}(X_1, U_2) \\ X_3 &= \phi^{(3)}(X_2, U_3) \\ &\vdots \end{aligned}$$

Exercise 3.2 p. 31 Häggström : the choice of the update function is not necessarily unique!

Consider the example of the Gothenburg weather. Show that we get another valid update function if we replace (*) with

$$(**) \quad \phi(s_2, x) = \begin{cases} s_2 & \text{for } x \in [0, 0.75) \\ s_1 & \text{for } x \in [0.75, 1] \end{cases}$$

Solution:

Clearly $\phi(s_i, x)$ is piecewise constant for each s_i. Let's check if (ii) is valid:

Consider s_1:

$\phi(s_1, x) = s_1$ if $x \in I_1 = [0, 0.75)$ $\ell(I_1) = 0.75 = P_{11}$

$\phi(s_1, x) = s_2$ if $x \in I_2 = [0.75, 1]$ $\ell(I_2) = 0.25 = P_{12}$

ok!

Consider s_2 in (**):

$\phi(s_2, x) = s_1$ if $x \in I_2 = [0.75, 1]$ $\ell(I_2) = 0.25 = P_{22}$

$\phi(s_2, x) = s_2$ if $x \in I_1 = [0, 0.75)$ $\ell(I_1) = 0.75 = P_{21}$

ok!

Note that :